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Finite Difference Scheme for a Coupled System of Singularly Perturbed Time-Dependent Delay Initial Value Problems with same perturbation parameter

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Abstract

Objectives: This article focuses on the development of finite difference scheme for solving a coupled system of singularly perturbed time-dependent Robin-type initial value problems with delay and same perturbation parameter. **Methods:** A mesh refinement-based finite difference scheme is constructed on uniform and piecewise uniform Shishkin mesh on the temporal and spatial domains, respectively. In order to get a better analysis of the solution's layer phenomena, the solution component is further decomposed into regular and layer components. Analytical and stability results are derived. Using bounds of solution components, the resulting error estimate is established. **Findings:** The proposed scheme proved to have almost first-order convergence. To support the theoretical findings, an example problem is constructed which validates the proposed scheme. **Novelty:** The present work focuses on the study of a solution's layer behavior when the perturbation parameter is the same, rather than different, and provides detailed analytical and numerical analysis.

Keywords: Coupled system; Singular perturbation problems; Robin initial conditions; Delay; Shishkin mesh

1 Introduction

Singular perturbation problems (SPPs) have a wide range of applications in the field of applied mathematics and engineering due to their boundary layer phenomena. The boundary layer emerges in the region where the solution encounters rapid and drastic variation. The emergence of the boundary layer happens due to the presence of the perturbation parameter (ε), which multiplies the term that involves higher-order derivatives. Classical numerical schemes on uniform mesh often fail to intercept the solution's layer phenomena. Researchers developed a specialised numerical scheme on a piecewise uniform Shishkin mesh to address the difficulties that arise in resolving the layer phenomena, which provides a better solution approximation. In recent years, several researchers have investigated finite difference method[(1)]-[(2)], finite element method[(3)], virtual element method[(4)] for solving SPPs. Some of the recent works are given below. Finite Difference Method (FDM) was developed for Singularly Perturbed Initial Value Problems (SPIVPs) with Robin initial conditions[5]. SPIVPS with delay and Robin initial conditions were investigated using FDM on Shishkin mesh in[6]. Crank-Nicolson scheme was implemented for parabolic SPPs of reaction-diffusion type with delay and Discontinuous Source Term (DST)[7]. System of 1-D hyperbolic delay differential equations were addressed using the methods of lines and Runge-Kutta methods in[8]. Layer-resolving FDM was used to solve a system of two time-dependent SPPs with spatial delay and DST[9]. Motivated by the articles present in the existing literature and work proposed in[10], we have developed a finite difference scheme for a special case of the problem considered in[10]. Specifically, we consider a coupled system of singularly perturbed time-dependent delay initial value problems where the convection term is multiplied by the same perturbation parameter ε. The solution exhibits initial and interior layers owing to the presence of ε and delay terms. To tackle the difficulty involved in resolving the layer behavior, we have inherited a layer-adapted Shishkin mesh on finite difference scheme. The proposed scheme precisely captures the solution's layer behavior, ensuring first-order convergence in space and time.

2 Methodology

2.1 Problem Formation

For all $(x,t) \in \Psi = (0,2] \times (0,T]$

$$\vec{\mathcal{L}}\vec{\omega}(x,t) = \frac{\partial \vec{\omega}}{\partial t}(x,t) + E\,\frac{\partial \vec{\omega}}{\partial x}(x,t) + R(x,t)\vec{\omega}(x,t) + S(x,t)\vec{\omega}(x-1,t) = \vec{f}(x,t) \tag{1}$$

$$\vec{\beta}\vec{\omega}(x,t) = \vec{\omega}(x,t) - E\frac{\partial \vec{\omega}}{\partial x}(x,t) = \vec{\phi}_L(x,t) \; on \; \Psi^* = [-1,0] \times (0,T] \tag{2}$$

$$\vec{\omega}(x,0) = \vec{\phi}_B(x) \; on \; \xi_B = [0,2] \times 0. \tag{3}$$

For all $(x,t) \in \overline{\Psi}$,$\vec{\omega}(x,t) = (\omega_1(x,t),\omega_2(x,t))^T$ and $\vec{f}(x,t) = (f_1(x,t), f_2(x,t))^T$. Here, $E,\ R(x,t)$ and $S(x,t)$ are 2x2 matrices. Let $\mathcal{E} = diag(\vec{\varepsilon}) = (\varepsilon,\varepsilon)$ with $0 < \varepsilon << 1$ and $\mathcal{S} = diag(\vec{s}) = (s_1(x,t), s_2(x,t))$. Let $\vec{\mathcal{L}}\vec{\omega} = \vec{f}$ on Ψ be the operator form of (1), where $\vec{\mathcal{L}} = \frac{\partial}{\partial t} + E\frac{\partial}{\partial x} + R + S$.

The following constraints hold throughout this article:

$$r_{ii}(x,t) > \sum_{\substack{j \neq i \\ j=1}}^{2} \left| r_{ij}(x,t) + s_i(x,t) \right| \text{ and } s_i(x,t), r_{ij}(x,t) \leq 0 \text{ for } 1 \leq i \neq j \leq 2, \tag{4}$$

$$0 < \alpha < \min_{\substack{x \in \overline{\Psi} \\ 1 \leq i \leq 2}} \left\{ \sum_{j=1}^{2} r_{ij}(x,t) + s_i(x,t) \right\}, \text{ for any positive number } \alpha \tag{5}$$

The problem (1) is rewritten as

$$\vec{\mathcal{L}_1}\vec{\omega}(x,t) = \frac{\partial \vec{\omega}}{\partial t}(x,t) + E\frac{\partial \vec{\omega}}{\partial x}(x,t) + R(x,t)\vec{\omega}(x,t) = \vec{g}(x,t) \text{ on } \Psi^- = (0,1] \times (0,T], \tag{6}$$

$$\begin{aligned} \vec{\mathcal{L}_2}\vec{\omega}(x,t) &= \frac{\partial \vec{\omega}}{\partial t}(x,t) + E\frac{\partial \vec{\omega}}{\partial x}(x,t) + R(x,t)\vec{\omega}(x,t) + S(x,t)\vec{\omega}(x-1,t) \\ &= \vec{f}(x,t) \text{ on } \Psi^+ = (1,2] \times (0,T], \end{aligned} \tag{7}$$

where $\vec{g}(x,t) = \vec{f}(x,t) - S(x,t)\vec{\phi}_L(x-1,t)$.

2.2 Analytical Results

Lemma 1

Assume that $R(x,t)$ and $S(x,t)$ are functions satisfying the conditions (4) and (5). Consider a function $\vec{\zeta}(x,t)$ in the domain of $\vec{\mathcal{L}}$ such that $\vec{\beta}\vec{\zeta}(x,t) \geq \vec{0}$ on Ψ^* and $\vec{\zeta}(x,0) \geq \vec{0}$ on ξ_B. Then $\vec{\mathcal{L}_1}\vec{\zeta}(x,t) \geq \vec{0}$ on Ψ^- and $\vec{\mathcal{L}_2}\vec{\zeta}(x,t) \geq \vec{0}$ on Ψ^+ implies that $\vec{\zeta}(x,t) \geq \vec{0}$ on $\overline{\Psi}$.

Proof.

Consider $\zeta_{i^\star}(x^\star,t^\star) = \min_{\substack{(x,t)\epsilon\overline{\Psi} \\ i=1,2}} \zeta_i(x,t)$. If $\zeta_{i^\star} \geq 0$, the result is trivial. Assume the condition when $\zeta_{i^\star} < 0$, then,

$$\left(\vec{\beta}\vec{\zeta}\right)_{i^\star}(x^\star,t^\star) = \zeta_{i^\star}(x^\star,t^\star) - \varepsilon\frac{\partial\zeta_{i^\star}}{\partial x}(x^\star,t^\star) < 0,$$

which contradicts the initial assumption.

For $t^\star = 0, \zeta_{i^\star}(x^\star,0) < 0$, which also contradicts with the initial assumption.

For $(x^\star,t^\star) \in \Psi^-$ *and* $(x^\star,t^\star) \in \Psi^+$,

$$\begin{aligned}
\left(\vec{\mathcal{L}_1}\,\vec{\zeta}\right)_{i^\star} &= \tfrac{\partial\zeta_{i^\star}}{\partial t}(x^\star,t^\star) + \varepsilon\,\tfrac{\partial\zeta_{i^\star}}{\partial x}(x^\star,t^\star) + r_{i^\star1}(x^\star,t^\star)\zeta_1(x^\star,t^\star) + r_{i^\star2}(x^\star,t^\star)\zeta_2(x^\star,t^\star) < 0,\\
\left(\vec{\mathcal{L}_2}\,\vec{\zeta}\right)_{i^\star} &= \tfrac{\partial\zeta_{i^\star}}{\partial t}(x^\star,t^\star) + \varepsilon\,\tfrac{\partial\zeta_{i^\star}}{\partial x}(x^\star,t^\star) + r_{i^\star1}(x^\star,t^\star)\zeta_1(x^\star,t^\star) + r_{i^\star2}(x^\star,t^\star)\zeta_2(x^\star,t^\star)\\
&\quad + s_{i^\star}(x^\star,t^\star)\zeta_{i^\star}(x^\star-1,t^\star) < 0,
\end{aligned}$$

which contradicts the given premise. Then, $\zeta_{i^\star}(x^\star,t^\star) \geq 0$ and for all $(x,t) \in \overline{\Psi}$, we obtain $\vec{\zeta}(x,t) \geq \vec{0}$ on $\overline{\Psi}$. Therefore, the above lemma holds.

Lemma 2

Assume that $\mathcal{R}(x,t)$ and $\mathcal{S}(x,t)$ are functions satisfying the conditions (4) and (5). Let $\vec{\zeta}(x,t)$ be any function in the domain of $\vec{\mathcal{L}}$ such that for each $(x,t) \in \overline{\Psi}$, then

$$\left|\vec{\zeta}(x,t)\right| \leq \max\left\{\left\| \vec{\beta}\vec{\zeta}(0,t) \right\|, \left\| \vec{\zeta}(x,0) \right\|, \frac{1}{\alpha}\left\| \vec{\mathcal{L}_1}\vec{\zeta} \right\|, \frac{1}{\alpha}\left\| \vec{\mathcal{L}_2}\vec{\zeta} \right\|\right\}.$$

Proof.

Barrier function is defined below

$\vec{\chi}^{\pm} = \max\left\{\left\| \vec{\beta}\vec{\zeta}(0,t) \right\|, \left\| \vec{\zeta}(x,0) \right\|, \frac{1}{\alpha}\left\| \vec{\mathcal{L}_1}\vec{\zeta} \right\|, \frac{1}{\alpha}\left\| \vec{\mathcal{L}_2}\vec{\zeta} \right\|\right\} \pm \vec{\zeta}(x,t)$,

$\vec{\chi}^{\pm} = \mathcal{M} \pm \vec{\zeta}(x,t)$, where $\mathcal{M} = \max\left\{\left\| \vec{\beta}\vec{\zeta}(0,t) \right\|, \left\| \vec{\zeta}(x,0) \right\|, \frac{1}{\alpha}\left\| \vec{\mathcal{L}_1}\vec{\zeta} \right\|, \frac{1}{\alpha}\left\| \vec{\mathcal{L}_2}\vec{\zeta} \right\|\right\} \pm \vec{\zeta}(x,t)$.

It is obvious that, $\vec{\beta}\vec{\chi}^{\pm}(x,t) \geq \vec{0}$, $\vec{\chi}^{\pm}(x,t) \geq \vec{0}$, $\vec{\mathcal{L}_1}\vec{\chi}^{\pm}(x,t) \geq \vec{0}$ *on* Ψ^- *and* $\vec{\mathcal{L}_2}\vec{\chi}^{\pm}(x,t) \geq \vec{0}$ *on* Ψ^+. Using the result of the above Lemma 1, we get $\vec{\chi}^{\pm}(x,t) \geq \vec{0}$. Therefore, the above lemma holds.

2.3 Solution Components

The exact solution $\vec{\omega}$ of (1) – (3) is defined as the sum of two components $\vec{a} + \vec{b}$.

The regular component $\vec{a}$ satisfies

$$\vec{\mathcal{L}_1}\vec{a}(x,t) = \frac{\partial\vec{a}}{\partial t}(x,t) + E\frac{\partial\vec{a}}{\partial x}(x,t) + R(x,t)\vec{a}(x,t) = \vec{f}(x,t) - S(x,t)\vec{\phi}_L(x-1,t) \text{ on } \Psi^-, \tag{8}$$

$$\vec{\mathcal{L}_2}\vec{a}(x,t) = \frac{\partial\vec{a}}{\partial t}(x,t) + E\frac{\partial\vec{a}}{\partial x}(x,t) + R(x,t)\vec{a}(x,t) + S(x,t)\vec{a}(x-1,t) = \vec{f}(x,t) \text{ on } \Psi^+, \tag{9}$$

with, $\vec{\beta}\,\vec{a}(0,t) = \vec{\beta}\,\vec{\omega}_0(0,t)$, $\vec{a}(x,0) = \vec{\omega}_0(x,0)$.

The layer component $\vec{b}$ satisfies

$$\vec{\mathcal{L}_1}\vec{b}(x,t) = \frac{\partial\vec{b}}{\partial t}(x,t) + E\frac{\partial\vec{b}}{\partial x}(x,t) + R(x,t)\vec{b}(x,t) = \vec{0} \text{ on } \Psi^-, \tag{10}$$

$$\vec{\mathcal{L}_2}\vec{b}(x,t) = \frac{\partial\vec{b}}{\partial t}(x,t) + E\frac{\partial\vec{b}}{\partial x}(x,t) + R(x,t)\vec{b}(x,t) + S(x,t)\vec{b}(x-1,t) = \vec{0} \text{ on } \Psi^+, \tag{11}$$

with $\vec{b} = \vec{\omega} - \vec{a}$, $\vec{\beta}\,\vec{b}(0,t) = \vec{\beta}\,(\vec{\omega} - \vec{a})(0,t)$, $\vec{a}(x,0) = \vec{0}$.

2.4 Bounds

Lemma 3

Assume that $R(x,t)$ and $S(x,t)$ are functions satisfying the conditions (4) – (5) and $\vec{\omega}$ is the solution of the problem (1) – (3). Consequently, $\forall (x,t) \in \overline{\Psi}$ and $i = 1,2$, there exists a generic constant C such that

$$\left|\frac{\partial^l \omega_i}{\partial t^l}(x,t)\right| \leq C\left\{\left\|\vec{\phi}_L(t)\right\|,\left\|\vec{\phi}_B(x)\right\|,\sum_{q=0}^{l}\left\|\frac{\partial^q \vec{f}}{\partial t^q}\right\|\right\}, l = 0,1,2,$$

$$\left|\frac{\partial \omega_i}{\partial x}(x,t)\right| \leq C\varepsilon^{-1}\left\{\left\|\vec{\phi}_L(t)\right\|,\left\|\vec{\phi}_B(x)\right\|,\sum_{q=0}^{1}\left\|\frac{\partial^q \vec{f}}{\partial t^q}\right\|\right\},$$

$$\left|\frac{\partial^2 \omega_i}{\partial x \partial t}(x,t)\right| \leq C\varepsilon^{-1}\left\{\left\|\vec{\phi}_L(t)\right\|,\left\|\vec{\phi}_B(x)\right\|,\sum_{q=0}^{2}\left\|\frac{\partial^q \vec{f}}{\partial t^q}\right\|\right\},$$

$$\left|\frac{\partial^2 \omega_i}{\partial x^2}(x,t)\right| \leq C\varepsilon^{-2}\left\{\left\|\vec{\phi}_L(t)\right\|,\left\|\vec{\phi}_B(x)\right\|,\left\|\frac{\partial \vec{f}}{\partial x}\right\|,\sum_{q=0}^{2}\left\|\frac{\partial^q \vec{f}}{\partial t^q}\right\|\right\}.$$

Proof. Using an analogous procedure outlined in [10], we obtain the above estimates.

Lemma 4

Assume that $R(x,t)$ and $S(x,t)$ are functions satisfying the conditions (4) – (5). Consequently, $\forall (x,t) \in \overline{\Psi}$ and $i = 1,2$, there exists a generic constant C such that

$$\left|\frac{\partial^l a_i}{\partial t^l}(x,t)\right| \leq C, \text{ for } l = 0,\ 1,\ 2,$$

$$\left|\frac{\partial^2 a_i}{\partial x \partial t}(x,t)\right| \leq C,$$

$$\left|\frac{\partial^2 a_i}{\partial x^2}(x,t)\right| \leq C\varepsilon^{1-l}, \text{ for } l = 1,2.$$

Proof. Using an analogous procedure outlined in [10], we obtain the above estimates.

2.5 Layer Functions

Let $G_{0,i}, G_{1,i}, i = 1,2$ be the associated layer functions for the solution ω_i

$$G_{\lambda,i}(x) = e^{-\alpha(x-\lambda)/\varepsilon},\ \lambda = 0,1,\ i\ = 1,2 \text{ on } (x,t) \in \overline{\Psi}, \tag{12}$$

where, $G_{0,i}(x) = e^{-\alpha x/\varepsilon}$ on $\overline{\Psi}^-$, $G_{1,i}(x) = e^{-\alpha(x-1)/\varepsilon}$ on $\overline{\Psi}^+$.

Lemma 5

Assume that $R(x,t)$ and $S(x,t)$ are functions satisfying the conditions (4) – (5). Consequently, $\forall (x,t) \in \overline{\Psi}$, $i = 1,2$ and $\lambda = 0,1$, there exists a generic constant C such that

$$\left|\frac{\partial^l b_i}{\partial t^l}(x,t)\right| \leq C\ G_{\lambda,2}(x), \text{ for } l = 0,\ 1,\ 2,$$

$$\left|\frac{\partial^l b_i}{\partial x^l}(x,t)\right| \leq C\ \varepsilon^{-l}\ G_{\lambda,2}(x), \text{ for } l = 1,\ 2.$$

Proof. Using an analogous procedure outlined in [10], we obtain the above estimates.

2.6 Numerical Method

2.6.1. Discretization

Discretization of the temporal domain is carried out using a uniform mesh. While the discretization of the spatial domain is implemented by using a piecewise uniform Shishkin mesh, which allows us to precisely capture the layer phenomena of the solution across the domain. The domain $(0,2]$ is split into 4 sub-domains as given below

$$(0,\varsigma]\cup(\varsigma,1]\cup(1,1+\varsigma]\cup(1+\varsigma,2],$$

where,

$$\varsigma=\min\left\{\frac{1}{2},\frac{\varepsilon}{\alpha}\ln N\right\},\ \ 0<\varsigma\leq\frac{1}{2}.$$

The layer regions $(0,\varsigma]$ and $(1,1+\varsigma]$ have $\frac{N}{4}$ mesh points each, while the outer regions $(\varsigma,1]$ and $(1+\varsigma,2]$ also have $\frac{N}{4}$ mesh points each.

Let us consider the following notation for the domain of the discretized problem

$\Psi_t^M=\{t_k\}_{k=1}^M,\ \ \overline{\Psi}_t^M=\{t_k\}_{k=0}^M,\quad \Psi_x^N=\{x_j\}_{j=1}^N,\ \ \overline{\Psi}_x^N=\{x_j\}_{j=0}^N,$

$\Psi_x^{-N}=\{x_j\}_{j=1}^{\frac{N}{2}-1},\ \ \Psi_x^{+N}=\{x_j\}_{j=\frac{N}{2}+1}^{N},\ \ \overline{\Psi}_x^{-N}=\{x_j\}_{j=0}^{\frac{N}{2}},\ \ \overline{\Psi}_x^{+N}=\{x_j\}_{j=\frac{N}{2}}^{N},$

$\Psi^{-M,N}=\Psi_t^M\times\Psi_x^{-N},\ \ \overline{\Psi}^{-M,N}=\overline{\Psi}_t^M\times\overline{\Psi}_x^{-N},\ \ \Psi^{+M,N}=\Psi_t^M\times\Psi_x^{+N},\ \ \overline{\Psi}^{+M,N}=\overline{\Psi}_t^M\times\overline{\Psi}_x^{+N},$

$\Psi_x^N=\Psi_x^{-N}\cup\Psi_x^{+N},\ \ \overline{\Psi}_x^N=\overline{\Psi}_x^{-N}\cup\overline{\Psi}_x^{+N},\ \ \Psi^{M,N}=\Psi_t^M\times\Psi_x^N,\ \ \overline{\Psi}^{M,N}=\overline{\Psi}_t^M\times\overline{\Psi}_x^N,\ \ \xi^{M,N}=\xi\cap\overline{\Psi}^{M,N}.$

2.6.2. Finite Difference Scheme

Layer Resolving Finite Difference Scheme is applied to the problem (1) – (3).

$$\begin{aligned}\vec{\mathcal{L}}^{M,N}\overrightarrow{W}(x_j,t_k) &= D_t^-\overrightarrow{W}(x_j,t_k)+E\,D_x^-\overrightarrow{W}(x_j,t_k)+R(x_j,t_k)\overrightarrow{W}(x_j,t_k)+S(x_j,t_k)\overrightarrow{W}(x_j-1,t_k)\\ &= \vec{f}(x_j,t_k)\ \text{on}\ \Psi^{M,N},\end{aligned}\tag{13}$$

$$\vec{\beta}^{M,N}\overrightarrow{W}(0,t_k)=\overrightarrow{W}(0,t_k)-E\,D_x^+\overrightarrow{W}(0,t_k)=\vec{\phi}_L(t_k),\overrightarrow{W}(x_j,0)=\vec{\phi}_B(x_j).\tag{14}$$

The problem (13) – (14) can be reformulated as

$$\begin{aligned}\vec{\mathcal{L}}_1^{M,N}\overrightarrow{W}(x_j,t_k) &= D_t^-\overrightarrow{W}(x_j,t_k)+E\,D_x^-\overrightarrow{W}(x_j,t_k)+R(x_j,t_k)\overrightarrow{W}(x_j,t_k)\\ &= \vec{f}(x_j,t_k)-S(x_j,t_k)\vec{\phi}_L(x_j-1,t_k)\ \text{on}\ \Psi^{-M,N},\end{aligned}\tag{15}$$

$$\begin{aligned}\vec{\mathcal{L}}_2^{M,N}\overrightarrow{W}(x_j,t_k) &= D_t^-\overrightarrow{W}(x_j,t_k)+E\,D_x^-\overrightarrow{W}(x_j,t_k)+R(x_j,t_k)\overrightarrow{W}(x_j,t_k)+S(x_j,t_k)\overrightarrow{W}(x_j-1,t_k)\\ &= \vec{f}(x_j,t_k)\ \text{on}\ \Psi^{+M,N},\end{aligned}\tag{16}$$

$$\vec{\beta}^{M,N}\overrightarrow{W}(0,t_k)=\vec{\phi}_L(t_k),\ \overrightarrow{W}(x_j,0)=\vec{\phi}_B(x_j).\tag{17}$$

Here,

$D_t^-\overrightarrow{W}(x_j,t_k)=\frac{\overrightarrow{W}(x_j,t_k)-\overrightarrow{W}(x_j,t_{k-1})}{t_k-t_{k-1}},\quad D_x^-\overrightarrow{W}(x_j,t_k)=\frac{\overrightarrow{W}(x_j,t_k)-\overrightarrow{W}(x_{j-1},t_k)}{x_j-x_{j-1}}$ *and* $D_x^+\overrightarrow{W}(x_j,t_k)=\frac{\overrightarrow{W}(x_{j+1},t_k)-\overrightarrow{W}(x_j,t_k)}{x_{j+1}-x_j}.$

The discrete solution $\overrightarrow{W}$ of (13) is defined as the sum of two components $\vec{A}+\vec{B}$.

The regular component $\vec{A}$ satisfies

$$\begin{aligned}\vec{\mathcal{L}}_1^{M,N}\vec{A}(x_j,t_k) &= D_t^-\vec{A}(x_j,t_k)+E\,D_x^-\vec{A}(x_j,t_k)+R(x_j,t_k)\vec{A}(x_j,t_k)\\ &= \vec{f}(x_j,t_k)-S(x_j,t_k)\vec{\phi}_L(x_j-1,t_k)\ \text{on}\ \Psi^{-M,N},\end{aligned}\tag{18}$$

$$\begin{aligned}\mathcal{L}_2^{M,N}\vec{A}(x_j,t_k) &= D_t^-\vec{A}(x_j,t_k)+E\,D_x^-\vec{A}(x_j,t_k)+R(x_j,t_k)\,\vec{A}(x_j,t_k)+S(x_j,t_k)\,\vec{A}(x_j-1,t_k)\\ &= \vec{f}(x_j,t_k)\ \text{on}\ \Psi^{+M,N},\end{aligned} \tag{19}$$

$$\vec{\beta}^{M,N}\vec{A}(0,t_k)=\vec{\beta}\vec{a}(0,t_k),\ \ \vec{A}(x_j,0)=\vec{a}(x_j,0). \tag{20}$$

The singular component $\vec{B}$ satisfies

$$\mathcal{L}_1^{M,N}\vec{B}=\vec{0}\ \text{on}\ \Psi^{-M,N},\ \ \mathcal{L}_2^{M,N}\vec{B}=\vec{0}\ \text{on}\ \Psi^{+M,N}, \tag{21}$$

$$\vec{\beta}^{M,N}\vec{B}(0,t_k)=\vec{\beta}\,\vec{b}(0,t_k),\quad \vec{B}(x_j,0)=\vec{b}(x_j,0). \tag{22}$$

Lemma 6

Assume that $R(x_j,t_k)$ and $S(x_j,t_k)$ are functions satisfying the conditions (4) – (5). Then for any mesh function $\vec{\Phi}$, the inequalities $\vec{\beta}^{M,N}\vec{\Phi}(0,t_k)\geq\vec{0}$, $\vec{\Phi}(x_j,0)\geq\vec{0}$, $\mathcal{L}_1^{M,N}\vec{\Phi}\geq\vec{0}$ on $\Psi^{-M,N}$, $\mathcal{L}_2^{M,N}\vec{\Phi}\geq\vec{0}$ on $\Psi^{+M,N}$ imply that $\vec{\Phi}\geq\vec{0}\ on\ \overline{\Psi}^{M,N}$.

Proof.

The proof of this lemma is obtained by adopting a similar approach to that used in Lemma 1.

Lemma 7

Assume that $R(x_j,t_k)$ and $S(x_j,t_k)$ be functions satisfying the conditions (4) – (5). Then for any mesh function $\vec{\Phi}$ on $\overline{\Psi}^{M,N}$,

$$\left|\vec{\Phi}(x_j,t_k)\right|\leq\max\{\,\|\,\vec{\beta}^{M,N}\vec{\Phi}(0,t_k),\|\,\vec{\Phi}(x_j,0)\,\|,\ \frac{1}{\alpha}\,\|\,\mathcal{L}_1^{M,N}\vec{\Phi}\,\|,\ \frac{1}{\alpha}\,\|\,\mathcal{L}_2^{M,N}\vec{\Phi}\,\|\,\}.$$

Proof.

The proof of this lemma is obtained by adopting a similar approach to that used in Lemma 2.

2.6.3. Error Estimate

The error $\left|\vec{W}-\vec{\omega}\right|$ is obtained using the bounds of $\left|\vec{\beta}^{M,N}\left(\vec{W}-\vec{\omega}\right)(0,t_k)\right|,\left|\left(\vec{W}-\vec{\omega}\right)(x_j,0)\right|\ and\ \left|\vec{\mathcal{L}}^{M,N}\left(\vec{W}-\vec{\omega}\right)\right|$.

Let the error $\left|\vec{W}-\vec{\omega}\right|$ is defined as the sum of $\left|\vec{A}-\vec{a}\right|$ and $\left|\vec{B}-\vec{b}\right|$, as given below

$$\left|\vec{W}-\vec{\omega}\right|\leq\left|\vec{A}-\vec{a}\right|+\left|\vec{B}-\vec{b}\right|.$$

Then,

$$\begin{gathered}\left|\left(\vec{\beta}^{M,N}\left(\vec{W}-\vec{\omega}\right)\right)_i(0,t_k)\right|\leq\left|\left(\vec{\beta}^{M,N}\left(\vec{A}-\vec{a}\right)\right)_i(0,t_k)\right|+\left|\left(\vec{\beta}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(0,t_k)\right|,\\ \left|\left(\vec{W}-\vec{\omega}\right)_i(x_j,0)\right|\leq\left|\left(\vec{A}-\vec{a}\right)_i(x_j,0)\right|+\left|\left(\vec{B}-\vec{b}\right)_i(x_j,0)\right|,\\ \left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{W}-\vec{\omega}\right)\right)_i(x_j,t_k)\right|\leq\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{A}-\vec{a}\right)\right)_i(x_j,t_k)\right|+\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(x_j,t_k)\right|.\end{gathered}$$

Here,

$$\left|\left(\vec{\beta}^{M,N}\left(\vec{A}-\vec{a}\right)\right)_i(0,t_k)\right|=\left|\varepsilon\left(\frac{\partial}{\partial x}-D_x^+\right)a_i(0,t_k)\right|, \tag{23}$$

$$\left|\left(\vec{\beta}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(0,t_k)\right|=\left|\varepsilon\left(\frac{\partial}{\partial x}-D_x^+\right)b_i(0,t_k)\right|, \tag{24}$$

$$\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{A}-\vec{a}\right)\right)_i(x_j,t_k)\right|\leq\left|\left(\frac{\partial}{\partial t}-D_t^-\right)a_i(x_j,t_k)\right|+\varepsilon\left|\left(\frac{\partial}{\partial x}-D_x^-\right)a_i(x_j,t_k)\right|, \tag{25}$$

$$\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(x_j,t_k)\right| \leq \left|\left(\frac{\partial}{\partial t}-D_t^-\right)b_i(x_j,t_k)\right| + \varepsilon\left|\left(\frac{\partial}{\partial x}-D_x^-\right)b_i(x_j,t_k)\right|. \tag{26}$$

The error estimates for $\left|\vec{A}-\vec{a}\right|$ and $\left|\vec{B}-\vec{b}\right|$ are obtained in the subsequent theorems.

Theorem 1

Assume that $R(x,t)$ and $S(x,t)$ be functions satisfying the conditions (4) – (5). Let $\vec{a}$ represent the solution of (8) – (9) and $\vec{A}$ represent the solution of (18) – (19), then

$$\parallel \vec{A}-\vec{a} \parallel \leq C\left(M^{-1}+N^{-1}\right).$$

Proof.

Using the estimates from Lemma 4, we get

$$\begin{aligned}
\left|\left(\vec{\beta}^{M,N}\left(\vec{A}-\vec{a}\right)\right)_i(0,t_k)\right| &= \left|\varepsilon\left(\tfrac{\partial}{\partial x}-D_x^+\right)a_i(0,t_k)\right| \\
&\leq C\,\varepsilon\,(x_1-x_0)\max_{p\in[x_0,x_1]}\left|\tfrac{\partial^2 a_i}{\partial x^2}(p,t_k)\right| \\
&\leq C\,N^{-1},\\
\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{A}-\vec{a}\right)\right)_i(x_j,t_k)\right| &\leq \left|\left(\tfrac{\partial}{\partial t}-D_t^-\right)a_i(x_j,t_k)\right| + \varepsilon\left|\left(\tfrac{\partial}{\partial x}-D_x^-\right)a_i(x_j,t_k)\right| \\
&\leq C\,\mu_k\,\max_{p\in[t_{k-1},t_k]}\left|\tfrac{\partial^2 a_i}{\partial t^2}(x_j,p)\right| + C\,\varepsilon\,h_j\max_{p\in[x_{j-1},x_j]}\left|\tfrac{\partial^2 a_i}{\partial x^2}(p,t_k)\right| \\
&\leq C\,\left(M^{-1}+N^{-1}\right).
\end{aligned}$$

Therefore, the above theorem holds.

Theorem 2

Assume that $R(x,t)$ and $S(x,t)$ be functions satisfying the conditions (4) – (5). Let $\vec{b}$ represent the solution of (10) – (11) and $\vec{B}$ represent the solution of (21), then

$$\parallel \vec{B}-\vec{b} \parallel \leq C\left(M^{-1}+N^{-1}\ln N\right).$$

Proof. The error estimated here depends upon the choice of the transition parameter $\varsigma = \frac{1}{2}\ or\ \varsigma = \frac{\varepsilon}{\alpha}\ ln\ N$.

Case 1:$\varsigma = \frac{1}{2}$.

Here, the mesh will be uniform and $\frac{1}{2} \leq \frac{\varepsilon}{\alpha}\ ln\ N$. Then, by using the estimates from Lemma 5, we get

$$\begin{aligned}
\left|\left(\vec{\beta}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(0,t_k)\right| &= \left|\varepsilon\left(\tfrac{\partial}{\partial x}-D_x^+\right)b_i(0,t_k)\right| \\
&\leq C\,\varepsilon\,(x_1-x_0)\max_{p\in[x_0,x_1]}\left|\tfrac{\partial^2 b_i}{\partial x^2}(p,t_k)\right| \\
&\leq C\,N^{-1}\ln N,\\
\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(x_j,t_k)\right| &\leq \left|\left(\tfrac{\partial}{\partial t}-D_t^-\right)b_i(x_j,t_k)\right| + \varepsilon\left|\left(\tfrac{\partial}{\partial x}-D_x^-\right)b_i(x_j,t_k)\right| \\
&\leq C\,\mu_k\,\max_{p\in[t_{k-1},t_k]}\left|\tfrac{\partial^2 b_i}{\partial t^2}(x_j,p)\right| + C\,\varepsilon\,h_j\max_{p\in[x_{j-1},x_j]}\left|\tfrac{\partial^2 b_i}{\partial x^2}(p,t_k)\right| \\
&\leq C\,\left(M^{-1}+N^{-1}\ln N\right).
\end{aligned}$$

Case 2:$\varsigma = \frac{\varepsilon}{\alpha}\ ln\ N$.

Here, the mesh will be piecewise uniform and $\frac{\varepsilon}{\alpha}\ ln\ N \leq \frac{1}{2}$.

Layer Region:$[0,\varsigma]\ and\ [1,1+\varsigma]$

The mesh length of $[0,\varsigma]\ and\ [1,1+\varsigma]$ be $\frac{4\varsigma}{N}$. Then by using the estimates from Lemma 5, we get

$$\begin{aligned}
\left|\left(\vec{\beta}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(0,t_k)\right| &= \left|\varepsilon\left(\tfrac{\partial}{\partial x}-D_x^+\right)b_i(0,t_k)\right| \\
&\leq C\,\varepsilon\,(x_1-x_0)\max_{p\in[x_0,x_1]}\left|\tfrac{\partial^2 b_i}{\partial x^2}(p,t_k)\right| \\
&\leq C\,N^{-1}\ln N,\\
\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(x_j,t_k)\right| &\leq \left|\left(\tfrac{\partial}{\partial t}-D_t^-\right)b_i(x_j,t_k)\right| + C\,\varepsilon\left|\left(\tfrac{\partial}{\partial x}-D_x^-\right)b_i(x_j,t_k)\right| \\
&\leq C\,\mu_k\,\max_{p\in[t_{k-1},t_k]}\left|\tfrac{\partial^2 b_i}{\partial t^2}(x_j,p)\right| + C\,\varepsilon\,h_j\max_{p\in[x_{j-1},x_j]}\left|\tfrac{\partial^2 b_i}{\partial x^2}(p,t_k)\right| \\
&\leq C\,\left(M^{-1}+N^{-1}\ln N\right).
\end{aligned}$$

Coarse Region:$[\varsigma,1]\ and\ [1+\varsigma,2]$

The mesh length of $[\varsigma,1]$ and $[1+\varsigma,2]$ be $\frac{4(1-\varsigma)}{N}$. Then by using the estimates from Lemma 5, we get

$$\left|\left(\vec{\beta}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(0,t_k)\right|\leq C\,N^{-1},$$

$$\left|\left(\vec{\mathcal{L}}^{M,N}\left(\vec{B}-\vec{b}\right)\right)_i(x_j,t_k)\right|\leq C\,(M^{-1}+N^{-1}).$$

Therefore, the above theorem holds.

Theorem 3

Let $\vec{\omega}$ represent the solution of (1) – (3) and $\overrightarrow{W}$ represent the solution of (13) – (14). Then,

$$\|\,\overrightarrow{W}(x_j,t_k)-\vec{\omega}(x_j,t_k)\,\|\leq C\,N^{-1}\ln N.$$

Proof.

By combining the estimates obtained from Theorems 1 and 2, the above stated result is obtained. This concludes the proof of the theorem.

3 Results and Discussion

3.1 Numerical Illustration

We have formulated finite difference scheme on a layer-adapted Shishkin mesh for a coupled system of singularly perturbed time-dependent delay initial value problems with same perturbation parameter and Robin initial conditions. The robustness and effectiveness of the proposed scheme are validated by solving an example problem. The two-mesh algorithm is used to estimate the convergence rate P^* and error constants C_{P^*} for the approximate solution. The results obtained are then presented in Table 1. The solution profiles of ω_1 and ω_2 are given in Figures 1 and 2.

Example 1

$$\frac{\partial\omega_1}{\partial t}(x,t)+\varepsilon\,\frac{\partial\omega_1}{\partial x}(x,t)+\left(6+e^{\frac{t}{2}}\right)\,\omega_1(x,t)-\omega_2(x,t)-\,\omega_1(x-1,t)=3,$$

$$\frac{\partial\omega_2}{\partial t}(x,t)+\varepsilon\,\frac{\partial\omega_2}{\partial x}(x,t)-0.5\,\omega_1(x,t)+\left(5+e^{\frac{-t}{2}}\right)\,\omega_2(x,t)-0.5\,\omega_2(x-1,t)=1,$$

$$\omega_1(0,t)-\varepsilon\frac{\partial\omega_1}{\partial x}(0,t)=\omega_2(0,t)-\varepsilon\frac{\partial\omega_2}{\partial x}(0,t)=1\;and\;\omega_1(x,0)=\omega_1(x,0)=1.$$

Table 1. For $\varepsilon=\frac{\varrho}{8}$, values of P^* and C_{P^*} are computed

ϱ	N			
	32	64	128	256
2^{-4}	9.705×10^{-2}	6.90245×10^{-2}	4.50627×10^{-2}	2.77917×10^{-2}
2^{-8}	9.705×10^{-2}	6.90245×10^{-2}	4.50627×10^{-2}	2.77917×10^{-2}
2^{-12}	9.705×10^{-2}	6.90245×10^{-2}	4.50627×10^{-2}	2.77917×10^{-2}
2^{-16}	9.705×10^{-2}	6.90245×10^{-2}	4.50627×10^{-2}	2.77917×10^{-2}
2^{-20}	9.705×10^{-2}	6.90245×10^{-2}	4.50627×10^{-2}	2.77917×10^{-2}
D^N	9.705×10^{-2}	6.90245×10^{-2}	4.50627×10^{-2}	2.77917×10^{-2}
P^N	4.9162×10^{-1}	6.1517×10^{-1}	6.9728×10^{-1}	
$C^N_{P^*}$	1.313429	1.313429	1.205628	1.04545
		$P^*=4.9162\times10^{-1}$		
		$C_{P^*}=1.313429$		

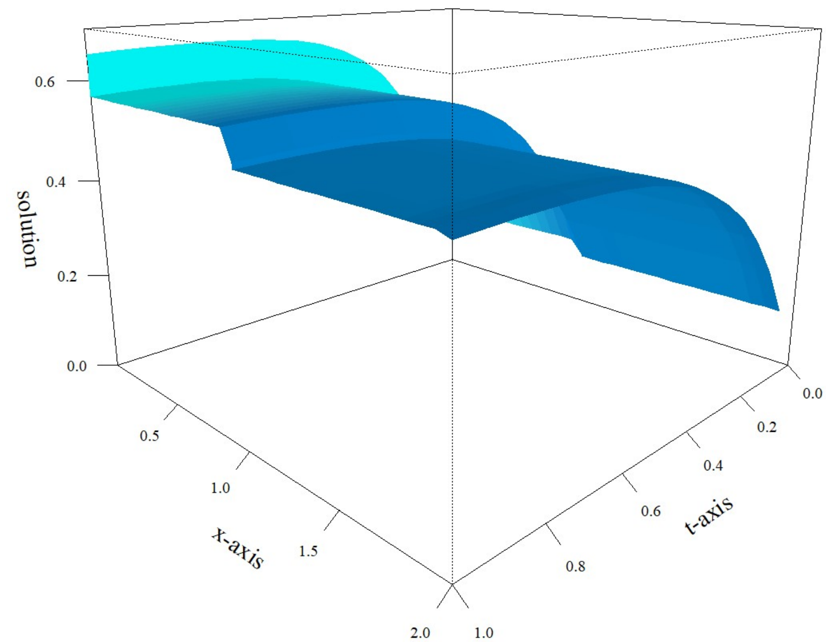

Fig 1. Solution profile of ω_1 for Example 1

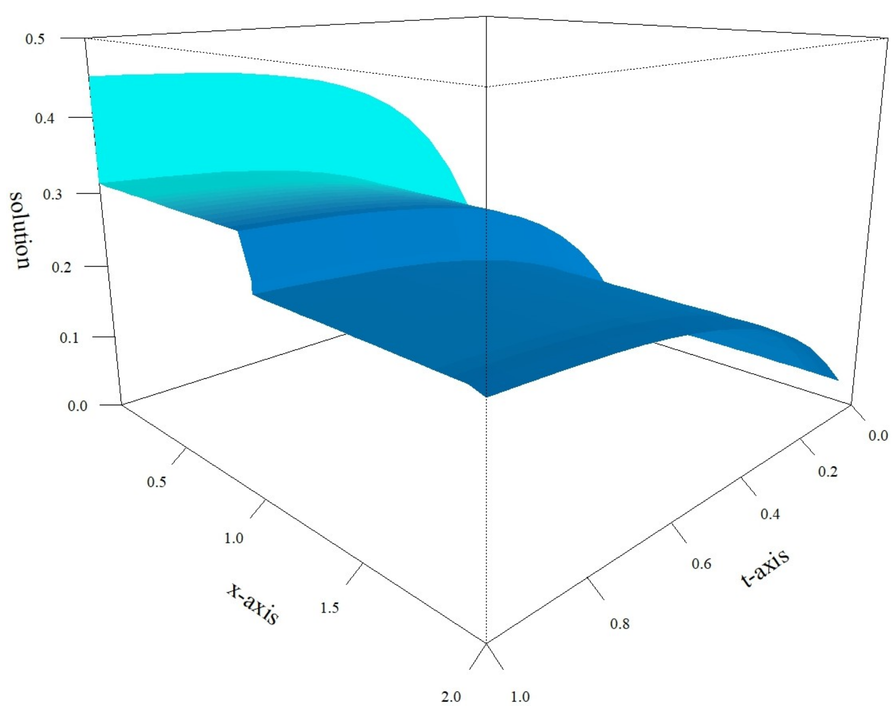

Fig 2. Solution profile of ω_2 for Example 1

4 Conclusion

This study has formulated a layer-resolving finite difference scheme on a piecewise uniform Shishkin mesh. This scheme is applied to a coupled system of singularly perturbed time-dependent delay initial value problems with Robin initial conditions. Due to the presence of ε and delay terms, initial and interior layers appear in the solution domain at $(0,t)$ and $(1,t)$, respectively. The proposed scheme is validated through an example problem and the obtained numerical result shows that the scheme achieves first order convergence. Future work will focus on extending the proposed scheme to more complex and higher-dimensional problems.

5 Contributory statement

Conceptualization of study problem: Dr. M. Joseph Paramasivam & K. Ramiya Bharathi, Software or algorithm: K. Ramiya Bharathi & Dr. M. Joseph Paramasivam, Research Guidance: Dr. M. Joseph Paramasivam, Original Drafting: K. Ramiya Bharathi & Dr. M. Joseph Paramasivam, Editing & Reviewing: Dr. M. Joseph Paramasivam.

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