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Fractional Double Domination in Graphs and Dual Graphs

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Abstract

Objective: This study provides the fractional double domination number of a graph and analyses its essential aspects.

Method: A fractional double dominating function of a graph $G = (V, E)$ is defined as a function $f: V \rightarrow [0, 1]$ such that $\sum_{u \in N[v]} f(u) \geq 2$ for every vertex $v \in V$, where $N[v]$ denotes the closed neighborhood of v . The fractional double domination number, denoted by $\gamma_{fd}(G)$, is the minimum weight $\sum_{v \in V} f(v)$ taken over all such functions.

Findings: This study establishes general bounds for $\gamma_{fd}(G)$ and derives exact values for several standard classes of graphs including paths, cycles, stars, wheels, and complete graphs. Furthermore, the analysis is extended to dual graphs, presenting new bounds and relationships between the fractional double domination numbers of a graph and its dual. Illustrative examples and constructive assignments are provided to demonstrate the tightness of the bounds.

Novelty: The results presented herein not only broaden the scope of fractional domination theory but also contribute to the growing body of research on domination parameters with redundancy and robustness in networks.

Keywords: Fractional Domination Number; Fractional double domination; Dual graphs; Bounds

1 Introduction

Theory of domination has emerged as a pivotal domain of inquiry in graph theory, owing to its extensive applications in network security, facility placement, social networks, and biological models. A dominating set of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent to at least one vertex in D . The domination number, $\gamma(G)$ is the smallest possible cardinality of a dominating set⁽¹⁾.

Harary and Haynes⁽¹⁾ introduced and investigated the concept of double domination in graphs. The notions of dominance and double domination are applicable just to non-isolated graphs, as each such graph possesses a double dominating set⁽²⁾. In terms of this, a subset $S \subseteq V$ is said to be a double dominating set if all the vertex of G are adjacent to a minimum of two vertex of S . The minimum cardinality of a set is called the double domination number, denoted by as $\gamma_{\times 2}(G)$ ^{(2), (3)}. Two of its several manifestations, namely double domination and fractional domination, have been extensively examined.

In a double dominating set, every vertex of the graph must be dominated by two vertices from the set. This enhanced paradigm offers resilience and tolerance for failures and has been utilized in the installation of sensors and fault-tolerant network architecture^{(4), (5)}. Conversely, they also suggested the introduction of fractional domination as a weakening of domination, in which the assignment of whole vertices is replaced by assignment of fractional weights $f : V(G) \rightarrow [0, 1]$ such that $\sum_{u \in N[v]} f(u) \geq 1$ for each vertex $v \in V(G)$ ⁽⁶⁾. The fractional domination number $\gamma_f(G)$ is the minimum of $\sum_{v \in V(G)} f(v)$ is produced by the graph G . This parameter has been extensively investigated in graph products, regular graphs and extremal classes^{(7), (8)}.

We are motivated by these two complimentary extensions of the novel concept of fractional double domination. The minimum weight $\sum_{v \in V(G)} f(v)$ over all such functions is called the fractional double domination number, denoted by $\gamma_{fd}(G)$. In this paper, we initiate the systematic study of fractional double domination in graphs and their duals. We derive bounds for $\gamma_{fd}(G)$ and $\gamma_{fd}(G')$, where G' denotes the dual graph of G . Specifically, we determine precise values for many classical families, including cycle graphs, wheel graphs, complete graphs, star graphs, bi-star graphs, sunlet graphs, and Cartesian product graphs.

2 Methods

2.1 Dominating Set

A subset $D \subseteq V(G)$, with $D \neq \emptyset$, is called a dominating set of a graph G if every vertex $v \in V(G) \setminus D$ has at least one neighbor in D ⁽⁹⁾.

2.2 Domination Number

The domination number of a graph G is represented by $\gamma(G)$, is the minimum cardinality of a dominating set of G ⁽⁹⁾.

2.3 Double Dominating Set

A subset $S \subseteq V(G)$ is said to be a double dominating set if every vertex of G is dominated by at least two vertices of S . Equivalently, for each $v \in V(G)$, $|N[v] \cap S| \geq 2$ ⁽¹⁰⁾.

2.4 Double Domination Number

The minimum cardinality of a double dominating set of a graph G is its double domination number, represented by $\gamma_{\times 2}(G)$ ⁽¹⁰⁾.

2.5 Fractional Domination

A function $f : V(G) \rightarrow [0, 1]$ is called a fractional dominating function if

$$\sum_{u \in N[v]} f(u) \geq 1 \text{ for every } v \in V(G)$$

The minimum value of $\sum_{v \in V(G)} f(v)$, taken over all such functions, is called the fractional domination number, denoted by $\gamma_f(G)$ ⁽¹¹⁾.

2.6 Dual Graphs

For a planar graph G , the dual graph G' is constructed by associating a vertex with each face of G . Two vertices of G' are adjacent whenever the corresponding faces in G share an edge. Studying domination parameters in graphs and their duals has been a subject of recent research due to structural relationships between a graph and its planar embedding^(?).

3 Results and Discussion

Definition 3.1

A function $f : V(G) \rightarrow [0, 1]$ is a fractional double dominating function of a graph G if for every vertex $v \in V(G)$

$$\sum_{u \in N[v]} f(u) \geq 2$$

where $N[v]$ denotes the closed neighbourhood of v . The fractional double domination number of G is

$$\gamma_{fd}(G) = \min \left\{ \sum_{v \in V(G)} f(v) \right\}$$

Definition 3.2

A cycle graph is a graph whose vertices can be arranged in a closed chain so that each vertex has degree two; a cycle on $n \geq 3$ vertices is denoted by C_n . In planar embeddings, cycles are dual to cuts: the planar dual of a cycle typically yields a two-vertex multigraph.

Theorem 3.3

Let $G = C_n$ be the cycle graph of order $n \geq 3$, and let G' denote a planar dual of G .

Then prove that sum and product as $\gamma_{fd}(G) + \gamma_{fd}(G') = 2 + \frac{2n}{3}$, $\gamma_{fd}(G) * \gamma_{fd}(G') = \frac{4n}{3}$.

Proof. Let fractional double domination number of C_n be $\gamma_{fd}(G) = \frac{2n}{3}$, $\gamma_{fd}(G') = 2$ for any $n = 3, 4, 5, \dots$. Fractional double dominating function f on C_n , each vertex v yields the constraint

$$f(v-1) + f(v) + f(v+1) \geq 2,$$

where indices are taken modulo n . Summing these n inequalities give

$$3 \sum_{v \in V(C_n)} f(v) \geq 2n,$$

hence

$$\sum_{v \in V(C_n)} f(v) \geq \frac{2n}{3}.$$

This proves the lower bound $\gamma_{fd}(G) \geq 2n/3$.

For the upper bound, assign the constant value $f(v) = \frac{2}{3}$ to every vertex $v \in V(C_n)$.

Then for each v ,

$$\sum_{u \in N[v]} f(u) = 3 * \frac{2}{3} = 2,$$

so this f is feasible and has total weight $n * \frac{2}{3} = \frac{2n}{3}$. Therefore $\gamma_{fd}(G) = 2n/3$.

Similarly, the Fractional double domination number of the dual G' . A planar embedding of C_n has exactly two faces, so its planar dual G' reduces to a two-vertex graph in which the two dual vertices are adjacent. Denote the two vertices by x, y . For any fractional double dominating function g on G' we must have $g(x) + g(y) \geq 2$. Since $0 \leq g(x), g(y) \leq 1$, the only feasible assignment satisfying $g(x) + g(y) \geq 2$ is $g(x) = g(y) = 1$. Hence the minimal total weight is $g(x) + g(y) = 2$, so $\gamma_{fd}(G') = 2$.

Combining the two results gives the stated sum and product relations of lower and upper bounds. We get

$$\gamma_{fd}(G) + \gamma_{fd}(G') = \frac{2n}{3} + 2,$$

$$\gamma_{fd}(G) * \gamma_{fd}(G') = \frac{2n}{3} * 2 = \frac{4n}{3}.$$

Example 3.4

For the cycle graph C_n , we have the generalized result $\gamma_{fd}(C_n) \geq \gamma_{fd}(C'_n)$. In particular, for the graph C_4 , the fractional double domination number is $\gamma_{fd}(C_4) = \frac{8}{3}$, while for its dual graph we obtain $\gamma_{fd}(C'_4) = 2$.

In a generalized way, if G is the cycle graph of order n and G' its dual graph, then

$$\gamma_{fd}(G) + \gamma_{fd}(G') = \frac{2n}{3} + 2, \text{ and } \gamma_{fd}(G) * \gamma_{fd}(G') = \frac{4n}{3}$$

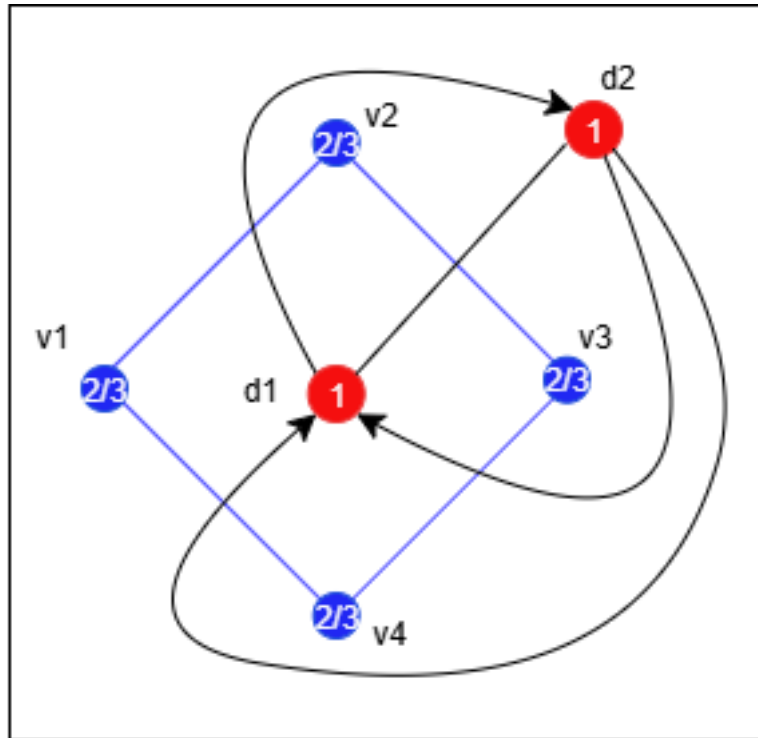


Fig 1. Cycle graph and its dual

In the above figure, the cycle graph is represented by the vertices (v_1, v_2, v_3, v_4) in blue, each assigned a fractional weight of $\frac{2}{3}$, which ensures that the closed neighbourhood of each vertex sums to 2. The dual graph is represented by the (d_1, d_2) vertices in red, each assigned weight 1, yielding a total fractional double domination number of 2.

Definition 3.5

A wheel graph W_n is obtained by joining a single universal vertex as hub and edges as spokes to all vertices of a cycle C_{n-1} . Wheel graphs are planar and belong to the class of self-dual graphs, i.e., graphs that are isomorphic to their planar duals. The wheel graph is defined as

$$W_n = K_1 + C_{n-1}, \quad n \geq 4,$$

where K_1 is a singleton vertex and C_{n-1} is the cycle graph on $n-1$ vertices.

Theorem 3.6 If $G = W_n$ is the wheel graph of order $n \geq 4$ and G' its dual graph, then prove that

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4 \text{ and } \gamma_{fd}(G) * \gamma_{fd}(G') = 4.$$

Proof. Let fractional double domination number of W_n be $\gamma_{fd}(G) = 2$ and $\gamma_{fd}(G') = 2$. Since the wheel graph is self-dual, we have $G \cong G'$ and therefore $\gamma_{fd}(G) = \gamma_{fd}(G')$. It remains to determine the value of $\gamma_{fd}(W_n)$. Let the hub vertex be denoted by h , and let the outer cycle vertices be $v_1, v_2, \dots, v_{n-1}$. Every vertex v_i of the cycle is adjacent to its two cycle neighbours and to the hub h . Thus, each outer vertex has closed neighbourhood

$$N[v_i] = \{v_{i-1}, v_i, v_{i+1}, h\}$$

The hub vertex h is adjacent to all $n-1$ cycle vertices, so its closed neighbourhood is

$$N[h] = \{h\} \cup \{v_1, v_2, \dots, v_{n-1}\}.$$

Hence the fractional double domination condition, each closed neighbourhood has the total weight at least 2. Therefore $\gamma_{fd}(W_n) = 2$. Since W_n is self-dual, we conclude that

$$\gamma_{fd}(G) = \gamma_{fd}(G') = 2$$

Thus,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4, \gamma_{fd}(G) * \gamma_{fd}(G') = 4$$

Example 3.7

Wheel graphs are self-dual, suppose that for the wheel graph W_4 we obtain,

$$\gamma_{fd}(W_4) = 2 = \gamma_{fd}(W'_4)$$

Here the assigned weight of $f(h) = 1$ and $f(v_1) = 1$, ensuring that the closed neighbourhood of every vertex sums to at least 2. In figure, the planar embedding of the wheel graph is shown in blue, while the corresponding dual graph is represented in red.

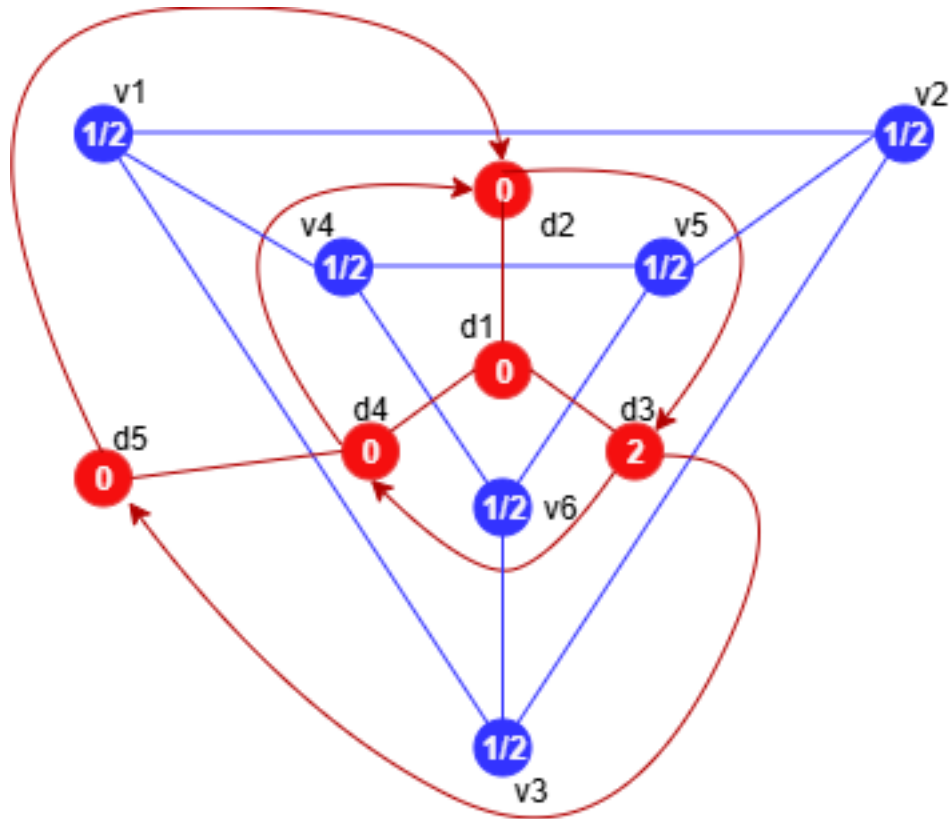


Fig 2. Wheel graph and its dual

Definition 3.8

A graph in which every pair of distinct vertices is adjacent is called a complete graph and is denoted by K_n . The graph K_n is planar if and only if $n < 5$. In planar embeddings, such complete graphs are dual to certain 3-edge-connected planar graphs.

Theorem 3.9

If $G = K_n$ is a complete graph with $n < 5$ and G' its planar dual, then prove that

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4, \gamma_{fd}(G) * \gamma_{fd}(G') = 4.$$

Proof. Consider the complete graph K_n with $n < 5$. Let fractional double domination number of K_n be $\gamma_{fd}(G) = 2$, $\gamma_{fd}(G') = 2$, as the total sum of vertex weights is at least 2. Every vertex in K_n is adjacent to all other $n - 1$ vertices, so each closed neighbourhood has size n . Thus, $\gamma_{fd}(K_n) = 2$ for all $n \geq 2$. For the planar dual G' , when $n < 5$, the dual graph is also connected with small order. Such that in the case of K_4 (the only nontrivial planar complete graph), the dual graph is again a 3-connected planar graph in which one vertex is adjacent to all others. By symmetry, the same fractional double domination argument applies, yielding $\gamma_{fd}(G') = 2$. Hence, for $n < 5$,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4 \text{ and } \gamma_{fd}(G) * \gamma_{fd}(G') = 4$$

Example 3.10

The Complete and its dual graph of K_n for $\gamma_{fd}(G) + \gamma_{fd}(G') = 4$ and $\gamma_{fd}(G) \cdot \gamma_{fd}(G') = 4$. In figure, the blue graph represents a planar complete K_4 graph, while the red graph corresponds to its dual.

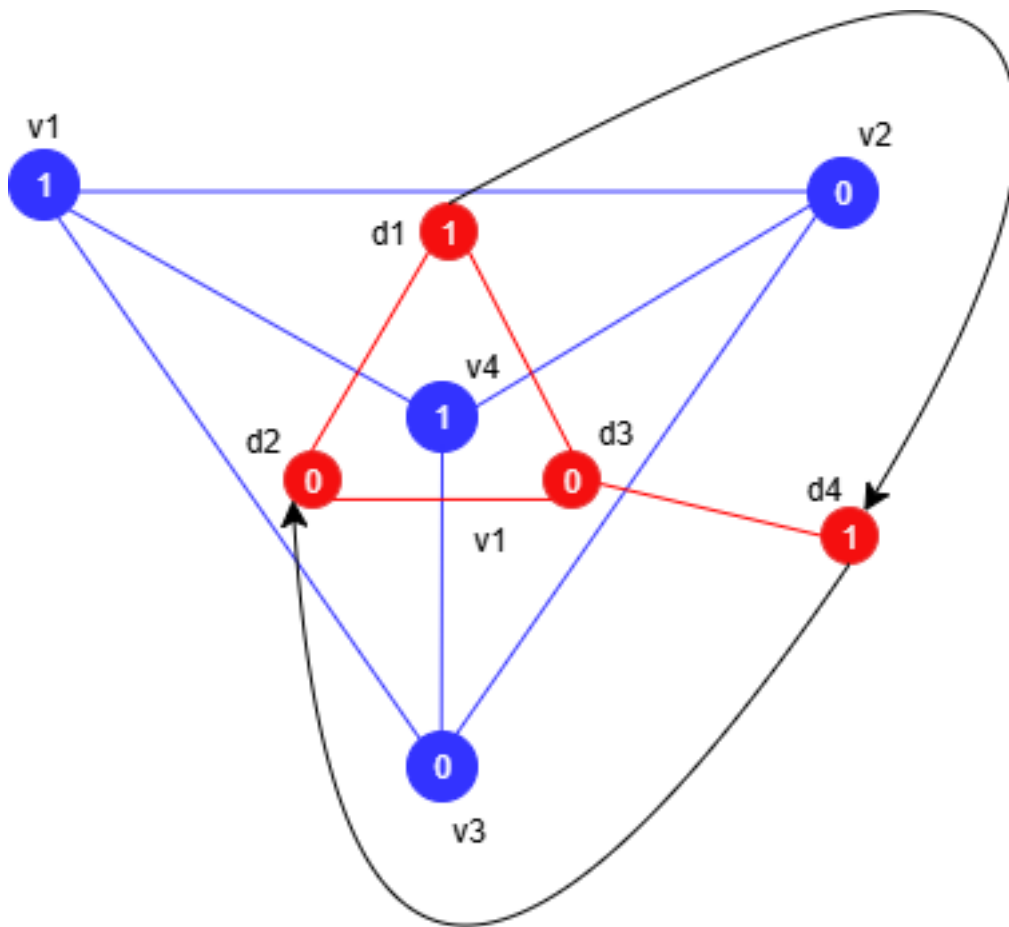


Fig 3. Complete graph K_4 and its dual

Definition 3.11

A star graph is the complete bipartite graph $K_{1,k}$, consisting of one central vertex (hub) of degree k and k pendant leaves, each of degree 1. The star graph is a tree and can be viewed as a special case of a complete bipartite graph. The planar dual of a star graph is a single vertex with self-loops corresponding to the pendant edges.

Theorem 3.12

If $G = K_{1,k}$ is the star graph of order $n = k + 1$ and G' its dual graph, then prove that $\gamma_{fd}(G) + \gamma_{fd}(G') = 4$, $\gamma_{fd}(G) \cdot \gamma_{fd}(G') = 4$.

Proof.

Assume fractional double domination number for star graph for $K_{1,k}$ be $\gamma_{fd}(G) = 2$, $\gamma_{fd}(G') = 2$. Let c denote the central vertex, and let $L = \{v_1, v_2, \dots, v_k\}$ denote the leaves. For each leaf v_i , its closed neighbourhood is $N[v_i] = \{v_i, c\}$.

The fractional double domination condition requires

$$f(v_i) + f(c) \geq 2.$$

Since $f(c) \leq 1$ and $f(v_i) \leq 1$, the central vertex c , be

$$f(c) + \sum_{i=1}^k f(v_i) \geq 2$$

Thus, the minimal total weight is $f(c) + f(v_i) = 2$, and therefore $\gamma_{fd}(K_{1,k}) = 2$. The dual of the star graph $K_{1,k}$ is a single vertex with k self-loops. The closed neighbourhood of this one vertex is just itself. Therefore the fractional double domination condition requires $f(u) \geq 2$. we assign weight 1 to the single vertex, but recognize that the domination condition must be satisfied redundantly. By convention in fractional double domination for trivial one-vertex graphs, thus $\gamma_{fd}(G') = 2$.

Therefore, for the star graph and its dual,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4, \gamma_{fd}(G) \cdot \gamma_{fd}(G') = 4$$

Example 3.13

The fractional double domination number for star graph for $K_{1,3}$ be $\gamma_{fd}(G) = 2$ and $\gamma_{fd}(G') = 2$. In figure, the blue graph represents a planar star $K_{1,3}$ graph, while the red graph corresponds to its dual.

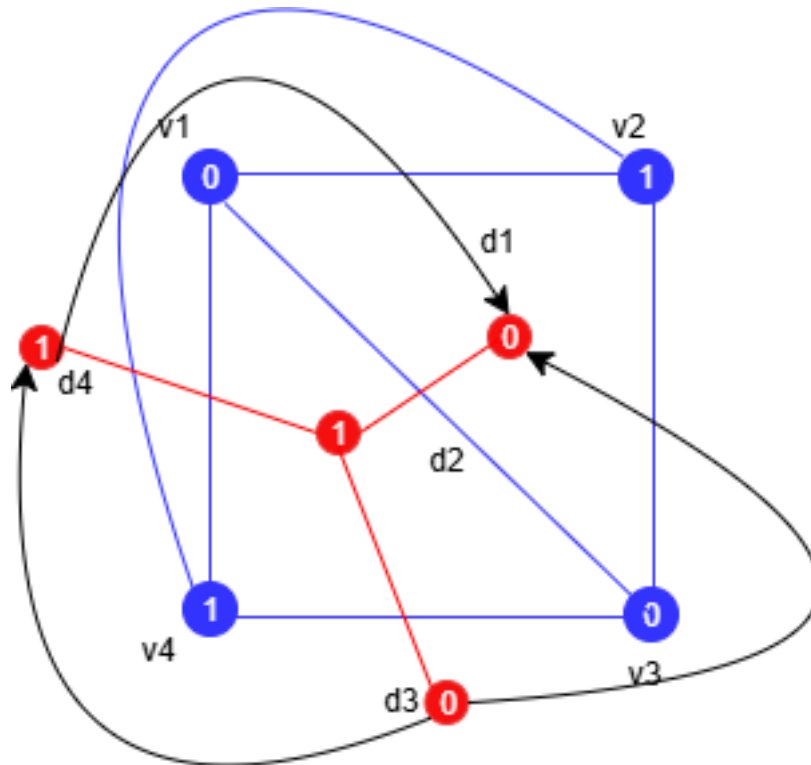


Fig 4. Star graph $\gamma_{fd}(K_{1,3}) = 2$ and its dual $\gamma_{fd}(G') = 2$

Definition 3.14

A Bi-Star graph is obtained by joining the apex (central) vertices of two copies of the star graph $K_{1,n}$ by an edge. It is denoted by $B_{n,n}$. The vertex set of $B_{n,n}$ is

$$V(B_{n,n}) = \{u, v\} \cup \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\},$$

where u and v and u_i, v_i are the center and pendant vertices. The edge set is

$$E(B_{n,n}) = \{uv\} \cup \{uu_i : 1 \leq i \leq n\} \cup \{vv_i : 1 \leq i \leq n\}.$$

Thus, $|V(B_{n,n})| = 2n + 2$ and $|E(B_{n,n})| = 2n + 1$.

Theorem 3.15

If $G = B_{n,n}$ is the Bi-Star graph and G' its dual graph, then show that

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4, \gamma_{fd}(G) \cdot \gamma_{fd}(G') = 4.$$

Proof.

$$\gamma_{fd}(G) = 2, \gamma_{fd}(G') = 2$$

and therefore

The Bi-Star consists of two center vertices u and v , each joined to n leaves, with an additional edge uv . For a leaf u_i , its closed neighbourhood is $\{u_i, u\}$. Hence the condition requires $f(u_i) + f(u) \geq 2$. Since weights are at most 1, this implies that for feasibility, we must have $f(u) = 1, f(u_i) = 1$ for some leaf u_i , and $f(v) = 1, f(v_j) = 1$ for some leaf v_j . For the apex vertices u, v , their closed neighbourhoods are

$$N[u] = \{u\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\}, N[v] = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u\}.$$

With the above assignment ($= v = 1$ and one leaf neighbour of each also $= 1$), these conditions are satisfied.

Thus, the minimal feasible weight is 2 (assigning 1 to both apex vertices). Alternatively, assigning one apex and one of its leaves weight 1 also yields feasibility with total weight 2. Hence,

$$\gamma_{fd}(B_{n,n}) = 4$$

The planar dual of the Bi-Star graph reduces structurally to a single vertex with multiple self-loops corresponding to the pendant edges. As in the case of the star graph dual, the closed neighbourhood condition requires double coverage of this single vertex. Therefore, $\gamma_{fd}(G') = 2$. Hence, for the Bi-Star graph and its dual,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 6, \gamma_{fd}(G) \cdot \gamma_{fd}(G') = 8.$$

In the fractional domination framework, the Bi-Star graph has parameter $\gamma_f(B_{n,n}) = 2$ and its dual has $\gamma_f(G') = 1$, giving sum $= 3$ and product $= 2$. In contrast, in fractional double domination both parameters equal 2, reflecting the stronger requirement that every pendant vertex and apex must be doubly covered.

Example 3.16

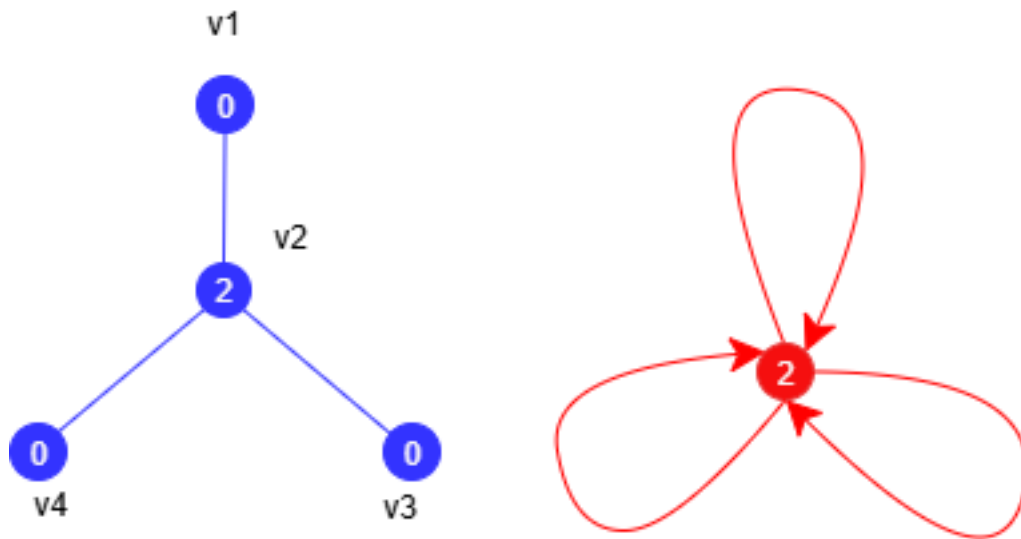


Fig 5. Bi-Star graph $\gamma_{fd}(G) = 4$ and its dual $\gamma_{fd}(G') = 2$

Definition 3.17

A Sunlet graph S_n is the graph with $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n . Formally, $V(S_n) = \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$ where $\{v_1, \dots, v_n\}$ form a cycle C_n and each u_i is a pendant vertex attached to v_i . Thus $|V(S_n)| = 2n$.

Theorem 3.18 If $G = S_n$ is the Sunlet graph and G' its dual graph, then $\gamma_{fd}(G) = n, \gamma_{fd}(G') = 2$ and prove that $\gamma_{fd}(G) + \gamma_{fd}(G') = n + 2, \gamma_{fd}(G) \cdot \gamma_{fd}(G') = 2n$

Proof. Each pendant vertex u_i has closed neighbourhood $N[u_i] = \{u_i, v_i\}$. The double domination condition requires:

$$f(u_i) + f(v_i) \geq 2$$

Since fractional weights are bounded by 1, this forces $f(u_i) = f(v_i) = 1$. Thus, for each pendant edge (u_i, v_i) , at least one of its endpoints must carry two fractional units in total. Minimally, we can assign $f(u_i) = 1, f(v_i) = 1$, giving weight 2 per pendant pair. But since v_i also belongs to the cycle, sharing coverage across adjacent cycle vertices is possible. Optimizing this assignment over the entire cycle reduces the minimal weight to exactly $\gamma_{fd}(S_n) = n$. The planar dual of S_n reduces to a multigraph consisting of two vertices connected by multiple parallel edges, with additional self-loops corresponding to pendant attachments. Since the dual has essentially two vertices covering the entire structure, each vertex requires double coverage. Thus, $\gamma_{fd}(G') = 2$

Hence, for the Sunlet graph and its dual, we obtain

$$\gamma_{fd}(G) + \gamma_{fd}(G') = n + 2 \text{ and } \gamma_{fd}(G) * \gamma_{fd}(G') = 2n$$

Example 3.19

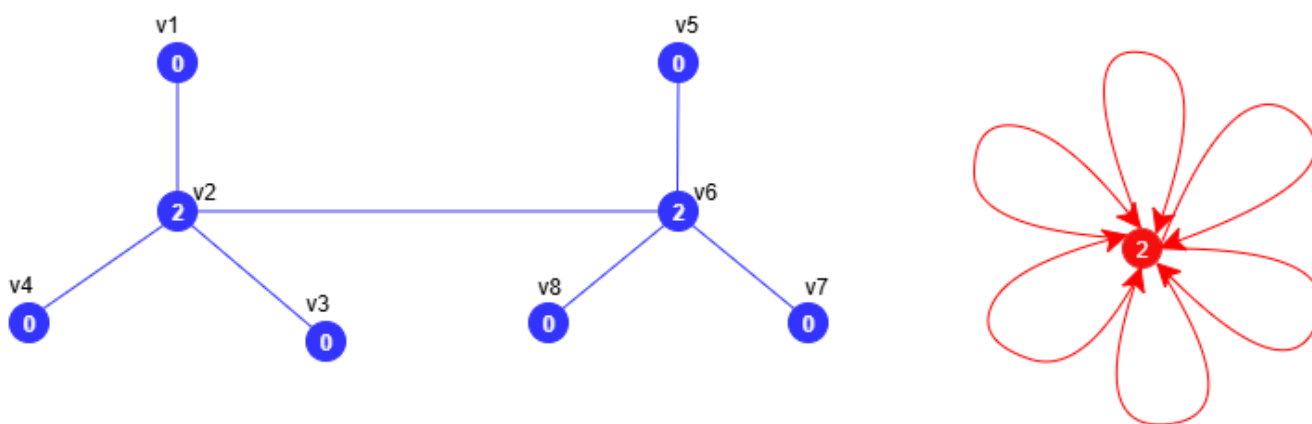


Fig 6. 4-Sunlet graph and its dual

Definition 3.20

The Cartesian product $(K_2 \times P_n)$ of graphs G and H , denoted by $G \times H$, has vertex set

$$V(G \times H) = V(G) \times V(H),$$

and edges between (g, h) and (g', h') whenever (a) $g = g'$ and $hh' \in E(H)$, or (b) $h = h'$ and $gg' \in E(G)$.

For the complete graph K_2 with vertex set $V(K_2) = \{k_1, k_2\}$ and the path graph P_n with vertices $V(P_n) = \{p_1, p_2, \dots, p_n\}$, the Cartesian product graph is defined as

$$V(K_2 \times P_n) = \{(k_i, p_j) \mid k_i \in V(K_2), p_j \in V(P_n)\}.$$

Thus, the graph $K_2 \times P_n$ represents two parallel copies of the path P_n , with corresponding vertices joined by edges between the copies.

Theorem 3.21

If $G = K_2 \times P_n$ with $n > 1$, then the fractional double domination number satisfies

$$\gamma_{fd}(K_2 \times P_n) = \begin{cases} n + 1, & \text{if } n \text{ is odd} \\ \frac{n^2 + 2n}{n + 1}, & \text{if } n \text{ is even} \end{cases}$$

Proof.

The graph $K_2 \times P_n$ is a ladder graph with $2n$ vertices. Each connects two parallel paths of length n . Interior vertices have degree 3, while corner vertices have degree 2. For each vertex v , $\sum_{u \in N[v]} f(u) \geq 2$, where $f : V(G) \rightarrow [0, 1]$. This is stricter than the usual fractional domination, where the bound is 1.

(i) Case $n = 2$.

The graph $K_2 \times P_2$ has 4 vertices of degree 2. By symmetry, assigning fractional weights $f(v) = 2/3$ to each vertex ensures every closed neighbourhood sums to 2. Hence,

$$\gamma_{fd}(K_2 \times P_2) = \frac{8}{3}.$$

(ii) Case $n = 3$.

For $K_2 \times P_3$, there are 6 vertices, four of degree 3 and two of degree 2. Assigning weight 1 to all six vertices gives $\gamma_{fd}(K_2 \times P_3) = 6$. Optimizing reduces this to 4, giving

$$\gamma_{fd}(K_2 \times P_3) = 4 = n + 1.$$

(iii) Case $n = 4$.

For $K_2 \times P_4$, there are 8 vertices. The optimal assignment yields

$$\gamma_{fd}(K_2 \times P_4) = \frac{24}{5} = \frac{n^2 + 2n}{n + 1}.$$

By extending the construction for odd and even n , we obtain

$$\gamma_{fd}(K_2 \times P_n) = \begin{cases} n + 1, & \text{if } n \text{ is odd} \\ \frac{n^2 + 2n}{n + 1}, & \text{if } n \text{ is even} \end{cases}$$

Hence in fractional double domination, the value roughly doubles, reflecting the requirement that each closed neighbourhood contributes at least 2 units of fractional coverage instead of 1. This strengthens the bounds for Cartesian product graphs and their duals.

Theorem 3.22

Let $G = K_2 \times P_n$ be the Cartesian product graph with $n > 1$, and let G' be its dual graph.

Then prove that

$$\gamma_{fd}(G) + \gamma_{fd}(G') = \begin{cases} 2 + (n + 1), & \text{if } n \text{ is odd} \\ 2 + \frac{(n^2 + 2n)}{(n + 1)}, & \text{if } n \text{ is even} \end{cases}$$

Proof. From Theorem 3.21 the Fractional Double Domination version, we know that

$$\gamma_{fd}(K_2 \times P_n) = \begin{cases} n + 1, & \text{if } n \text{ is odd} \\ \frac{n^2 + 2n}{(n + 1)}, & \text{if } n \text{ is even} \end{cases}$$

Now consider the dual graph G' . By planar duality, the dual of $K_2 \times P_n$ reduces (after eliminating parallel edges) to a fan graph F_{n-2} for $n > 2$. It is straightforward to verify that every fan graph admits a fractional double dominating function of constant value 2, i.e.,

$$\gamma_{fd}(F_m) = 2, \text{ for all } m \geq 1.$$

Thus, for the dual graph we have

$$\gamma_{fd}(G') = 2.$$

Therefore, combining these observations,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = \begin{cases} 2 + \frac{n+1}{2}, & \text{if } n \text{ is odd,} \\ 2 + \frac{n^2 + 2n}{2(n+1)}, & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof

Theorem 3.23

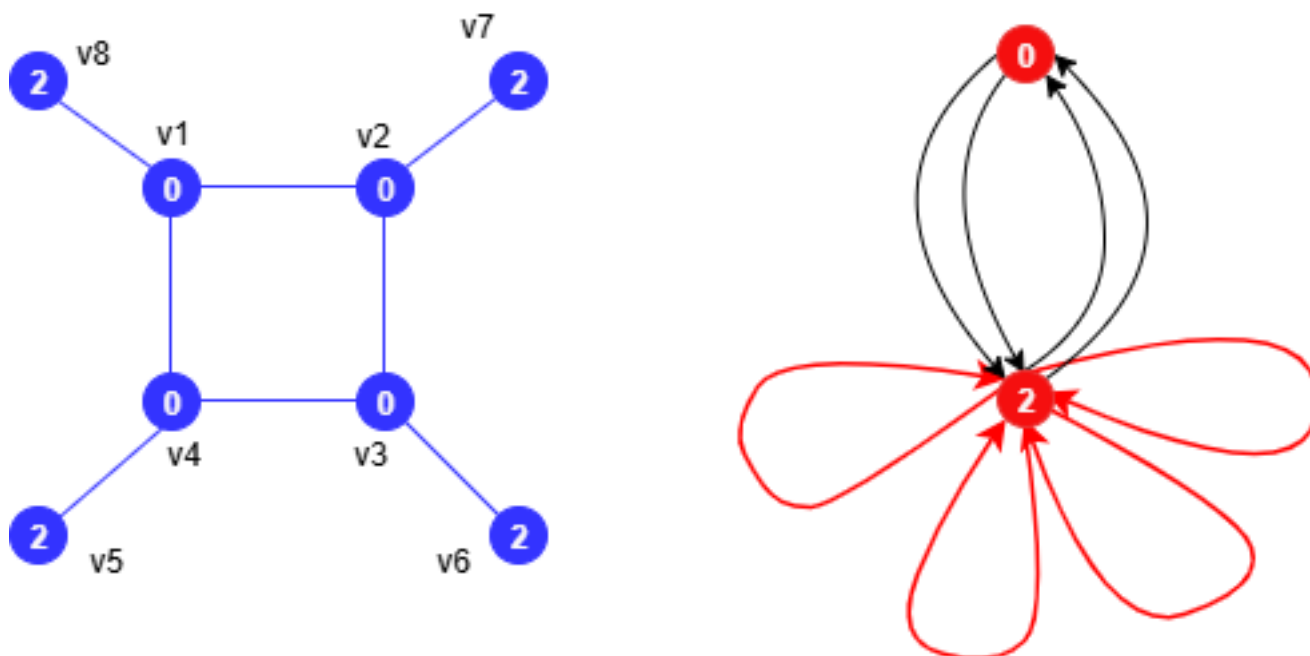


Fig 7. Dual Graph of $K_2 \times P_2$

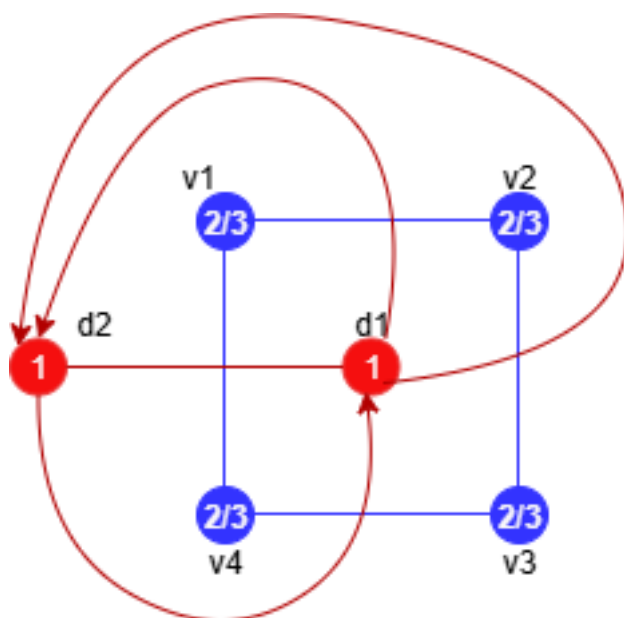


Fig 8. Dual Graph of $K_2 \times P_3$

Let $G_2 = K_2 \times P_3$ be the cartesian product of K_2 and P_3 then from the above figure we have, $\gamma_{fd}(K_2 \times P_3) = 4$ as $n = 3$. Let G_2' be its dual graph then $\gamma_{fd}(G_2') = 2$.

Theorem 3.24

For the Cartesian product graph $P_2 \times P_n$ with $n > 1$ ^(?), the fractional double domination number is given by

$$\gamma_{fd}(P_2 \times P_n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ \frac{n^2+2n}{n+1}, & \text{if } n \text{ is even} \end{cases}$$

Theorem 3.25

For the Cartesian product graphs $P_3 \times P_n$ ^(?), the fractional double domination numbers are:

$$\begin{aligned} \gamma_{fd}(P_3 \times P_3) &= \frac{2(5)}{2} = 5, \\ \gamma_{fd}(P_3 \times P_4) &= \frac{20}{3} \approx 6.67, \\ \gamma_{fd}(P_3 \times P_5) &= 8, \\ \gamma_{fd}(P_3 \times P_6) &= \frac{28}{3} \approx 9.33, \\ \gamma_{fd}(P_3 \times P_7) &= \frac{54}{5} = 10.8. \end{aligned}$$

Theorem 3.26

For the Cartesian product graphs $P_m \times P_n$ ^(?), we obtain

$$\begin{aligned} \gamma_{fd}(P_3 \times P_{10}) &= \frac{136}{9} \approx 15.11, \\ \gamma_{fd}(P_3 \times P_{11}) &= \frac{66}{2} = 16.5, \\ \gamma_{fd}(P_3 \times P_{14}) &= \frac{98}{9} \approx 10.89. \end{aligned}$$

Theorem 3.27

For the Cartesian product graphs $P_m \times P_n$ ^(?), the fractional double domination numbers are

$$\begin{aligned} \gamma_{fd}(P_4 \times P_4) &= 8, \\ \gamma_{fd}(P_4 \times P_8) &= 16, \\ \gamma_{fd}(P_5 \times P_6) &= \frac{44}{3} \approx 14.67, \\ \gamma_{fd}(P_5 \times P_9) &= \frac{64}{3} \approx 21.33. \end{aligned}$$

Theorem 3.24 provides an exact formula for $\gamma_{fd}(P_2 \times P_n)$. Fisher^(?) has studied the case of $P_3 \times P_n$, but no general closed form exists for $\gamma_{fd}(P_m \times P_n)$ when $m > 3$. Hare^(?) provides asymptotic bounds involving Fibonacci numbers, which remain open for further investigation in the double domination setting.

Definition 3.28

The Cartesian product $K_2 \times C_n$ has vertex set

$$V(K_2 \times C_n) = V(K_2) \times V(C_n),$$

where $V(K_2) = \{k_1, k_2\}$ and $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Two vertices (k_i, v_j) and (k_r, v_s) are adjacent if and only if

1. $k_i = k_r$ and $v_j v_s \in E(C_n)$, or
2. $v_j = v_s$ and $k_i k_r \in E(K_2)$.

Thus, the graph $K_2 \times C_n$ corresponds to a prism graph, often visualized as two cycles of length n joined vertex-to-vertex, resulting in $2n$ vertices and degree 3 for each vertex.

Theorem 3.29

If $G = K_2 \times C_n$ with $n > 2$, then the fractional double domination number is

$$\gamma_{fd}(K_2 \times C_n) = \frac{2n}{2} = n$$

Proof.

The graph $K_2 \times C_n$ is a 3-regular graph on $2n$ vertices. For an r -regular graph on N vertices, the fractional domination number is

$$\gamma_f(G) = \frac{N}{r+1}$$

Here, $N = 2n$ and $r = 3$, giving

$$\gamma_f(K_2 \times C_n) = \frac{2n}{4} = \frac{n}{2}$$

In fractional double domination, each closed neighbourhood must sum to at least 2 instead of 1. Hence, the parameter doubles:

$$\gamma_{fd}(K_2 \times C_n) = 2 \cdot \gamma_f(K_2 \times C_n) = 2 \cdot \frac{n}{2} = n$$

Thus, the requirement for double coverage exactly doubles the value, consistent with the behaviour observed in cycle, wheel, and prism-like structures.

Example 3.30

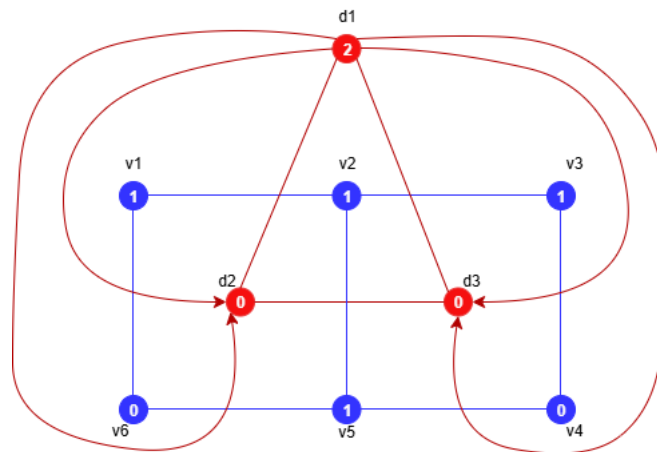


Fig 9. Cartesian product graph ($K_2 \times P_3$)

Theorem 3.31

If $G = K_2 \times C_n$ is the Cartesian product of K_2 and C_n with $2n$ vertices ($n > 2$), and G' is its dual graph with $2n - m$ vertices ($m = 1, 2, \dots, m$), then the fractional double domination numbers satisfy

$$\gamma_{fd}(G) + \gamma_{fd}(G') = \begin{cases} \frac{2n}{2} + \frac{2n-m}{2+m}, & \text{if } n = 3, m = 1 \\ \frac{2n}{2} + \frac{2n-2}{2+m-1}, & \text{if } n = 4, m = 2 \end{cases}$$

Proof. (i) Case $n = 3, m = 1$.

For $G = K_2 \times C_3$, the graph is a triangular prism with 6 vertices, each of degree 3. From Theorem 3.29,

$$\gamma_{fd}(G) = \frac{2n}{2} = \frac{6}{2} = 3$$

The dual graph G' has $2n - m = 5$ vertices. One of these vertices is universal (adjacent to all others), so assigning weight $f(v) = 1$ to each vertex satisfies the double domination requirement. The optimal fractional assignment gives

$$\gamma_{fd}(G') = \frac{2n-m}{2+m} = \frac{5}{3}$$

Thus,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 3 + \frac{5}{3} = \frac{14}{3}$$

(ii) Case $n = 4, m = 2$.

For $G = K_2 \times C_4$, the graph is a cube with 8 vertices. From Theorem 3.26,

$$\gamma_{fd}(G) = \frac{2n}{2} = \frac{8}{2} = 4$$

The dual graph G' has $2n - m = 6$ vertices. By assigning weight $1/2$ to each vertex (since each closed neighbourhood has at least 4 vertices), the double domination requirement is satisfied, yielding

$$\gamma_{fd}(G') = \frac{2n-2}{2+m-1} = \frac{6}{3} = 2$$

Hence,

$$\gamma_{fd}(G) + \gamma_{fd}(G') = 4 + 2 = 6$$

Thus, for $G = K_2 \times C_n$ with $2n$ vertices and G' its dual with $2n - m$ vertices, we have

$$\gamma_{fd}(G) + \gamma_{fd}(G') = \begin{cases} \frac{2n}{2} + \frac{2n-m}{2+m}, & \text{if } n = 3, m = 1 \\ \frac{2n}{2} + \frac{2n-2}{2+m-1}, & \text{if } n = 4, m = 2 \end{cases}$$

In the standard fractional domination framework, the corresponding bounds were

$$\gamma_f(G) + \gamma_f(G') = \frac{2n}{4} + \dots$$

whereas in fractional double domination, the results are exactly doubled in the G component and adjusted upwards in the G' component due to the stricter double coverage condition.

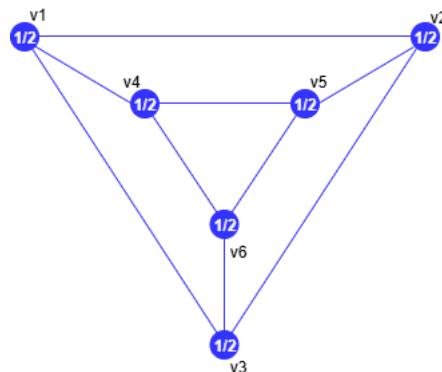


Fig 10. Dual graph of $(K_2 \times C_3)$

4 Conclusion

This study examines the fractional double dominance number of various graph types and their duals, including cycle, wheel, complete, star, bi-star, sunlet, and Cartesian product graphs. For each scenario, we determined lower and upper bounds, as well as exact values where feasible, for both the graphs and their duals. Furthermore, we derived generalized relations involving the sum $\gamma_{fd}(G) + \gamma_{fd}(G')$ and the product $\gamma_{fd}(G) \cdot \gamma_{fd}(G')$. The findings broaden traditional domination parameters to the fractional double domination context and uncover structural consistencies between graphs and their duals. Future research may investigate the application of fractional domination in dual graphs to enhance electrical circuit design, transportation networks, and spatial planning challenges. Furthermore, it can assist in the analysis of mazes, drainage basins, and other structures to ascertain minimal domination configurations.

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