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RESEARCH ARTICLE

Performance Evaluation of Erlang Loss Model With Encouraged Arrival and Reverse Reneging

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Abstract

Objectives: To study the stochastic modeling of an Erlang loss model incorporating features to capture realistic customer behavior: encouraged arrival and reverse reneging. **Methods:** The Markov process method and probability-generating functions are used to establish steady-state probabilities and closed-form expressions of various performance measures for this model. R programming software is employed to perform the sensitivity analysis, cost analysis, and calculations for the numerical example. **Findings:** Closed-form expressions of various performance measures are derived, providing insights into the causes of queuing issues from customer impatience and opportunities for service delivery improvement. Beyond traditional performance metrics, this study introduces several new measures to enhance the evaluation of the system's effectiveness. To the best of our knowledge, in literature, the determination of optimum number servers concerning cost as well as sensitivity analysis for the Erlang loss model under the assumed restrictions is not available. Hence a cost analysis is performed to obtain an optimum number of servers, which provides a clear understanding of the financial implications of implementing and operating the system. A numerical sensitivity analysis observes variations in performance measures with system parameter changes. **Novelty:** No previous study has modeled an Erlang loss system performing cost analysis that takes into account encouraged arrival and reverse reneging together. Also, closed-form expressions of system performance measures are rarely available in the existing work. This study is an attempt to address these gaps by providing closed-form expressions of the model's performance measures and a cost analysis, which are key to the effective management of any system.

Mathematics Subject Classification (2020): 60K25, 68M20, 90B22.

Keywords: Erlang; Markovian; Markov Process; Reverse Reneging; Encouraged Arrival

1 Introduction

People standing in queues are common at various business outlets, whether shopping malls, shops, restaurants, airports, railway stations, etc. Customer impatience and rush over billing as well as receiving services play a pivotal role in organizing and shaping the overall performance of the system. Customers are said to be impatient if they decide to join a queue only when expecting a short wait and choose to stay in the queue. There are two forms of customer impatience: balking and reneging. Balking occurs when a customer approaches a queue, deems it too long, and opts not to join, leaving the system instead. On the other hand, reneging happens when customers in the queue lose patience and exit without receiving the service. Many works have been done in the queuing literature addressing customer impatience. An analysis in [1] investigates reverse balking in the M/M/1/k queue model deriving steady-state probabilities and performance metrics to enhance understanding customer behavior and its impact on trust-based industries. The concept of reverse reneging and encouraged arrival was not considered. However, sensitivity analysis was not explored in that study, which we address in our work to determine how changes in parameters affect the results.

In the study of queueing systems and customer conduct, the concepts of reneging and reverse reneging play a critical role in understanding service dynamics and streamlining functional productivity. Reneging occurs when customers leave the queue before being served due to impatience or dissatisfaction with wait times. This behavior of customers can altogether negatively impact service efficiency and customer loyalty, creating difficulties for organizations that intend to keep up with proficiency and hold customers. In contrast, Reverse reneging is something contrary to typical reneging conduct, where customers are motivated to stay in the queue as the queue length extends. A few potential reasons are that a long queue could indicate to customers that the service is top-notch or popular, growing their eagerness to stand by. Typical reneging happens because of various mental and social elements. Additionally, seeing other customers waiting can act as social proof, proposing that the service is worth waiting for. For example, we can see this reverse reneging conduct in customers choosing insurance organizations. Customers often prefer organizations with the biggest number of existing policyholders. The purpose of this is that an organization with numerous customers is seen as more dependable, trustworthy, and offering top-notch service. As a result, they are more willing to wait longer or go through more work to turn into a customer of such an organization. Reverse reneging adds a layer of intricacy to queue management and service operations, as it requires systems to be flexible enough to accommodate customers who change their minds. Understanding the factors leading to both reneging and reverse reneging is essential for designing effective queue management strategies and upgrading the general client experience. An interconnected M/M/c/N queueing model with reverse balking, reneging, and adjustable arrival rates was carried out in [2]. Steady-state conditions and performance metrics are established. However, the study does not consider the assumptions of reverse reneging and encouraged arrival. A finite-buffer feedback queue with modified reverse balking and retention of impatient customers is carried out in [3] by limiting to the single-server scenario and deriving steady-state probabilities. However, the study fails to address the scenario in the case of extending the analysis to multi-server systems. Even though reverse reneging and encouraged arrival are considered together [4], these restrictions were applied to a heterogeneous M/M/2/K model, not to the Erlang loss model. Also, cost analysis has not been carried out.

In today's highly competitive environment, companies face tough competition and strive to understand and deal with customers' changing ways of behaving while at the same time fulfilling industry standards, satisfying customer needs, and achieving service goals. Customers often look for appealing offers from various businesses before making a purchase decision. To ensure sustainable growth, businesses regularly offer a variety of promotions and discounts to retain existing customers and attract new ones. These strategies lead customers to spend more time in queues, a phenomenon known as encouraged arrival of customers. The concepts of encouraged and discouraged arrival have become increasingly significant in Queueing Theory. Although much work has been done on customer impatience in various types of queueing systems, we could not perhaps find a solitary study that considers reverse reneging and encouraged arrival together in an Erlang loss model. This paper expects to address these gaps. The study [5] inspects an FM/FM/1/N queue with encouraged or discouraged arrivals and modified reneging under fuzzy circumstances, using α-cut and Zadeh's principle to derive crisp performance intervals. However, a notable research gap is that this study fails to obtain the optimum number of servers with respect to cost. A recent study [6] analyses an M/M/C queue with multiple and single working vacations under encouraged arrivals and customer impatience. Despite considering encouraged arrivals, the study does not address the reverse reneging circumstances. Again, a study [7] uses Chapman-Kolmogorov techniques to optimize car assembly line performance by analyzing a model with multiple working vacations and encouraged arrivals. The assumption of particular server states (busy, idle, regular, breakdown) represents a research gap since it fails to capture the complexity and variability of real-world server conditions. The study [8] develops a finite-server Markovian queue with heterogeneous services with encouraged arrivals, recursive probabilities, and performance metrics. A study [9] on the M/M/1/N model which considers encouraged arrivals, feedback, balking, and maintaining reneged customers has derived performance measures whereas the study [10] also analyzed the same model considering modified reneging and derived steady-state equations. However, both the study do not consider the scenarios of multiple servers. Also, cost analysis was not performed in both. This research gap is addressed in the present study by extending it to a multi-server model along with encouraged arrivals and reverse reneging.

Even though research has been carried out on customers' impatience across various queuing patterns, the stochastic modeling of the Erlang loss model with reverse reneging and encouraged arrival- an innovative concept in queueing literature, appears to remain unexplored in the literature. Our paper is motivated by these works and thus the objective is to explore reverse reneging along with encouraged arrival on the Erlang loss model. Furthermore, closed-form expressions for various performance measures are not available and sensitivity analysis to see the effect of model parameters on different performance measures for the proposed model is lacking. Also cost analysis for the proposed model is not available. In our research work cost analysis is used to determine the optimal number of servers concerning minimum cost which has a wide range of applications in business. We aim to address these research gaps in the literature. This work hunts for valuable insights into managing queueing systems effectively in environments where customer behavior can significantly impact service delivery and customer satisfaction.

The designed model is formulated under the following assumptions:

- Arrivals follow Poisson distribution with parameter λ. The encouraged arrival rate is taken as $\lambda(1+\eta)$, where 'η' represents the percentage increase in the arrival rate of customers, calculated from past or observed data.
- Service rate follows an exponential distribution with parameter μ.
- The system consists of c servers (c >1).
- The system capacity is restricted to c i.e. the system's capacity is finite.
- Reneging rate follows an exponential distribution with parameter ν and the reverse reneging rate is (c-n+1) ν, n=1,2,…,c

The rest of the model's assumptions are the same as the traditional M/M/1 queueing system.

Application of our proposed model can be observed in a car washing center. Suppose a car wash center, "Sparkle Clean," operates with 4 washing bays, here bays being the server, (c= 4). Only 4 cars are allowed to be in the bays. Suppose Sparkle Clean offers a 20% discount on car washes during peak hours or festive sessions (e.g., Vishwakarma Puja). They initiate a loyalty program where frequent customers get every 10th wash for free. Such offers/discounts encourage more customers to come in. Customers visit the wash center as they find a greater number of customers gathering in Sparkle Clean due to trusted services, this attracts newer customers to put their cars in Sparkle Clean. This resembles our assumption of reverse reneging.

The rest of the paper is organized as follows: Section 2 gives a detailed insight into the methodology used in this research work. Section 3 presents the result and discussion divided into 5 sub-sections. Section 3.1 presents the derivation of steady-state probabilities, Section 3.2 includes various effective performance measures to demonstrate the practical utility of the proposed model, and Section 3.3 includes a numerical example. Section 3.4 conducts a sensitivity analysis. In Section 3.5, a cost analysis is carried out to clarify the financial implications of implementing and operating the system, aiding decision-makers in evaluating its economic feasibility. Section 4 offers a concise conclusion summarizing the key findings and their implications. Lastly, Section 5 lists the References.

2 Methodology

The Markov process method is employed to derive the steady-state probabilities of the assumed model. However, the same has been obtained in [7] using Chapman Kolmogorov equations. Several techniques viz: differential equation method [7], matrix geometric method [11], etc. are available in the literature to derive closed-form expression of various performance measures. To avoid mathematical complexity probability-generating function method has been used to derive some traditional as well as newly designed performance measures in this paper. Calculations for numerical examples have been done using R programming. The same software is also employed to perform the sensitivity analysis and the cost analysis.

3 Result and Discussion

3.1 Steady State Probabilities

The state transition diagram is given by Figure 1.

The steady-state equations are:

$$\lambda\,(1+\eta)\,p_0 = (\mu + c\,\nu)\,p_1,\; n=0 \tag{3.1}$$

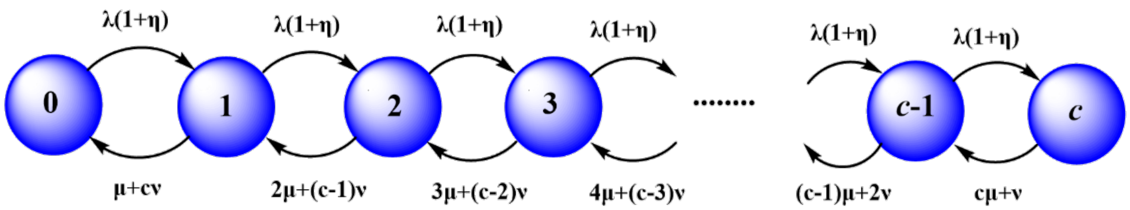

Fig 1. State Transition diagram

$$\lambda\,(1+\eta)\,p_{n-1}+\{(n+1)\mu+(c-n)\,\nu\}p_{n+1}=\{\,\lambda(1+\eta)+n\mu+(c-n+1)\nu\}\,p_n,\,n=1,2,...c-1 \tag{3.2}$$

$$\lambda\,(1+\eta)\,p_{c-1}=(c\mu+\,\nu)p_c\,,n=c \tag{3.3}$$

where p_n denotes the probability that there are n customers in the system.

Solving the above equations, we get

$$p_n=\frac{\lambda^n(1+\eta)^n}{\prod_{s=1}^{n}\ s\mu+(c-s+1)\,\nu}p_0,\,where\,n=\,1,\,2,\,\ldots,\,c-1,c \tag{3.4}$$

Using the normalizing condition, $\sum_{n=0}^{c}p_n=1$, we obtain p_0

$$p_0=\left\{1+\sum_{n=1}^{c}\frac{\lambda_n(1+\eta)^n}{\underset{s=1}{\overset{n}{\pi}}s\mu+(c-s+1)\nu}\right\}^{-1} \tag{3.5}$$

Special Case:

I. Putting c=1 in Equation (3.4) and Equation (3.5) respectively, we get the expression of p_n and p_0 for single server one capacity Markovian queueing model with encouraged arrival and reverse reneging.

$$p_1=\frac{\lambda(1+\eta)}{\mu+\nu}p_0 \tag{3.6}$$

And

$$p_0=\left(1+\frac{\lambda(1+\eta)}{\mu+\nu}\right)^{-1} \tag{3.7}$$

II. Putting $\eta=0$ and ν=0 in Equation (3.4) and Equation (3.5) respectively, we get the expression of p_n and p_0 for traditional M/M/c/c queueing model.

$$p_n=\frac{\lambda^n}{\prod_{s=1}^{n}\ s\mu}p_0,\,where\,n=\,1,2,\,\ldots,\,c-1,\,c \tag{3.8}$$

$$p_0=\left(1+\sum_{n=1}^{c}\frac{\lambda^n}{\prod_{s=1}^{n}s\mu}\right)^{-1} \tag{3.9}$$

3.2 Performance Measures

Performance measures are quantitative indicators used to evaluate the efficiency and effectiveness of a system or process. Common metrics include average system size, average queue size, average waiting time in the system, average waiting time in the queue, Average rate of reneging, etc. These measures help in ensuring the system meets desired performance standards. Having calculated probabilistic measures, the following measures of performance have been derived:

I. Average System Size (L)

The average system size in a queuing model refers to the expected number of customers present in the system at any given time usually denoted by L and is given by.

$L = \frac{1}{\{\mu-\nu\}}[\ \lambda(1+n)\ (1-p_c) - \nu(1-p_0)(c+1)]$

(derivation of L is given in Appendix I)

II. Average Reverse Reneging Rate ($\mathbf{Avg_{rr}}$)

In reverse reneging, as the queue length increases, the customers are more likely to wait longer. Since the reverse reneging rate is a function of the state of the system, it would vary according to the state of the system. The average reverse reneging rate is given by,

$Avg_{rr} = \sum_{n=1}^{c}(c-n+1)\nu p_n$

$= \frac{\nu}{\{\mu-\nu\}}[\mu\ (1\text{-}p_0)(c+1) - \lambda(1+\eta)(1-p_c)]$

III. Effective Arrival Rate λ_e

The effective arrival rate in the queuing model is the rate at which customers enter the system and remain until served, excluding those who balk and turn away because of finite buffer restrictions. Thus, the effective arrival rate is given by,

$\lambda_e = \lambda(1+\eta)\sum_{n=0}^{c-1} p_n$

$= \lambda(1+\eta)(1-p_c)$

where $p_c = \frac{\lambda^c(1+\eta)^c}{\prod_{s=1}^{c} s\mu+(c-s+1)\nu}$

IV. Proportion of Total Customer Lost ($\mathbf{P^T}$)

Customer loss can happen in three ways viz: reneging, balking, and due to finite buffer restriction. So, we can determine the proportion of total customers lost by combining the proportion of customers lost through these three mechanisms. The closed form expression for the proportion of total customer lost is given by,

$P^T = \frac{\lambda-\lambda_e+Avg_{rr}}{\lambda}$

$= 1 - \frac{\mu}{\mu-\nu}(1+\eta)(1-p_c) + \frac{\nu\mu(1-p_0)(c+1)}{\lambda(\mu-\nu)}$

V. Actual load of the server ($\boldsymbol{\lambda^s}$)

The actual load of a server represents the fraction of time it is actively processing tasks. It is typically expressed as a ratio of the server's busy time to its total available time. A high load indicates the server is frequently busy, while a low load suggests underutilization. Proper load management ensures efficient resource use and optimal performance.

$\lambda^s = \lambda_e(1 - \frac{Avg_{rr}}{\lambda_e})$

$= \lambda(1+\eta)(1-p_c)\frac{\mu}{\mu-\nu} - \frac{\nu}{\{\mu-\nu\}}\mu\ (1\text{-}p_0)(c+1)$

3.3 Numerical Illustration

To illustrate the usefulness of our results, we consider the following example from [12].

"A cell tower can support c simultaneous calls in its coverage area. Demand for calls is Poisson with rate of 30 per hour. Calls are exponential with a mean of 4 minutes. Calls are blocked when the cell tower is busy with c calls in progress.

(a) If c=4, determine the fraction of blocked calls.

(b) Each completed call generates revenue of $0.50. Each blocked calls incurs a cost of $1.00. Which of the following cell towers has the shortest break-even time (the time at which the cumulative revenue equals the cost of the cell tower): A cell tower with c=2, costing $10,000; a cell tower with c=4, costing $20,000; a cell tower with c=6, costing $30,000".

This is a design problem. A reading of the problem statement tells us that the arrival rate is 30 per hour and the service rate is 15 per hour. Hence here, λ = 30/h and μ = 15/h. Let us consider a possible Markovian reneging rate of ν = 0.3/h.

Encouraged arrival and Reverse reneging can be fitted into this problem as follows:

The telecom company Reliance Jio began its operations in 2016 in India. Jio offered free services to its customers for an entire year, which served as a key strategy to attract users. The provision of unlimited high-speed 4G data and unlimited calling was unprecedented and highly appealing to customers. Within just 2.5 years, Jio amassed over 300 million active subscribers. This aligns with our assumption of encouraged arrival. (The reason behind choosing n=0.014 and n=0.071in this numerical example is given in Appendix II)

As Jio's customer base grew rapidly, the company had to establish additional cell towers to meet the increasing demand. This expansion improved connectivity, enhancing customer satisfaction and loyalty to the Jio network. The growing number of satisfied customers also attracted new users, as the perception of better service spread. This phenomenon, where satisfied customers attract more users, can be described as reverse reneging.

Various performance measures of interest computed are given in Table 1 and Table 2. These measures were arrived at by using R programming codes. Different choices of c were considered (all rates in the tables are per-hour rates).

Table 1. Performance measures assuming λ = 30/h, μ = 15/h, ν = 0.3/h, η=0.014

Performance Measures	Number of calls	
	c =2	c =4
Fraction of time that all servers are idle (p_o)	0.2038	0.1519
Fraction of blocked calls	0.3989	0.0945
Average length of system	1.1951	1.7872
Mean reneging rate	0.3581	0.7359
Effective arrival rate	18.2853	27.5446
Proportion of customers lost due to reneging	0.0196	0.02672
Proportion of total customers lost	0.4024	0.1064

Table 2. Performance measures assuming λ = 30/h, μ = 15/h, and ν = 0.3/h, η=0.071

Performance Measure	Number of calls	
	c =2	c =4
Fraction of time that all server is idle (p_o)	0.1907	0.1375
Fraction of blocked calls	0.4165	0.1064
Average length of system	1.2258	1.8650
Mean reneging rate	0.3606	0.7342
Effective arrival rate	18.7476	28.7095
Proportion of customers lost due to reneging	0.0192	0.0256
Proportion of total customers lost	0.3871	0.0674

Some interesting observations from Table 1 and Table 2 are

- The Fraction of blocked calls for η=1.4% when c=2 is 0.3989 and when c=4 is 0.0945. The fraction of blocked calls decreases by 76.3% as we increase the maximum supported simultaneous calls in its coverage area from 2 to 4.
- The Fraction of blocked calls for η=7.1% when c=2 is 0.4165 and when c=4 is 0.1064. The fraction of blocked calls decreases by 74.45% as we increase the maximum supported simultaneous calls in its coverage area from 2 to 4.
- By increase in encouraged arrival rate from η=1.4% to η=7.1%, the fraction of blocked calls increases by 4.22% when c=2.
- By increase in encouraged arrival rate from η=1.4% to η=7.1%, the fraction of blocked calls increases by 11.18% when c=4.
- For η=1.4% the percentage decrease in the Proportion of total customers lost when the server increases from C=2 to c=4 is 29.6%.
- For η=7.1% the percentage decrease in the Proportion of total customers lost when the server increases from C=2 to c=4 is 82.58%.

3.4 Sensitivity Analysis

It is interesting to investigate how server utilization changes when system parameters are altered. Let $p_n(\lambda, \mu, \nu)$ denotes the probability that there are 'n' customers in a system with parameters λ, μ, ν in a steady state.

(i) If $\lambda_1 > \lambda_o, then\ \frac{P_o(\lambda_1,\mu,\nu)}{P_o(\lambda_0,\mu,\nu)} < 1$

$$\Rightarrow \frac{1+\sum_{n=1}^{c} \frac{\lambda_0 n(1+\eta)^n}{\prod_{s=1}^{n} \frac{1}{s\mu+c-s+1\nu}}}{1+\sum_{n=1}^{c} \frac{\lambda_1^n (1+\eta)^n}{\prod_{s=1}^{n} \frac{1}{s\mu+c-s+1\nu}}} < 1$$

$$\Rightarrow 1+\sum_{n=1}^{c} \frac{\lambda_o^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\nu} < 1+\sum_{n=1}^{c} \frac{\lambda_1^n\ \ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\nu}$$

$$\Rightarrow \sum_{n=1}^{c} \frac{\lambda_o^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\nu} < \sum_{n=1}^{c} \frac{\lambda_1^n\ \ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\nu}$$

$$\Rightarrow \frac{(\lambda_o-\lambda_1)\ (1+\eta)}{(\mu+c\nu)} + \frac{(\lambda_o^2-\lambda_1^2)\ (1+\eta)^{\ 2}}{(\mu+c\nu)\ (\mu+c-1)\nu} + \cdots < 0$$

Hence, $p_o \downarrow$ as $\lambda \uparrow$.

(ii) If $\mu_1 > \mu_o, then\ \frac{P_0(\lambda,\mu_1,\nu)}{P_0(\lambda,\mu_0,\nu)} > 1$

$$\Rightarrow \frac{1+\sum_{n=1}^{c} \frac{\lambda^n(1+\eta)^n}{\prod_{s=1}^{n} \frac{1}{s\mu_0+c-s+1\nu}}}{1+\sum_{n=1}^{c} \frac{\lambda^n(1+\eta)^n}{\prod_{s=1}^{n} \frac{1}{s\mu_1+c-s+1\nu}}} > 1$$

$$\Rightarrow 1+\sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu_o+(c-s+1)\nu} > 1+\sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu_1+(c-s+1)\nu}$$

$$\Rightarrow \sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu_o+(c-s+1)\nu} > \sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu_1+(c-s+1)\nu}$$

$$\Rightarrow \frac{\lambda\ (1+\eta)}{\mu_o+c\nu} + \frac{\lambda^2\ (1+\eta)^2}{(\mu_o+c\nu)\ \{2\mu_o+(c-1)\nu\}} + \cdots + \frac{\lambda^c\ (1+\eta)^c}{(\mu_o+c\nu)\ldots.\{c\mu_o+\nu\}} > \frac{\lambda\ (1+)}{\mu_1+c\nu} +$$
$$\frac{\lambda^2\ (1+\eta)^2}{(\mu_1+c\nu)\ \{2\mu_1+(c-1)\nu\}} + \cdots + \frac{\lambda^c\ (1+\eta)^c}{(\mu_1+c\nu)\ldots.\{c\mu_1+\nu\}}$$

$$\lambda\,(1+\eta)\left(\frac{1}{\mu_o+c\nu} - \frac{1}{\mu_1+c\nu}\right\} + \lambda^2\,(1+\eta)^2\left(\frac{1}{(\mu_o+c\nu)\ \{2\mu_o+(c-1)\nu\}} - \frac{1}{(\mu_1+c\nu)\ \{2\mu_1+(c-1)\nu\}}\right\} +$$
$$\ldots\lambda^c\,(1+\eta)^c\left(\frac{1}{(\mu_o+c\nu)\ldots\ldots(c\mu_o+\nu\}} - \frac{1}{(\mu_1+c\nu)\ \{c\mu_1+\nu\}}\right\} > 0$$

Hence, $p_o \uparrow$ as $\mu \uparrow$.

(iii) If $\nu_1 > \nu_o, then\ \frac{P_0(\lambda,\mu,\nu_1)}{P_0(\lambda,\mu,\nu_0)} > 1$

$$\Rightarrow \frac{1+\sum_{n=1}^{c} \frac{\lambda^n(1+\eta)^n}{\prod_{s=1}^{n} \frac{1}{s\mu+(c-s+1)\nu_0}}}{1+\sum_{n=1}^{c} \frac{\lambda^n(1+\eta)^n}{\prod_{s=1}^{n} \frac{1}{s\mu+(c-s+1)\nu_1}}} > 1$$

$$\Rightarrow 1+\sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\ \nu_o} > 1+\sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\ \nu_1}$$

$$\Rightarrow \sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\ \nu_o} > \sum_{n=1}^{c} \frac{\lambda^n\ (1+\eta)^{\ n}}{\pi_{s=1}^{n}\ s\mu+(c-s+1)\nu_1}$$

$$\Rightarrow \frac{\lambda(1+\eta)}{\mu+c\nu_o} + \frac{\lambda^2\,(1+\eta)^2}{(\mu+c\nu_o)\ \{2\mu+(c-1)\nu_o\}} + \ldots \frac{\lambda^c(1+\eta)^c}{(\mu+c\nu_o)+\cdots+\{c\mu+\nu_o\}} > \frac{\lambda(1+\eta)}{\mu+c\nu_1} +$$
$$\frac{\lambda^2\ \ (1+\eta)^2}{(\mu+c\nu_1)\ \{2\mu_1+(c-1)\nu_1\}} + \cdots + \frac{\lambda^c(1+\eta)^c}{(\mu+c\nu_1)\ldots\ldots\{c\mu+\nu_1\}}$$

$$\Rightarrow (1+\eta)\left(\frac{1}{\mu+c\nu_o} - \frac{1}{\mu+c\nu_1}\right\} + \lambda^2\,(1+\eta)^2\left(\frac{1}{(\mu+c\nu_o)\ \{2\mu+(c-1)\nu_o\}} - \frac{1}{(\mu+c\nu_1)\ \{2\mu_1+(c-1)\nu_1\}}\right\} +$$
$$\ldots\lambda^c\,(1+\eta)^c\left(\frac{1}{(\mu+c\nu_o)\ldots\ldots\{c\mu+\nu_o\}} - \frac{1}{(\mu+c\nu_1)\{c\mu+\nu_1\}}\right\} > 0$$

Hence, $p_o \uparrow$ as $\nu \uparrow$.

We have also carried out sensitivity analysis numerically to observe the variation in other performance measures with respect to the system parameters and plotted graphically. The analysis has been carried out considering the parameter values from the numerical illustration.

From Figure 2, it can be observed that a rise in the arrival rate corresponds to an increase in the mean system size and the proportion of total customers lost. The arrival of more customers results in an expanded system size and places a greater workload on the server. This increased workload results in longer waiting times for customers, ultimately leading to a greater proportion of customers lost. Conversely, there is a decrease in the probability of the server remaining idle with higher arrival rates. It is because as the number of arriving customers increases, the server has to work more and so the probability of the server being idle decreases.

From Figure 3, it is evident that as the service rate increases, there's a corresponding reduction in the system size. Consequently, the proportion of total customers lost decreased. This happens because faster service means shorter wait times, so fewer people get frustrated and leave the system. As the service rate rises, there is an accompanying increase in the probability of the system being idle, which is quite evident.

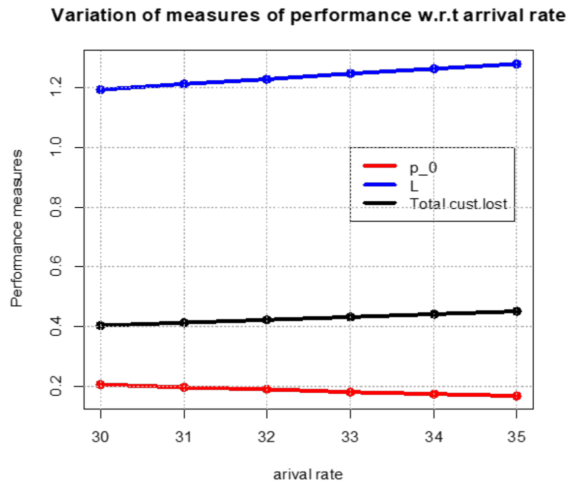

Fig 2. Variation of measures of performance w.r.t arrival rate

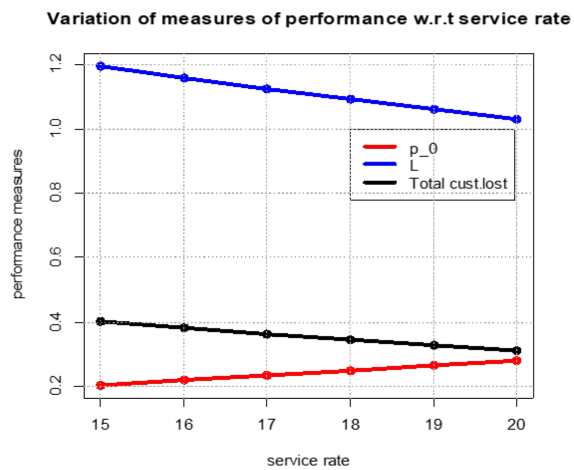

Fig 3. Variation of measures of performance w.r.t service rate

From Figure 4, it is evident that as the reneging rate rises, there is a corresponding increase in the number of customers leaving the queue, consequently leading to a decrease in the system size.

As the reneging rate increases, the proportion of total customers lost increases. This phenomenon is quite straightforward. The probability of the server being idle rises up. This is because as more customers leave the queue, the server has to work less.

3.5 Cost Analysis

We perform cost analysis for a multi-server Markovian queuing model considering encouraged arrival and reverse reneging.

In order to carry out the analysis we convert the cost currency from US dollar ($) to Indian rupee (Rs). We also interchange the cost of revenue generated from each completed call and the cost incurred due to each blocked call.

The improvised numerical problem mentioned in section 5 is:

“A cell tower can support c simultaneous calls in its coverage area. Demand for calls is Poisson with rate 30 per hour. Calls are exponential with mean of 4 minutes. Calls are blocked when the cell tower is busy with c calls in progress.

(a) If c=4, determine the fraction of blocked calls.

(b) Each completed call generates revenue of Rs 83.40. Each blocked calls incurs a cost of Rs 41.70. Which of the following cell towers has the shortest break-even time (the time at which the cumulative revenue equals the cost of the cell tower): A cell tower with c=2, costing Rs 8,34,000; a cell tower with c=4, costing Rs 16,68,000; a cell tower with c=6, costing Rs 25,02,000.”[12]

We first define some cost elements which are given below:

R: Revenue gained by providing service per customer

C: Loss due to blocking of calls

The total loss incurred by the system due to the restriction of supporting only c simultaneous calls is given by

Total loss=C* λ* $p_{\mathbf{c}}$

The total revenue earned by the queuing system by providing service to the customer is given by

Total Revenue= λ^{s}***(1-**p_0**)*R**

The total expected profit is given by

Profit=Total Revenue- Total loss

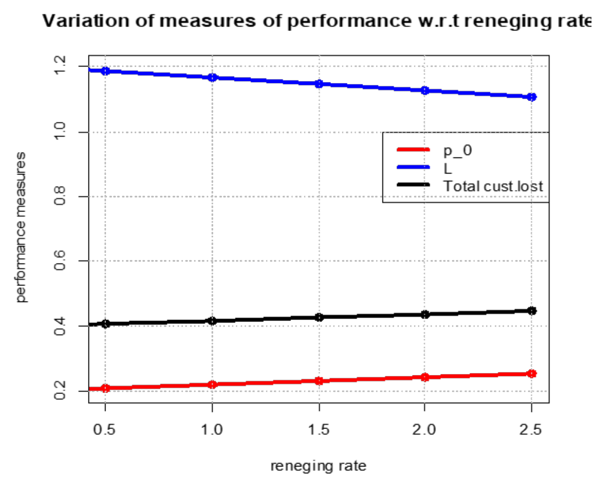

Fig 4. Variation of measures of performance w.r.t reneging rate

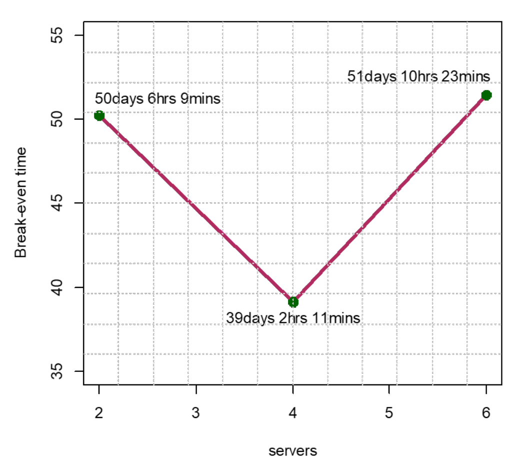

Fig 5. Variation of Break-even time with Number of servers

On comparing the break-even time for servers; c=2, c=4, c=6 we observe that cell towers c=4 have the shortest break-even time of 39 days 2 hours, and 11 minutes (Figure 5).

4 Conclusion

This study presents an analysis of a multi-server finite buffer Markovian queue with distinctive features such as encouraged arrival and reverse reneging. With the assumed restrictions, cost and sensitivity analysis on the Erlang loss model have not been carried out in the literature. Cost analysis is performed to elucidate the financial implications of system implementation and operation, assisting decision-makers in assessing its economic viability. The analysis focuses on steady-state considerations, resulting in explicit formulas for both traditional and innovative performance metrics. To validate the theoretical framework, a numerical example is presented along with sensitivity analysis, showcasing its applicability in real-world scenarios for designing operational strategies. The results indicate potential avenues for expansion, particularly in integrating encouraged arrival and threshold limit concepts. Moreover, this paper emphasizes steady-state probabilities, but time-dependent studies of the model could also be performed. Exploring this model in a non-Markovian setting is a promising direction for future research.

5 APPENDIX I. DERIVATION OF P'(1)

From Equation (3.2), we have

λ (1+n) p_{n-1}+{(n+1)μ+(c-n) ν } p_{n+1} ={ λ(1+n)+nμ+(c-n+1) ν } p_n ; n=1,2….c-1

Multiplying both sides of equation (4.2) by s^n and summing over n, we get-

- $\sum_{n=1}^{c-1}$ $[\lambda(1+n)\,p_{n-1}+\{(n+1)\mu+(c-n)\,\nu\}\,p_{n+1}]\,s^n =$ $\sum_{n=1}^{c-1}[\,\{\lambda(1+n)+n\mu+(c-n+1)\ \nu\}\,p_n]\,s^n$
- λ (1+n) $\sum_{n=1}^{c-1}\ p_{n-1}s^n$+ $\sum_{n=1}^{c-1}\ \{(n+1)\ \mu+(c-n)\ \nu\}\,p_{n+1}s^n$= λ (1+n)$\sum_{n=1}^{c-1}\ p_n s^n$+ $\mu\sum_{n=1}^{c-1}\ n\,p_n s^n$+$\sum_{n=1}^{c-1}\ \{(c-n+1)\ \nu\}\,p_n s^n\}$
- λ (1+n)$s\sum_{n=1}^{c-1}\ p_{n-1}s^{n-1}$+$\frac{1}{S}\sum_{n=1}^{c-1}((n+1)\,\mu+(c-n)\ \nu\}\ \ p_{n+1}s^{n+1}$=

$$\lambda\ (1+\eta)\ \sum_{n=1}^{c-1}\ p_n s^n+\mu\ \sum_{n=1}^{c-1}\ n\,p_n s^n+\sum_{n=1}^{c-1}\ \{(c-n+1)\ \nu\}\ \ p_n s^n \tag{9.1}$$

From Equation (3.3), we have

λ(1+n)p_{c-1}= (cμ+ ν) p_c , n=c

Multiplying both sides of equation (4.3), by s^c,we get-

$\Rightarrow$ λ (1+n) p_{c-1} s^c= (cμ+ ν) $p_c s^c$

$$\Rightarrow\lambda\ (1+\eta)s\,p_{c-1}s^{c-1}=\ (c\mu+\nu)\ p_c s^c \tag{9.2}$$

Adding Equation (9.1) and Equation (9.2), we get

- λ(1+n)$s\sum_{n=1}^{c-1}\ p_{n-1}s^{n-1}$+$\lambda$(1+n)$s\,p_{c-1}s^{c-1}+\frac{1}{S}\sum_{n=1}^{c-1}\ ((n+1)\,\mu+\ (c--n)\,\nu\}\ \ p_{n+1}s^{n+1}$= λ(1+n) $\sum_{n=1}^{c-1}\ p_n\ s^n$+$\mu\sum_{n=1}^{c-1} n\,p_n s^n+\sum_{n=1}^{c-1}\ \{(c-n+1)\,\nu\}\,p_n s^{\ n}$+ (c$\mu$+ ν) $p_c s^c$
- λ(1+n) $s\sum_{n=1}^{c}\ p_{n-1}s^{n-1}$ + $\frac{1}{S}$ $\sum_{n=1}^{c-1}\ \{(n+1)\mu+(c-n)\,\nu\}\,p_{n+1}s^{n+1}$= λ(1+n) $\sum_{n=1}^{c-1}\ p_n\ s^n$ + $\mu\sum_{n=1}^{c-1} n\ \ p_n\ s^n$+c$\nu\sum_{n=1}^{c-1}\ p_n\ s^n-\nu\sum_{n=1}^{c-1} n\,p_n\ s^n\ \nu\sum_{n=1}^{c-1} p_n\ s^n$+c $\mu\,p_c s^c$+ $\nu\,p_c s^c$
- λ(1+n) $s\sum_{n=1}^{c}\ p_{n-1}\ s^{n-1}$+ $\frac{\mu}{S}\ \sum_{n=1}^{c-1}\ (n+1)\ p_{n+1}s^{n+1}$+ $\frac{\nu}{S}\sum_{n=1}^{c-1}\ (c-n)\,p_{n+1}s^{n+1}$ = λ(1+n) $\sum_{n=1}^{c-1}\ p_n s^n$+ $\mu\ \sum_{n=1}^{c} n\,p_n s^n$+ c $\nu\sum_{n=1}^{c-1}\ p_n s^n-\nu\ \sum_{n=1}^{c-1}\ n\,p_n s^n$+ $\nu\sum_{n=1}^{c} p_n s^n$
- λ(1+n) $s\sum_{n=1}^{c} p_{n-1}s^{n-1}$+ $\frac{\mu}{S}\sum_{n=1}^{c-1}(n+1)\,p_{n+1}\ s^{n+1}$+$\frac{c\,\nu}{S}\sum_{n=1}^{c-1} p_{n+1}s^{n+1}-\frac{\nu}{S}\sum_{n=1}^{c-1} n\,p_{n+1}s^{n+1}$ =λ(1+n) $\sum_{n=1}^{c-1} p_n\ s^n$ +$\mu\sum_{n=1}^{c} n\,p_n s^n$+ $c\nu\sum_{n=1}^{c-1}\ p_n s^n-\nu\sum_{n=1}^{c-1} np_n s^n$+ $\nu\sum_{n=1}^{c} p_n s^n$
- λ(1+n) s {P(s)- p_c s^c }+ μ\{P$'$ (s)-P_1}+$\frac{c\,\nu}{S}$\{P(s)- $p_0-p_1 s$\}- [$\frac{\nu}{S}\sum_{n=1}^{c-1}(n+1)\ p_{n+1}s^{n+1}-\frac{\nu}{S}\sum_{n=1}^{c-1}\ \ p_{n+1}\ s^{n+1}$]= λ(1+n) {P(s)- p_0- p_c s^c}+ μ $S\sum_{n=1}^{c} n\,p_n s^{n-1}$+ c ν\{P(s)- p_0-$p_c s^c$\}- $\nu s\ \sum_{n=1}^{c-1}\ n\,p_n\ s^{n-1}$ + ν\{P(s)- p_0\}
- λ(1+n) s P(s)- λ(1+n) s $p_c s^c$+μ P$'$(s)-μp_1+$\frac{c\,\nu}{S}$ P(s)$-\frac{c\,\nu}{S}\,p_0-c\,\nu p_1-\nu\sum_{n=1}^{c-1}\ (n+1)p_{n+1}s^n$ + $\frac{\nu}{S}\sum_{n=1}^{c-1}\ p_{n+1}s^{n+1}$= λ(1+n)P(s)- λ(1+n) p_0- λ(1+n) $p_c s^c$+ μsP$'$(s)+ cνP(S)- c ν p_0- c ν $p_c s^c$- νs {P$'$(s)-c $p_c s^{c-1}$}+ ν P(s)- νp_0
- λ(1+n) s P(s)- λ(1+n) s p_c s^c+ μP$'$ (s)- $p_1\{\mu+c\ \nu\}$+ $\frac{c\nu}{S}$ P(s)- $\frac{c\,\nu}{S}$ p -ν\{P$'$ (s)- p_1\}+$\frac{\nu}{S}$\{ P(s)- p_0- p_1 s\}= λ(1+n) P(s)- λ(1+n) p_0- λ(1+n) p_c s^c+ μ s P$'$(s)+ c νP(s)- c νp_0- c ν p_c s^c- νs P$'$(s)+ $\nu sc\,p_c s^{c-1}$+ ν P(s)- νp_0

- λ(1+n) P(s) {s-1}- λ(1+n) p_c s^c{s-1}- μP$'$(s){s-1}+ c ν P(s){ $\frac{1}{S}-1$}- c $\nu p_0\{\frac{1}{S}-1\}$+ νP$'$(s) {s-1}+ νP(s) { $\frac{1}{S}-1$}$-\nu p_0\{\frac{1}{S}-1\}$=$p_1$\{ $\mu+c\,\nu$\}- λ(1+n) $\frac{\mu+c\,\nu}{\lambda(1+n)}\,p_1$
- (s-1){ λ(1+n)P(s)- λ(1+n) p_c s^c- μP$'$ (s)+ ν P$'$(s)}+($\frac{1}{S}-1$){ $c\,\nu$ P(s)+ νP(s)- $\nu p_0 - c\,\nu p_0$}=0
- (s-1){ λ(1+n)P(s)- λ(1+n) $p_c s^c$- μP$'$ (s)+ νP$'$(s)}=-($\frac{1-S}{S}$)\{ $c\,\nu$P(s)+ νP(s)- $\nu p_0 - c\nu p_0$\}
- λ(1+n) P(s)- λ(1+n) $p_c s^c -$ P$'$(s){ $\mu-\ \nu$}=$\frac{1}{S}$\{ $c\,\nu$ P(s)+ νP(s)- $\nu p_0 - c\nu p_0$\}
- P$'$(s){ $\mu-\ \nu$}= λ(1+n) P(s)- λ(1+n) $p_c s^c - \frac{1}{S}$\{ $c\,\nu$ P(S)+ νP(s)-$\nu p_0 - c\nu p_0$\}
- P$'$(s)= $\frac{1}{\{\ \mu-\ \nu\}}$[λ(1+n) P(s)- λ(1+n) $p_c s^c - \frac{1}{S}$\{ $c\,\nu$ P(s)+ νP(s)- $\nu p_0 - c\nu p_0$\}]

Now,

Taking limit as $s \to 1$

- $P'(1)$= $\frac{1}{\{\ \mu-\ \nu\}}$[λ(1+n)-λ(1+n) $p_c - \{c\,\nu + \nu - \nu p_0 - c\nu p_0\}$]
- $P'(1)$= $\frac{1}{\{\ \mu-\nu\}}$[λ(1+n) {1$-p_c$} $-c\nu${1$-p_0$} $-\nu${1$-p_0$}]
- $P'(1)$= $\frac{1}{\{\ \mu-\nu\}}$[λ(1+n) (1$-p_c$) $-\nu$(1$-p_0$)(c+1)]

6 APPENDIX II. CALCULATION TO DETERMINE THE VALUE OF n

We observe various recharge plans offered by Jio Networks, particularly those that are popular and widely used. We consider the plans that offer 1.5 GB data per day, 100 SMS, and unlimited voice calls.

Table 3.

Plan (in Rs.)	Days	Cost per day (in Rs.)
199	23	8.65
239	28	8.53
666	84	7.92

a.) Comparing Rs 239 Plan with Rs 199 Plan:
The Percentage of offers can be calculated as :
Offer percentage = $\frac{8.65-8.53}{8.65}\times 100$
= 1.38 %
Therefore, η= $\frac{1.38}{100} = 0.0138 \approx 0.014$
b.) Comparing Rs 666 with Rs 239 Plan:
The Percentage of offers can be calculated as:
Offer percentage = $\frac{8.53-7.92}{8.53}\times 100$
= 7.15 %
Therefore, η = $\frac{7.15}{100} = 0.0715 \approx 0.071$

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