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Enclave Domination Number of Semi Total Graphs $T_1(G)$ and $T_2(G)$

M Santhosh Priya[1]*, A Mydeen Bibi[2]

1 Research Scholar, PG & Research Department of Mathematics, The Standard Fireworks Rajaratnam College for Women, Sivakasi, 626123, Tamil Nadu, India
2 Assistant Professor, PG & Research Department of Mathematics, The Standard Fireworks Rajaratnam College for Women, Sivakasi, 626123, Tamil Nadu, India

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* **Corresponding author**.

santhoshpriya97@gmail.com

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Abstract

Objectives: This research seeks to explore the concept of enclave domination in semi total line graphs, semi total point graphs, and to establish the sharp bounds and key properties of the enclave domination number in these graphs. **Methods:** This investigation explores enclave domination in semi-total graphs by identifying dominating sets that meet the specific requirements of enclave domination. The process involves determining vertex subsets where each vertex outside the set is adjacent to at least one vertex within it, with particular attention to the unique structural characteristics of semi-total graphs. Utilizing standard definitions, key parameters, and specialized techniques, we achieved significant results. **Findings:** Minimum enclave dominating sets in semi-total graphs have been characterized. Exact values are obtained through the analysis of lower and upper bounds in different standard graphs. Furthermore, the link between the enclave domination number in semi-total point graphs and semi-total line graphs was examined. **Novelty :** This study introduces new definitions and explanations of enclave domination for semi total graphs, expanding the field of domination theory. By looking at how these domination parameters work in semi-total point graphs, these findings provide a new understanding of their properties and how they relate to other graph parameters.

Keywords: Enclave Dominating Vertex; Enclave Dominating Set; Enclave Domination Number; Semi-Total Point Graph; Semi-Total Line Graph

1 Introduction

In the realm of graph theory, domination is a vibrant area of investigation. We concentrate on simple, finite, undirected, nontrivial, and connected graphs. For graph theoretic terminology, we refer [(1)]. Graphs have various special patterns like path, cycle, star, complete graph, bipartite graph, complete bipartite graph, regular graph, tree etc. We refer to Harary [(2)] for the definitions of all such graphs. The neighborhood of a vertex u in G, denoted by $N_G(u)$ is the set of all vertices adjacent to u, that is $N_G(u) = \{v \in V / uv \in E\}$.The closed neighborhood of a vertex u in G is denoted by $N_G[u]$, is defined as$N_G[u] = \{u\} \cup N_G(u)$. The degree of a vertex u , denoted by $d(u)$ is the number of vertices in its neighborhood, or equivalently, the number of edges incident to u , A vertex of degree one is called as a pendant vertex or leaf, whereas a vertex that is adjacent to a leaf is referred to as a supporting vertex. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\triangle(G)$ respectively. A path comprising n vertices is denoted by P_n, and its length is defined as the number of edges it contains. The minimum distance between two vertices *u* and *v* denoted by *d*(*u*,*v*), is the length of the shortest path connecting them. The greatest distance between any two vertices of a connected graph *G* is called the diameter of *G* and is denoted by *diam*(*G*). The corona product of two graphs G and H is $G \circ H$is defined as the graph obtained by taking one copy of G and for each vertex $v \in V(G)$ connecting a disjoint copy of *H* such that *v* is connected to every vertex of the copy of *H*. Given a vertex set $S \subseteq V$ in a graph *G*, a vertex $u \in S$ is called an enclave if $N[u] \subseteq S$, which is to say that the vertex *u* has no neighbors in $V - S$. A set *S* is called enclaveless if it contains no enclaves. From [3,4] a vertex cover of *G* is a set of vertices that covers all the edges of *G*. The minimum cardinality of a vertex cover in a graph *G* is called the vertex covering number of *G* and is denoted by $\beta(G)$. From [5], A dominating set *D* in a graph *G* is a subset of vertices where every vertex not in *D* is adjacent to a vertex in *D*. That is, for every $v \in V$, we find $N[v] \cap D \neq \emptyset$. The domination number denoted by $\gamma(G)$ is the minimum cardinality of the dominating set of *G*. The enclave domination number was introduced in [6] . A dominating set $E_u \subseteq V(G)$ is said to be an enclave dominating set if the set $E_u$has exactly one enclave vertex *u* in it. And the vertex *u* is called enclave dominating vertex. The minimum cardinality on all the enclave dominating sets, known as enclave domination number of *G*It is denoted by $\gamma_\varepsilon(G)$ The semi-total line graph $T_1(G)$, and the semi-total point graph $T_2(G)$ of a graph G was studied in [7–9] and is defined as follows. For any graph $G = (V, E)$, The semi-total line graph $T_1(G)$ is the graph whose vertex set is the union of vertices and edges in which two vertices are adjacent if and only if they are adjacent edges of *G* or one is a vertex of *G* and the other is an edge of *G* incident with it. The semi-total point graph $T_2(G)$ is the graph whose vertex set is the union of vertices and edges, in which two vertices are adjacent if and only if they are adjacent vertices of G or one is a vertex and the other is an edge of G incident with it. The study of split and non-split two domination numbers of semi-total point graphs [8] motivated us to introduce the enclave domination number in semi-total graphs.

2 Methodology

In this work, we define the specific domination parameters studied, such as enclave domination, and outline the conditions necessary for calculating them. The concept of an enclave dominating set, which includes exactly one enclave dominating vertex, is introduced and analyzed for semi-total graphs.

3 Enclave Domination Number of Semi-Total Line Graphs

Theorem 3.1.

For any path P_n with $n \geq 2$ vertices, $\gamma_\epsilon(T_1(P_n)) = \begin{cases} \frac{n+3}{2}, & n\ is\ odd \\ \frac{n+2}{2}, & n\ is\ even \end{cases}$

Proof:

Let P_n be a path with $n \geq 2$ vertices, $V(P_n) = \{u_1, u_2, ..., u_n\}$, $E(P_n) = \{e_1, e_2, ..., e_{n-1}\}$ then $V(T_1(P_n)) = \{u_1, u_2, ..., u_n, e_1, e_2, ..., e_{n-1}\}$. To prove the equality, we have the following two cases,

Case(i): When n is odd

In this case, the vertices $u_1, u_n, u_{2i} (1 \leq i \leq \frac{n-1}{2})$ satisfy the definition of enclave domination and the corresponding enclave dominating sets are as follows. $E_{u_1} = \{u_1, u_n, e_i / 1 \leq i \leq n-1, i \equiv 1(mod2)\}$,

$E_{u_n} = \{u_n, u_1, e_i / 1 \leq i \leq n-1, i \equiv 0(mod2)\}$

$$\left|E_{u_1}\right| = \left|E_{u_n}\right| = 2 + \frac{n-1}{2} = \frac{n+3}{2} \tag{1}$$

$$E_{u_{2i}} = \{u_{2i}, e_{2i}, e_j, e_k / 1 \leq j < 2i, 2i < k \leq n-1, j \equiv 1(mod2), k \equiv 0(mod2)\}$$

$$|E_{u_{2i}}| = 2 + \frac{2i}{2} + \frac{n-1-2i}{2} = \frac{n+3}{2} \tag{2}$$

From Equations (1) and (2) we say that when n is odd $\gamma_\epsilon(T_1(P_n)) = \frac{n+3}{2}$.

Case(ii): When n is even

In this case, the vertices u_1, and u_n satisfies the definition of enclave domination and the corresponding enclave dominating sets are, $E_{u_1} = \{u_1, e_i / 1 \leq i \leq n, i \equiv 1(mod 2)\}$, and $E_{u_n} = \{u_n, e_i / 1 \leq i \leq n, i \equiv 1(mod 2)\}$.

$$|E_{u_1}| = |E_{u_n}| = 1 + \frac{n}{2} = \frac{n+2}{2} \tag{3}$$

From Equation (3), we say that when n is even $\gamma_\epsilon(T_1(P_n)) = \frac{n+2}{2}$.

Theorem 3.2.

For any cycle C_n with $n \geq 3$ vertices, $\gamma_\epsilon(T_1(C_n)) = \begin{cases} \frac{n+3}{2}, & n\, is\, odd \\ \frac{n+4}{2}, & n\, is\, even \end{cases}$

Proof:

Let C_n be a cycle with $n \geq 3$ vertices, $V(C_n) = \{u_1, u_2, ..., u_n\}$, $E(C_n) = \{e_1, e_2, ..., e_n\}$ then$V(T_1(C_n)) = \{u_1, u_2, ..., u_n, e_1, e_2, ..., e_n\}$. To prove the equality, we have the following cases,

Case(i): When n is oddIn this case, the vertices u_i for $1 \leq i \leq n$ satisfies the definition of enclave domination and the corresponding enclave dominating sets are in the following subcases,

Subcase(i)(a): For $i \equiv 1(mod 2)$

$E_{u_i} = \{u_i, e_j, e_k / 1 \leq j < i, i \leq k \leq n, j \equiv 0(mod 2), k \equiv 1(mod 2)\}$

$$|E_{u_i}| = 1 + \frac{i+1}{2} + \frac{n-i}{2} = \frac{n+3}{2} \tag{4}$$

Subcase(i)(b): For $i \equiv 0(mod 2) E_{u_i} = \{u_i, e_j, e_k / 1 \leq j < i, i \leq k < n, j \equiv 1(mod 2), k \equiv 0(mod 2)\}$

$$|E_{u_i}| = 1 + \frac{i}{2} + \frac{n-1-i}{2} = \frac{n+3}{2} \tag{5}$$

From Equations (4) and (5) , we say that when n is odd $\gamma_\epsilon(T_1(C_n)) = \frac{n+3}{2}$

Case(ii): When n is even

In this case, the vertices u_i for $1 \leq i \leq n$ satisfies the definition of enclave domination and the corresponding enclave dominating sets are in the following subcases,

Subcase(ii)(a): For $i \equiv 1(mod 2)$

$$E_{u_i} = \{u_i, e_n, e_j, e_k / 1 \leq j < i, i \leq k \leq n, j \equiv 0(mod 2), k \equiv 1(mod 2)\}$$

$$|E_{u_i}| = 2 + \frac{i-1}{2} + \frac{n-i+1}{2} = \frac{n+4}{2} \tag{6}$$

Subcase(ii)(b): For $i \equiv 0(mod 2)$

$E_{u_i} = \{u_i, e_j, e_k\} / 1 \leq j < i, i \leq k < n, j \equiv 1(mod 2), k \equiv 0(mod 2)\}$

$$|E_{u_i}| = 1 + \frac{n}{2} + \frac{n-i+2}{2} = \frac{n+4}{2} \tag{7}$$

From Equations (6) and (7), we say that when n is even $\gamma_\epsilon(T_1(C_n)) = \frac{n+4}{2}$.

Theorem 3.3 .

For any complete graph K_n with $n \geq 3$ vertices, $\gamma_\epsilon(T_1(K_n)) = n$.

Proof:

Let $V(K_n) = \{u_1, u_2, ..., u_n\}$ be the vertices and $E(K_n) = \{e_1, e_2, ..., e_{\frac{n(n-1)}{2}}\}$ be the edge set of K_n then $V(T_1(K_n)) = \{u_1, u_2, ..., u_n, e_1, e_2, ..., e_{\frac{n(n-1)}{2}}\}$. Each vertex u_i satisfies the enclave domination condition and the enclave dominating set E_{u_i} will consist of u_i along with its $n-1$ edge neighborhood vertices. The $(n-1)$ edge vertices in E_{u_i} dominate all vertices in $V(T_1(K_n)) - E_{u_i}$. Thus E_{u_i} forms the minimum enclave dominating set of $(T_1(K_n))$ with $|E_{u_i}| = 1 + n - 1 = n$. Therefore $\gamma_\epsilon(T_1(K_n)) = n$.

Theorem 3.4 .

For any complete bipartite graph $K_{m,n}$ with $m+n$ vertices, $\gamma_\epsilon(T_1(K_{m,n})) = m+n$.

Proof:

Let $V(K_{m,n}) = \{u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\}$ denote the vertex set, and $E(K_{m,n}) = \{e_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$ denote the edge set of $K_{m,n}$. Then the vertex set of $T_1(K_{m,n}))$ is $V(T_1(K_{m,n})) = \{u_i, v_j, e_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$. The vertices u_i, v_j satisfy the enclave domination conditions, and their respective enclave dominating sets given as follows, $E_{u_i} = \{u_1, u_2, ...u_m, e_{ij} / 1 \leq j \leq n\}$, and $E_{v_j} = \{v_1, v_2, ...v_n, e_{ij} / 1 \leq i \leq m\}$.

Clearly $|E_{u_i}| = |E_{v_j}| = m + n$. Thus $\gamma_\epsilon(T_1(K_{m,n})) = m + n$.

Theorem 3.5 .

For any star $K_{1,n}$ with $n+1$ vertices, $\gamma_\epsilon(T_1(K_{1,n})) = n+1$

Proof:

Let $V(K_{1,n}) = \{u, u_1, u_2, ..., u_n\}$, $E(K_{1,n}) = \{e_1, e_2, ..., e_n\}$ then the vertex set of $T_2(K_{1,n})$ is$\{u, u_1, u_2, ..., u_n, e_1, e_2, ..., e_n\}$. The vertices u, u_i satisfy the enclave domination conditions, the corresponding enclave dominating sets are as follows, $E_u = \{u, e_1, e_2, ...e_n\}$, and $E_{u_i} = \{e_i, u_1, u_2, ..., u_n\}$. Clearly $|E_u| = |E_{u_i}| = n+1$. Hence $\gamma_\epsilon(T_1(K_{1,n})) = n+1$.

Theorem 3.6 .

For any bistar $B_{m,n}$ with $m, n \geq 1$, $\gamma_\epsilon(T_1(B_{m,n})) = m+n+1$

Proof:

Let $V(B_{m,n}) = \{u, v, u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\}$ be the vertex set, and the edge set $E(B_{m,n}) = \{e, e_i, e'_j / 1 \leq i \leq m, 1 \leq j \leq n\}$ where the edge e adjacent with vertices u, v the edge e_i adjacent with the vertices u, u_i, and the edge e'_j adjacent with the vertices v, v_j. Then the $V(T_1(B_{m,n})) = \{u, v, e, u_i, v_j, e_i, e'_j / 1 \leq i \leq m, 1 \leq j \leq n\}$. The vertices u_i, v_j satisfy the enclave domination conditions and the corresponding enclave dominating sets are $E_{u_i} = \{e_i, u_1, u_2, ..., u_m, e'_1, e'_2, ..., e'_n\}$, and $E_{v_j} = \{e'_j, e_1, e_2, ..., e_m, v_1, v_2, ..., v_n\}$.

Clearly $|E_{u_i}| = |E_{v_j}| = m+n+1$, $\therefore \gamma_\epsilon(T_1(B_{m,n})) = m+n+1$.

Theorem 3.7 .

For any wheel W_n, $n \geq 3$, $\gamma_\epsilon(T_1(W_n)) = \begin{cases} 4, & n = 3 \\ 5, & n = 4 \\ \frac{n+5}{2}, & n \; is \; odd, \; n > 4 \\ \frac{n+6}{2}, & n \; is \; even, \; n > 4 \end{cases}$

Proof:

Let $u, u_1, u_2, ..., u_n$ be vertices of the wheel W_n with $deg(u) = n$, and the $2n$ edges are denoted by $e_1, e_2, ..., e_n, e'_1, e'_2, ..., e'_n$ where e_i is the edge on the outer cycle, e'_i is the edge adjacent with u and u_i. Then $V(T_1(W_n)) = \{u, u_i, e_i, e'_i / 1 \leq i \leq n\}$. For $T_1(W_n)$ the enclave dominating sets are in the following cases,

Case(i): When $n = 3$

In this case the vertices u, u_1, u_2, u_3 are enclave dominating vertices and their corresponding enclave dominating sets are, $E_u = \{u, e'_1, e'_2, e'_3\}, E_{u_1} = \{u_1, e'_1, e_2, e_3\}, E_{u_2} = \{u_2, e'_2, e_1, e_2\}$, and $E_{u_3} = \{u_3, e'_3, e_2, e_3\}$. The cardinality of all these sets are equal to 4. Hence $\gamma_\epsilon(T_1(W_3)) = 4$.

Case(ii): When $n = 4$

In this case the vertices $u, u_i (1 \leq i \leq 4)$ are enclave dominating vertices, their corresponding enclave dominating sets are,
$E_u = \{u, e'_1, e'_2, e'_3, e'_4\}$, $E_{u_1} = \{u_1, e'_1, e_1, e_3, e_4\}$,
$E_{u_2} = \{u_2, e'_2, e_1, e_2, e_4\}, E_{u_3} = \{u_3, e'_3, e_1, e_2, e_3\}$, and $E_{u_4} = \{u_4, e'_4, e_2, e_3, e_4\}$.
The cardinality of all these sets are equal to 5. Hence $\gamma_\epsilon(T_1(W_4)) = 5$.

Case(iii): When $n > 4, n$ is odd

In this case the vertices $u_i (1 \leq i \leq n)$ are enclave dominating vertices, their corresponding enclave dominating sets are in the following subcases,

Subcase(iii)(a): For i is odd

$E_{u_i} = \{u_i, e_i, e'_i, e_j, e_k / 1 \leq j < i, i < k \leq n, j \equiv 0(mod2), k \equiv 1(mod2)\}$

$$|E_{u_i}| = 3 + \frac{n-i}{2} + \frac{i-1}{2} = \frac{n+5}{2} \tag{8}$$

Subcase(iii)(b): For i is even

$E_{u_i} = \{u_i, e_i, e'_i, e_j, e_k / 1 \leq j < i, i < k \leq n, j \equiv 1(mod2), k \equiv 0(mod2)\}$

$$|E_{u_i}| = 3 + \frac{i}{2} + \frac{n-i-1}{2} = \frac{n+5}{2} \tag{9}$$

From Equations (8) and (9) we get that when $n > 4$ and odd, $\gamma_\epsilon(T_1(W_n)) = \frac{n+5}{2}$.

Case(iv): When $n > 4, n$ is even

In this case the vertices $u_i (1 \leq i \leq n)$ are enclave dominating vertices, their corresponding enclave dominating sets are in the following subcases,

Subcase(iv)(a): For i is odd

$$E_{u_i} = \{u_i, e_i', e_j / 1 \leq j \leq n, j \equiv 1(mod2)\}$$

$$|E_{u_i}| = 2 + \frac{n+2}{2} = \frac{n+6}{2} \tag{10}$$

Subcase(iv)(b): For i is even

$E_{u_i} = \{u_i, e_i', e_{i-1}, e_j / 1 \leq j < n, j \equiv 0(mod2)\}$

$$|E_{u_i}| = 3 + \frac{n}{2} = \frac{n+6}{2} \tag{11}$$

From Equations (10) and (11) we get that when $n > 4$ and even, $\gamma_\epsilon(T_1(W_n)) = \frac{n+6}{2}$.

Theorem 3.8 .

Edge vertices in $T_1(G)$ are not enclave dominating vertices.

Proof:

We will prove this theorem by contradiction. Assume that in $T_1(G)$ the edge vertex (say) e_{ij} adjacent to the original vertices u_i, u_j is an enclave dominating vertex. Let $E_{e_{ij}}$ represent the enclave dominating set corresponding to the vertex e_{ij}. According to the definition of $T_1(G)$, we have $N[u_i] \subset N[e_{ij}] \subseteq E_{e_{ij}}$, and $N[v_j] \subset N[e_{ij}] \subseteq E_{e_{ij}}$. This implies that the enclave dominating set $E_{e_{ij}}$ contains more than one enclave vertex, which contradicts the definition of enclave dominating set. Therefore, in $T_1(G)$ the edge vertices cannot serve as the enclave dominating vertices.

Theorem 3.9 .

For any graph G we have $\gamma(G) \leq \gamma_\epsilon(T_1(G))$

Proof:

Since every enclave dominating set of $T_1(G)$ is a dominating set of $T_1(G)$, we have $\gamma(T_1(G)) \leq \gamma_\epsilon(T_1(G))$. Additionally, the cardinality of every minimum dominating set of G is less than or equal to the cardinality of a minimum dominating set of $T_1(G)$, i.e., $\gamma(G) \leq \gamma(T_1(G))$. From these inequalities, we get $\gamma(G) \leq \gamma_\epsilon(T_1(G))$.

4 Enclave Domination Number of Semi-Total Point Graphs

Theorem 4.1 .

For any path P_n with $n \geq 2$ vertices, $\gamma_\epsilon(T_2(P_n)) = \begin{cases} \frac{n+3}{2}, & n \; is \; odd \\ \frac{n+2}{2}, & n \; is \; even \end{cases}$

Proof:

Let P_n be a path with vertices $\{u_1, u_2, ..., u_n\}$, edges $\{e_1, e_2, ..., e_{n-1}\}$ and $V(T_2(P_n)) = \{u_1, u_2, ..., u_n, e_1, e_2, ..., e_{n-1}\}$. To prove the equality, we consider the following two cases,

Case(i): When n is odd

Here, the vertices $e_i (1 < i < n-1)$ in $T_2(P_n)$ are the enclave dominating vertices and their corresponding enclave dominating sets are as follows,

Subcase(i)(a): For $i \equiv 0(mod2)$

$E_{e_i} = \{e_i\} \cup \{u_p / 1 < p \leq i, p \equiv 0(mod2)\} \cup \{u_q / i < q \leq n, q \equiv 1(mod2)\}$

$$|E_{e_i}| = 1 + \frac{i}{2} + \frac{n-i+1}{2} = \frac{n+3}{2} \tag{12}$$

Subcase(i)(b): For $i \equiv 1(mod2)$

$$E_{e_i} = \{e_i\} \cup \{u_p / 1 \leq p \leq i, p \equiv 1(mod2)\} \cup \{u_q / i < q < n, q \equiv 0(mod2)\}$$

$$|E_{e_i}| = 1 + \frac{i+1}{2} + \frac{n-i}{2} = \frac{n+3}{2} \tag{13}$$

Case(ii): When n is even

Here the vertices e_i such that $1 < i < n$ and $i \equiv 0(mod2)$ satisfies the enclave domination conditions and the corresponding enclave dominating set is,

$$E_{e_i} = \{e_i\} \cup \{u_p / 1 < p \le i, p \equiv 0(mod2)\} \cup \{u_q / i < q < n, q \equiv 1(mod2)\}$$

$$|E_{e_i}| = 1 + \frac{i}{2} + \frac{n-i}{2} = \frac{n+2}{2} \tag{14}$$

From Equations (12), (13) and (14) the equality proved.

Theorem 4.2 .

For any cycle C_n, with $n \ge 3$ vertices $\gamma_\epsilon(T_2(C_n)) = \begin{cases} \frac{n+3}{2}, & if\ n\ is\ odd \\ \frac{n+4}{2}, & if\ n\ is\ even \end{cases}$

Proof:

Let C_n be a cycle with vertices $\{u_1, u_2, ..., u_n\}$, edges $\{e_1, e_2, ..., e_n\}$ and $V(T_2(C_n)) = \{u_1, u_2, ..., u_n, e_1, e_2, ..., e_n\}$. To prove the equality we have the following cases,

Case(i): When n is odd

In this case, the vertices e_i for $1 \le i \le n$ satisfy the enclave domination condition, and the corresponding enclave dominating sets are as follows in the subcases,

Subcase(i)(a): Suppose $i \equiv 0(mod2)$

$E_{e_i} = \{e_i, u_i\} \cup \{u_p / 1 \le p < i, p \equiv 1(mod2)\} \cup \{u_q / i < q < n, q \equiv 0(mod2)\}$

Subcase(i)(b): Suppose $i \equiv 1(mod2) E_{e_i} = \{e_i, u_i\} \cup \{u_p / 1 \le p < i, p \equiv 0(mod2)\} \cup \{u_q / i < q < n, q \equiv 1(mod2)\}$From the above two subcases,

$$|E_{e_i}| = \frac{n-i}{2} + \frac{i-1}{2} + 2 = \frac{n-i+i-1+4}{2} = \frac{n+3}{2} \tag{15}$$

Case(ii): When n is even

In this case, the vertices e_i for $1 \le i < n$ satisfies the definition of enclave domination and the corresponding enclave dominating sets are in the following subcases,

Subcase(ii)(a): Suppose $i \equiv 0(mod2)$

$E_{e_i} = \{e_i, u_i\} \cup \{u_p / 1 \le p < n, p \equiv 1(mod2)\}$

Subcase(ii)(b): Suppose $i \equiv 1(mod2)$

$E_{e_i} = \{e_i, u_i\} \cup \{u_p / 1 \le p < n, p \equiv 0(mod2)\}$From the above two subcases,

$$|E_{e_i}| = \frac{n}{2} + 2 = \frac{n+4}{2} \tag{16}$$

From Equations (15) and (16) the equality proved.

Theorem 4.3 .

For any complete graph K_n with n vertices, $\gamma_\epsilon(T_2(K_n)) = n$.

Proof:

Let $V(K_n) = \{u_1, u_2, ..., u_n\}$ be the vertices and $E(K_n) = \{e_{ij} / 1 \le i, j \le n, i \le j\}$ be the edge set of K_n then $V(T_2(K_n)) = \{u_i, e_{ij} / 1 \le i, j \le n, i \le j\}$.

The edge vertices e_{ij} will satisfy the enclave domination condition and the enclave dominating set $E_{e_{ij}}$ will have the edge vertex e_{ij} and its adjacent vertices u_i, u_j then to satisfy the enclave domination conditions the set will have any other $(n-3)u_i's$ from the vertex set of K_n. Therefore $|E_{e_{ij}}| = 3 + n - 3 = n$. Hence $\gamma_\epsilon(T_2(K_n)) = n$.

Theorem 4.4 .

For any complete bipartite graph $K_{m,n}$ with $m+n$ vertices, $\gamma_\epsilon(T_2(K_{m,n})) = min(m,n) + 2$.

Proof:

Let the vertex set of $K_{m,n}$ be $V(K_{m,n}) = \{u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set be $E(K_{m,n}) = \{e_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$ then $V(T_2(K_{m,n})) = \{u_i, v_j, e_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$. The edge vertices e_{ij} satisfy the enclave domination conditions, and the corresponding enclave dominating sets are outlined in the following cases,

Case(i): If $m < n$ then

$$\begin{aligned} &E_{e_{ij}} = \{e_{ij}, v_j, u_p / 1 \leq p \leq m\} \\ &\left|E_{e_{ij}}\right| = m + 2 \end{aligned} \tag{17}$$

Case(ii): If $n < m$ then $E_{e_{ij}} = \{e_{ij}, u_i, v_q / 1 \leq q \leq n\}$

$$|E_{e_{ij}}| = n + 2 \tag{18}$$

From Equations (17) and (18) , we get $|E_{e_{ij}}| = min(m,n) + 2$. Thus, the equality holds.

Theorem 4.5.

For any star $K_{1,n}$ with $n+1$ vertices, $\gamma_\epsilon(T_2(K_{1,n}))$ does not exist.

Proof:

Let $V(K_{1,n}) = \{u, u_1, u_2, ..., u_n\}$, $E(K_{1,n}) = \{e_1, e_2, ..., e_n\}$ then the vertex set of $T_2(K_{1,n})$ is $\{u, u_1, u_2, ..., u_n, e_1, e_2, ..., e_n\}$. If the enclave dominating set for any vertex in the graph is found, it will contain more than one enclave vertex, which contradicts the definition, showing that the enclave domination number does not exist for $T_2(K_{1,n})$.

Theorem 4.6 .

For any bistar $B_{m,n}$ with $m, n \geq 1$, $\gamma_\epsilon(T_2(B_{m,n})) = 3$

Proof:

Let $V(B_{m,n}) = \{u, v, u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\}$, $E(B_{m,n}) = \{e, e_i, e'_j / 1 \leq i \leq m, 1 \leq j \leq n\}$ where the edge e adjacent with vertices u, v the edge e_i adjacent with the vertices u, u_i, and the edge e'_j adjacent with v, v_j then $V(T_2(B_{m,n})) = \{u, v, e, u_i, v_j, e_i, e'_j / 1 \leq i \leq m, 1 \leq j \leq n\}$. For bistar we get a unique enclave dominating set with respect to the vertex e is $E_e = \{e, u, v\}$. Hence $\gamma_\epsilon(T_2(B_{m,n})) = 3$.

Theorem 4.7 .

For any wheel W_n, $n \geq 3$, $\gamma_\epsilon(T_2(W_n)) = 3 + \lceil \frac{n-2}{2} \rceil$

Proof:

Let $u, u_1, u_2, ..., u_n$ be vertices of the wheel W_n with $deg(u) = n$, and the $2n$ edges are denoted by $e_1, e_2, ..., e_n, e'_1, e'_2, ..., e'_n$ where e_i is the edge adjacent with u and u_i, e'_i is the edge on the outer cycle. Then $V(T_2(W_n)) = \{u, u_i, e_i, e'_i / 1 \leq i \leq n\}$.

For $T_2(W_n)$ the vertices $e_i, e'_i (1 \leq i \leq n)$ will be the enclave domination vertex and their corresponding enclave dominating sets are in the following cases,

Case(i): When $n = 3$

$E_{e_i} = \{e_i, u, u_i, e'_{i+1}\}$ for $i = 1, 2$, $E_{e_3} = \{e_3, u, u_3, e'_1\}$, and $E_{e'_i} = \{e'_i, u_1, u_2, u_3\}$. $\left|E_{e_i}\right| = \left|E_{e_3}\right| = \left|E_{e'_i}\right| = 4$. So $\gamma_\epsilon(T_2(W_3)) = 4$.

Case(ii): When $n = 4$

Subcase(ii)(a): i is odd

$E_{e_i} = \{e_i, u, u_j / 1 \leq j \leq n, j \equiv 1(mod2)\}$, $E_{e'_i} = \{e'_i, u_i, u_j / 1 \leq j \leq n, j \equiv 0(mod2)\}$

Subcase(ii)(b): i is even

$E_{e_i} = \{e'_i, u, u_j / 1 \leq j \leq n, j \equiv 0(mod2)\}$, $E_{e'_i} = \{e'_i, u_i, u_j / 1 \leq j \leq n, j \equiv 1(mod2)\}$Thus $\gamma_\epsilon(T_2(W_4)) = 4$.

Case(iii): when n is odd, $n \geq 5$

Subcase(iii)(a): i is odd

$E_{e'_i} = \{e'_i, u, u_i, u_{i+1}, u_p, u_q / 1 \leq p < i, p \equiv 1(mod2), i+1 < q \leq n, q \equiv 0(mod2)\}$

Subcase(iii)(b): i is even

$E_{e_i'} = \{e_i', u, u_i, u_{i+1}, u_p, u_q / 1 \leq p$
$< i, p$
$\equiv 0(mod2),\ i+1$
$< q \leq n, q$
$\equiv 1(mod2)\}$

Case(iv): When *n* is even, $.n \geq 6$.

In this case, only the edge vertex $e_i (1 \leq i \leq n)$ will be the enclave domination vertex. The enclave dominating sets are as follows,

Subcase(iv)(a):i is odd

$E_{e_i} = \{e_i, u, u_j / 1 \leq j \leq n, j \equiv 1(mod2)\}$

Subcase(iv)(b):i is even

$E_{e_i} = \{e_i, u, u_j / 1 \leq j \leq n, j \equiv 0(mod2)\}$In all the cases

$|E_{e_i}| = |E_{e_i'}| = 3 + \lceil \frac{n-2}{2} \rceil$Thus $\gamma_\epsilon(T_2(W_n)) = 3 + \lceil \frac{n-2}{2} \rceil$.

Theorem 4.8 .

For any graph G we have, $\gamma(G) \leq \gamma_\epsilon(T_2(G))$

Proof:

As every enclave dominating set of $T_2(G)$ is a dominating set of $T_2(G)$, we obtain $\gamma(T_2(G)) \leq \gamma_\epsilon(T_2(G))$. Furthermore, the cardinality of any minimum dominating set of G is less than or equal to the cardinality of a minimum dominating set of $T_2(G)$, i.e., $\gamma(G) \leq \gamma(T_2(G))$. Combining these inequalities, we conclude that $\gamma(G) \leq \gamma_\epsilon(T_2(G))$.

Theorem 4.9 .

In [6] For any graph $G, \gamma_\epsilon(G) \neq n$. where n is the order of the graph.

Theorem 4.10 .

In a graph suppose a vertex u, with $deg(u) = 2$ and it is adjacent with v and w. If v and w are adjacent vertices then v and w will not be the enclave domination vertex.

Proof:

Let u be a vertex in G such that $deg(u) = 2$. Let v, w are adjacent vertices of u and there is an edge $e = vw$. If E_v, E_w are enclave dominating sets then $N[u] \subseteq N[v] \subset E_v$, $N[u] \subseteq N[w] \subset E_w$. Thus, the sets E_v, E_w has more than one enclave vertex in it. So, v and w will not be the enclave domination vertex.

Theorem 4.11 .

In $T_2(G)$, $n-$vertices which originated from G will not be the enclave dominating vertex.

Proof:

The proof follows from the above theorem.

Theorem 4.12 .

In $T_2(G)$, enclave domination exists if $diam(T_2(G)) \geq 2$.

Proof:

Let us prove this theorem by contradiction. Suppose $diam(T_2(G)) = 1$ and $|V(T_2(G))| = n$. Assume the vertex u be the enclave dominating vertex, then its respective enclave dominating set E_u will have the vertices adjacent with u. Since $diam(T_2(G)) = 1$, the largest path between any two vertices will be 1. So, $|E_u| = |V(T_2(G))| = n$ which is a contradiction to theorem.4.9. Thus $diam(T_2(G)) \neq 1$, hence $diam(T_2(G)) \geq 2$.

Theorem 4.13 .

For any graph G, if e is the pendant edge in G then e will not be the enclave dominating vertex in its semi-total point graph $T_2(G)$.

Proof:

Let G be any graph, and $e = uv$ be any pendant edge in G, where u is the pendant vertex, v is the support vertex in G. In $T_2(G)$, e is the edge vertex, u and v are its adjacent vertices such that $d(e) = d(u) = 2, d(v) \geq 2$. Suppose E_e is the enclave dominating set then $N[u] \subseteq N[e] \subseteq E_e$. Which contradicts our definition. So, the pendant edge e will not be the enclave dominating vertex in $T_2(G)$.

Note:

The graph $T_2(K_{1,n})$ is an example for the above theorem.

Theorem 4.14 .

If $T_2(G)$ has diameter 2 or 3, then $\gamma_\epsilon(T_2(G)) = 3$.

Proof:

Let $T_2(G)$ be any graph with diameter 2 or 3.

Case(i): Let $diam(T_2(G)) = 2$.

By theorem 4.11. the edge vertices will be the enclave dominating vertex. Let us choose one edge vertex say e_1 then the enclave dominating set E_{e_1} has e_1 and its neighborhoods say u_1, u_2. Since the diameter of $T_2(G)$ is 2, distance of e_1 to all other vertices in $V(T_2(G)) - E_{e_1}$ is 2 and distance of u_1, u_2 to all other vertices in $V(T_2(G)) - E_{e_1}$ is 1. Hence the set $E_{e_1} = \{e_1, u_1, u_2\}$ is the minimum enclave dominating set. Thus $\gamma_\epsilon(T_2(G)) = 3$.

Case(ii): Let $diam(T_2(G)) = 3$.

Choose the vertex say e_1 such that $d(e_1, e_i) = 2$ where e_i represents all other remaining edge vertices in $T_2(G)$. Then the enclave dominating set E_{e_1} will be $\{e_1, u_1, u_2\}$ where $u_, u_2$ are adjacent vertices of e_1. Since $d(e_1, e_i) = 2$, the vertices u_1, u_2 dominate all the edge vertices in $T_2(G)$. Then by the definition of $T_2(G)$, all the vertices also dominated by u_1, u_2.The minimum enclave dominating set is $E_{e_1} = \{e_1, u_1, u_2\}$. Thus $\gamma_\epsilon(T_2(G)) = 3$.

Conversely, suppose $T_2(G)$ be a graph such that $\gamma_\epsilon(T_2(G)) = 3$ we know that only edge vertex will be the dominating vertex in $T_2(G)$. Suppose e_i be the enclave dominating vertex and (say) u_i, u_j are its adjacent vertices then $E_{e_i} = \{e_i, u_i, u_j\}$. Since $\gamma_\epsilon(T_2(G)) = 3$, the set E_{e_i} is the enclave dominating set and it dominates all the vertices in G. Thus, all the vertices in $V(T_2(G)) - E_{e_i}$ will be adjacent to either u_i or u_j or both. Let us assume the following, v_i is the vertex adjacent with u_i, v'_i is the vertex adjacent with u_j, w_i is the vertex adjacent with both. Clearly $d(v_i, v'_i) = 3$ and $d(v_i, v_i) = d(v'_i, v'_i) = d(w_i, w'_i) = d(v_i, w'_i) = d(v'_i, w_i) = 2$. Thus $diam(T_2(G)) = 3$. If the vertex u_i, u_j adjacent to only one vertex say u_k then $diam(T_2(G)) = 2$.

Note:

The above theorem does not exist only if $G \cong C_3 \circ K_1$

Theorem 4.15 .

For any graph G we have $\left\lceil \frac{n+m}{1+\triangle(G)} \right\rceil \leq \gamma_\epsilon(T_2(G)) \leq n+1$

Proof:

Let $|V(G)| = n$, $|E(G)| = m$ then $|V(T_2(G))| = n+m$. From[5] we have $\left\lceil \frac{n+m}{1+\triangle(G)} \right\rceil \leq \gamma(G)$ where n is the number of vertices in G and by theorem 4.8. we have $\gamma(G) \leq \gamma_\epsilon(T_2(G))$. Thus, we get the lower bound as $\left\lceil \frac{n+m}{1+\triangle(G)} \right\rceil \leq \gamma_\epsilon(T_2(G))$. By theorem 4.11. only the edge vertices in $T_2(G)$ will be the enclave dominating vertex. Let e_i be any edge vertex in $T_2(G)$ and E_{e_i} be the corresponding enclave dominating set. $E_{e_i} = e_i \cup V(G)$ is the maximum set which satisfies the enclave dominating conditions. And $|E_{e_i}| = n+1$ thus we get the upper bound as $\gamma_\epsilon(T_2(G)) \leq n+1$.

5 Enclave Domination Number of $T_1(T)$ and $T_2(T)$

Theorem 5.1.

For any tree T, $\gamma_\epsilon(T) = \beta(T)+1$

Proof:

Let T be any tree, A be the minimum vertex cover set of T and $|A| = \beta(T)$. Let p be any pendant vertex in T, and its corresponding minimum enclave dominating set is E_p. From[4] $\gamma(T) = \beta(T)$, and so $E_p = A \cup \{p\}$, clearly the set A must have all the support vertex in T. $|E_p| = |A|+1$, thus $\gamma_\epsilon(T) = \beta(T)+1$.

Theorem 5.2.

Let T be any tree with n vertices then $p+1 \leq \gamma_\epsilon(T_1(T))$, where p is the number of pendant edges in T. The equality holds if every vertex in T is either a pendant vertex or adjacent to exactly one pendant vertex.

Proof:

Let T be any tree with n vertices, and p be the number of pendant edges in T. Any pendant vertex (say) u in T are enclave dominating vertex in $T_1(T)$. And the enclave dominating set E_u of $T_1(T)$ must have all the pendant edge vertices, and one pendant vertex (say) u as enclave vertex. This implies, $p+1 < \gamma_\epsilon(T_1(T))$. If every vertex in T is either a pendant vertex or adjacent to exactly one pendant vertex then E_u will be the minimum enclave dominating set of $T_1(T)$. And $|E_u| = p+1$, thus $p+1 = \gamma_\epsilon(T_1(T))$.

From the above inequalities we conclude that $p+1 \leq \gamma_\epsilon(T_1(T))$.

Theorem 5.3.

Let T be any tree with n vertices then $s+1 = \gamma_\epsilon(T_2(T))$, if every vertex in T is either a pendant vertex or adjacent to at least one pendant vertex. Otherwise $s+2 < \gamma_\epsilon(T_2(T))$, where s is the number of support vertex in T.

Proof:

Let T be any tree with n vertices, and s be the number of support vertices in T. In $T_2(G)$ the edge vertices are enclave dominating vertex. The pendant edge vertex (say) p is the enclave dominating vertex in $T_2(T)$. If every vertex in T is either a pendant vertex or adjacent to at least one pendant vertex then E_p will be the minimum enclave dominating set of $T_2(T)$.

And $|E_p| = s+1$ thus $s+1 = \gamma_\epsilon(T_2(T))$. If some vertex in T is neither a pendant vertex nor adjacent to at least one pendant vertex then the enclave dominating set E_p will have p and its neighborhood 2 vertices, in which one vertex is the support vertex, other is the pendant vertex in T. The set E_p will have all the support vertex of T. Thus $|E_p| > 3+s-1 = s+2$. So, $s+2 < \gamma_\epsilon(T_2(T))$.

Theorem 5.4.

For any tree T with $n \geq 4$ vertices, $3 \leq \gamma_\epsilon(T_2(T)) \leq n+1$

Proof:

Let T be a tree such that $|V(T)| = n$ and $|E(T)| = n-1$ then $|V(T_2(T)| = 2n-1$

Case(i): Let T be a tree with $n = 4$ vertices, then $|V(T_2)| = 7$. If we choose the edge vertex which is adjacent with $\Delta(T)$ and its adjacent vertices will satisfy the enclave domination condition. Hence $\gamma_\epsilon(T_2(T)) \geq 3$.

Case(ii): Let T be a tree with $n > 4$ vertices.$T_2(T)$ has $n-1$ closed paths, in-order to satisfy the enclave domination we need to choose the edge vertex which is adjacent with $\Delta(T)$ and its adjacent vertices. So, one closed path is selected. Out of the remaining $n-2$ closed path we can choose at most two vertices so that enclave condition is satisfied. Thus, in each $n-2$ closed path, there is one vertex that does not belong to the enclave dominating set.

Hence $\gamma_\epsilon(T_2(T)) \leq 2n-1-(n-2) = 2n-1-n+2 = n+1$.

From the above cases, we get $3 \leq \gamma_\epsilon(T_2(T)) \leq n+1$.

Theorem 5.5.

For any tree T with $n \geq 2$ vertices, $2 \leq \gamma_\epsilon(T_1(T)) \leq n$

Proof:

Let T be a tree such that $|V(T)| = n$ and $|E(T)| = n-1$ then $|V(T_1(T)| = 2n-1$

Case(i): Let T be a tree with $n = 2$ vertices, then $|V(T_1(T))| = 3$. If we choose the edge vertex and its one adjacent vertex will satisfy the enclave domination condition. Hence $\gamma_\epsilon(T_1(T)) \geq 2$.

Case(ii): Let T be a tree with $n > 2$ vertices. By theorem 3.8. the edge vertices not be the enclave dominating vertex. Let us construct the enclave dominating set E_{u_i} with respect to any vertex u_i in T. Suppose the set E_{u_i} have all the edge vertices then the domination condition will be satisfied. If we choose the vertex u_i belongs to the set, then $E_{u_i} = \{u_i, e_1, e_2, ..., e_{n-1}\}$ will be the satisfy the enclave domination condition. Hence $\gamma_\epsilon(T_1(T)) \leq 1+(n-1) = n$.

From the above cases, we have $2 \leq \gamma_\epsilon(T_1(T)) \leq n$

Theorem 5.6.

Let T be any tree then $\gamma_\epsilon(T) \leq \gamma_\epsilon(T_2(T))$. The equality holds if every vertex in T is either a pendant vertex or adjacent to at least one pendant vertex.

Proof:

Let us prove the inequality by the following two cases,

Case(i): Suppose every vertex in T is either pendant vertex or adjacent to at least one pendant vertex.

Let S be the set of all support vertices in T, $|S| = s$. By theorem 5.1. $\gamma_\epsilon(T) = \beta+1$. In this case $\beta = |S| = s$. Thus $\gamma_\epsilon(T) = s+1$. By theorem 5.5. for this case, we have $s+1 = \gamma_\epsilon(T_2(T))$. From the two equalities, we get $\gamma_\epsilon(T) = \gamma_\epsilon(T_2(T))$.

Case(ii): Suppose there exist a vertex in T that is neither pendant vertex nor adjacent to pendant vertex.

Let A be the set of minimum vertex cover in T, $|A| = \beta$. The minimum enclave dominating set of T is $E_u = A \cup \{u\}$ where u is any pendant vertex in T. And $|E_u| = |A|+1$, $\gamma_\epsilon(T) = \beta+1$. In $T_2(T)$ any support edge vertex (say) e is the enclave dominating vertex and its minimal enclave dominating set is $E_e = N[e] \cup A$. Clearly, there exists exactly one vertex in $N[e] \cap A$. And so $|E_e| = |A|+2$ thus $\gamma_\epsilon(T_2(T)) = \beta+2$. From the two equlaites, we get $\gamma_\epsilon(T) < \gamma_\epsilon(T_2(T))$.

From the above cases, we get $\gamma_\epsilon(T) \leq \gamma_\epsilon(T_2(T))$.

Theorem 5.7.

Let T be any tree then $\gamma_\epsilon(T_2(T)) \leq \gamma_\epsilon(T_1(T))$. The equality holds if $T \cong P_n$, or if every vertex in T is either pendant vertex or adjacent to exactly one pendant vertex.

Proof:

Let us prove the inequality from the following cases,

Case(i) If $T \cong P_n$ By theorem 3.1. and theorem 4.1. we say that $\gamma_\epsilon(T_2(P_n)) = \gamma_\epsilon(T_1(P_n))$.

Case(ii): Assume that every vertex in T is either pendant vertex or is adjacent to at least one pendant vertex.

In this case, for $T_2(T)$ every support vertex, one pendant edge vertex and its corresponding pendant vertex in T are included in the enclave dominating set. For $T_1(T)$, all the pendant edge vertices and any one pendant vertex in T belong to the enclave dominating set. Furthermore, the cardinality of minimum enclave dominating sets in both graphs are equal.

Therefore $\gamma_\epsilon(T_2(T)) = \gamma_\epsilon(T_1(T))$.

Case(iii): Assume that some vertex in T is neither pendant vertex nor is adjacent to the pendant vertex.

Let A and B be the set of all support vertex and support edges in T respectively. Clearly for this case $|A| \leq |B|$. The minimum enclave dominating set of $T_2(T)$ must have all the support vertex in T, whereas the minimum enclave dominating set in $T_1(T)$ must have all the support edge vertex in T. Also for the enclave vertex in $T_2(T), N[e] = 3$, whereas in $T_1(T), N[u] = 2$. And by the adjacency conditions specified in $T_1(G)$ and $T_2(G)$ we get $\gamma_\epsilon(T_2(T)) < \gamma_\epsilon(T_1(T))$.

From the above cases, we conclude that $\gamma_\epsilon(T_2(T)) \leq \gamma_\epsilon(T_1(T))$.

Theorem 5.8.

For any tree T, $\gamma_\epsilon(T) \leq \gamma_\epsilon(T_2(T)) \leq \gamma_\epsilon(T_1(T))$. The equality holds if every non-pendant vertex in T is adjacent to exactly one pendant vertex.

Proof:

The inequalities follow from theorem (5.6.) and (5.7.)

6 Conclusion

This paper presents a study on enclave dominating sets and explores the enclave domination number of semi-total graphs $T_1(G)$, $T_2(G)$ associated with certain standard and special graphs. And we characterize the enclave dominating sets in semi-total line graphs and semi-total point graphs. Future research aims to develop algorithms for determining the enclave domination number and examine potential applications of this concept.

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