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Sums of Finite Product of Chebyshev Polynomials of Third and Fourth Kind and Fibonacci and Lucas Numbers

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Abstract

Objective: Classical orthogonal polynomials are extensively used for the numerical solution of differential equations and numerical analysis. Various generalization problems involving Chebyshev polynomials have been studied and one such area is the classical linearization problem. In this paper, we will consider a generalized classical linearization problem and study some identities representing the sums of finite products of the 3rd and 4th kinds of Chebyshev polynomials, Fibonacci numbers, and Lucas numbers as a linear sum of the Chebyshev polynomials of 2nd kind and their derivatives. **Methods:** Differential calculus, combinatorial, and elementary algebraic computations are employed to yield the results. **Findings:** A new set of identities expressing the sums of finite products of the 3rd and 4th kinds of Chebyshev polynomials, Fibonacci numbers, and Lucas numbers as a linear sum of the 2nd kinds of Chebyshev polynomials and their derivatives are studied. **Novelty:** Earlier studies have established the sums of finite products of the Lucas numbers, Fibonacci numbers, Fibonacci and Pell polynomials, and Chebyshev polynomials of the 1st and 2nd kinds as a linear sum of other orthogonal polynomials. However, representations of sums of finite products of the 3rd and 4th kinds of Chebyshev polynomials, Fibonacci numbers, and Lucas numbers as a linear sum of the 2nd kinds of Chebyshev polynomials and their derivatives have not been studied, which is the prime focus of this manuscript.

Keywords: Linearization; Chebyshev Polynomials; Fibonacci Numbers; Lucas numbers; Orthogonality

1 Introduction

In number theory, the Fibonacci sequence is one of the most widely studied mathematical objects due to its wide spectrum of applications in nature[1,2] and in various fields of science and technology ranging from physical science to engineering[3]. Several authors have attempted to explore their numerous generalizations from diverse perspectives. One such generalization culminated in the development of the theory of Fibonacci polynomials and Chebyshev polynomials. Classical orthogonal polynomials are extensively used for the numerical solution of differential equations and numerical analysis[4,5]. Chebyshev polynomials form an intriguing class of these types of polynomials that attracted the attention of recent researchers. Among the four kinds of Chebyshev polynomials[6], a wide variety of literature is available on the Chebyshev polynomials of the 1st and 2nd kind[7], whereas limited work has been done on the Chebyshev polynomials of the 3rd and 4th kind[8,9], which opens up an intriguing field for prospective researchers.

Recent researchers have shown a keen interest in studying classical linearization problem representing the sums of finite products of Chebyshev polynomials as a linear sum of other similar orthogonal polynomials[10,11]. Here in this work, some new representations for the sums of finite products of the 3rd & 4th kinds of Chebyshev polynomials, Fibonacci, and Lucas numbers as linear sum of Chebyshev polynomials of 2nd kinds and their derivatives are considered.

Firstly, we will discuss the essential terminology and notations that will be used throughout the work, as well as the introduction of the theme of the manuscript by citing the earlier results by various authors on the sums of products of some special orthogonal polynomials.

Definition 1.1 . For any non-negative integer α, the sequence of *Fibonacci numbers*(F_α) is defined recursively as follows:

$$F_\alpha = \begin{cases} 0, & \alpha = 0, \\ 1, & \alpha = 1, \\ F_{(\alpha-1)} + F_{(\alpha-2)}, & \alpha \geq 2. \end{cases} \tag{1.1}$$

Definition 1.2. For any non-negative integer α, the sequence of *Lucas numbers*(L_α) is defined recursively as follows:

$$L_\alpha = \begin{cases} 2, & \alpha = 0, \\ 1, & \alpha = 1, \\ L_{(\alpha-1)} + L_{(\alpha-2)}, & \alpha \geq 2. \end{cases} \tag{1.2}$$

Definition 1.3. For any non-negative integer α, the sequence of the *Fibonacci polynomials*$(F_\alpha(\zeta))$ is defined recursively as follows:

$$F_\alpha(\zeta) = \begin{cases} 0, & \alpha = 0, \\ 1, & \alpha = 1, \\ \zeta F_{(\alpha-1)}(\zeta) + F_{(\alpha-2)}(\zeta), & \alpha \geq 2. \end{cases} \tag{1.3}$$

Definition 1.4. For any non-negative integer α, the *Chebyshev polynomials of the first kind*$(T_\alpha(\zeta))$ are defined by

$$T_\alpha(\zeta) = cos\alpha\theta$$

where $\zeta = cos\theta,\ \zeta \in [-1,1]$, and $\theta \in [0,\pi]$ and follows the recursive relation

$$T_\alpha(\zeta) = \begin{cases} 1, & \alpha = 0, \\ \zeta, & \alpha = 1, \\ 2\zeta T_{\alpha-1}(\zeta) - T_{\alpha-2}(\zeta), & \alpha \geq 2. \end{cases} \tag{1.4}$$

Definition 1.5. For any non-negative integer α, the *Chebyshev polynomials of the second kind*$(U_\alpha(\zeta))$, are defined as

$$U_\alpha(\zeta) = \frac{sin(\alpha+1)\theta}{sin\theta},$$

where $\zeta = cos\theta, \zeta \in [-1,1]$, and $\theta \in [0,\pi]$ and follows the recursive relation

$$U_\alpha(\zeta) = \begin{cases} 1, & \alpha = 0, \\ 2\zeta, & \alpha = 1, \\ 2\zeta U_{\alpha-1}(\zeta) - U_{\alpha-2}(\zeta), & \alpha \geq 2. \end{cases} \tag{1.5}$$

Definition 1.6. For any non-negative integer α, the *Chebyshev polynomials of the third kind* $(V_\alpha(\zeta))$ are defined recursively as

$$V_\alpha(\zeta) = \frac{cos(\alpha+1/2)\theta}{cos\left(\frac{\theta}{2}\right)},$$

where $\zeta = cos\theta$, $\zeta \in [-1,1]$ and $\theta \in [0,\pi]$ and follows the recursive relation

$$V_{\alpha}(\zeta) = \begin{cases} 1, & \alpha = 0, \\ 2\zeta - 1, & \alpha = 1, \\ 2\zeta\, V_{\alpha-1}(\zeta) - V_{\alpha-2}(\zeta), & \alpha \geq 2. \end{cases} \tag{1.6}$$

Definition 1.7. For any non-negative integer α, the Chebyshev polynomials of the fourth kind ($W_{\alpha}\ (\zeta)$) are defined recursively for as

$$W_{\alpha}\ (\zeta) = \frac{sin\,(\alpha+1/2)\,\theta}{sin\,(\frac{\theta}{2})},$$

where $\zeta = cos\theta$, $\zeta \in [-1,1]$and $\theta \in [0,\pi]$and follows the recursive relation

$$W_{\alpha}(\zeta) = \begin{cases} 1, & \alpha = 0, \\ 2\zeta + 1, & \alpha = 1, \\ 2\zeta\, W_{\alpha-1}(\zeta) - W_{\alpha-2}(\zeta), & \alpha \geq 2. \end{cases} \tag{1.7}$$

These polynomials are also represented by their explicit relation as follows[4,12]

$$F_{\alpha}(\zeta) = \sum_{\beta=0}^{\left[\frac{\alpha-1}{2}\right]} \binom{\alpha-\beta-1}{\beta} (\zeta)^{\alpha-2\beta-1}. \tag{1.8}$$

$$T_{\alpha}(\zeta) = \frac{\alpha}{2} \sum_{\beta=0}^{\left[\frac{\alpha}{2}\right]} \frac{(-1)^{\beta}\,(\alpha-\beta-1)!}{\beta!\,(\alpha-2\beta)!}(2\zeta)^{\alpha-2\beta}. \tag{1.9}$$

$$U_{\alpha}(\zeta) = \sum_{\beta=0}^{\left[\frac{\alpha}{2}\right]} \frac{(-1)^{\beta}\,(\alpha-\beta)!}{\beta!\,(\alpha-2\beta)!}(2\zeta)^{\alpha-2\beta}. \tag{1.10}$$

$$V_{\alpha}(\zeta) = \sum_{\beta=0}^{[\alpha]} \frac{2^{\beta}\,(\alpha+\beta)!}{(2\beta)!\,(\alpha-\beta)!}(\zeta-1)^{\beta}. \tag{1.11}$$

$$W_{\alpha}(\zeta) = \sum_{\beta=0}^{[\alpha]} \frac{2^{\beta}\,(\alpha+\beta)!}{(2\beta)!\,(\alpha-\beta)!}(\zeta-1)^{\beta}. \tag{1.12}$$

Several authors have investigated Chebyshev polynomials and their properties and obtained a wide spectrum of interesting results, which have created a niche in the existing domain of mathematical research. Several mathematicians have shown keen interest in the domain of classical linearization problems and their generalization[11,12]. For instance, Zhang[13] considered the representations of summations of finite products of 2^{nd} kind Chebyshev polynomials, Lucas, and Fibonacci numbers and derived many intriguing identities, particularly

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} U_{\rho_1}(\zeta)\cdot U_{\rho_2}(\zeta)\cdots U_{\rho_{r+1}}(\zeta) = \frac{1}{2^r\,r!}\,U^{r}_{\alpha+r}(\zeta), \tag{1.13}$$

where $U^{r}_{\alpha}(\zeta)$ represents r^{th} derivative of $U_{\alpha}(\zeta)$ w.r.t ζ, and the sum is taken over all non-negative $\rho_1, \rho_2, \ldots, \rho_{r+1}$ such that $\rho_1 + \rho_2 + \cdots + \rho_{r+1} = \alpha$. A similar study was conducted by Kim T., Kim D.S., Dolgy D. V., and Kim D[14] in 2019 and obtained interesting results, expressing the sums of the finite products of Chebyshev polynomials as a linear combination of other orthogonal polynomials like Hermite, Legendre, extended Laguerre, Gegenbauer, and Jacobi polynomials. Similar representations for several orthogonal polynomials were developed by Han and L. Xinging[15], Patra and G.K. Panda and A.

Patra[12], and J. Kishore, V. Verma, and A.K. Sharma[16,17], also deduced analogous identities for sums of finite products of Chebyshev polynomials of the 1st and 2nd kind, Lucas and Fibonacci polynomials, and Pell polynomials. Analogous results are also obtained in[18].

Based on the preceding literature, existing works have established identities on the sums of finite products of Lucas and Fibonacci numbers, Pell, and Chebyshev polynomials of 3rd and 4th kinds in terms of derivatives of Lucas polynomials, Pell polynomials, Fibonacci polynomials, but the results on the sums of finite products in terms of Lucas, Pell, and Fibonacci numbers, and Chebyshev polynomials of 3rd and 4th kinds in terms of 2nd kinds of Chebyshev polynomials and their derivatives have not been studied. With this motivation, we will consider new representations of sums of finite products of the Chebyshev polynomials of 3rd and 4th kinds, Lucas and Fibonacci numbers in terms of the 2nd kind of Chebyshev polynomials and their derivatives.

2 Methodology

The methods of differential calculus, combinatorial, and elementary algebraic computations are employed to yield the results.

3 Results and Discussion

In this section, we will discuss the primary findings of this manuscript. With the aid of elementary computational techniques, we will express the sums of finite products of the Chebyshev polynomials of third and fourth kind, Fibonacci numbers, and Lucas numbers in terms of the Chebyshev polynomials of 2nd kind and their derivatives. Firstly, we will consider the following lemmas which will be very helpful in the establishment of the key inferences of the manuscript.

Lemma 1. For any integer $\alpha \geq 0$,

$$V_\alpha(\zeta) = U_\alpha(\zeta) - U_{\alpha-1}(\zeta). \tag{2.1}$$

$$W_\alpha(\zeta) = U_\alpha(\zeta) + U_{\alpha-1}(\zeta). \tag{2.2}$$

$$W_\alpha(\zeta) = (-1)^\alpha\, V_\alpha(-\zeta). \tag{2.3}$$

Proof. This lemma can be established using the Equations (1.8), (1.9), (1.10), (1.11), (1.12) and (1.13) and using Binet's formula for Pell as well as Chebyshev polynomials of 2nd kind. Please refer[4,17].

Lemma 2. *For any integer $\alpha, r \geq 0$,*

$$\begin{aligned} &\sum\nolimits_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} V_{\rho_1}(\zeta)\cdot V_{\rho_2}(\zeta)\cdots V_{\rho_{r+1}}(\zeta) \\ &= \frac{1}{2^r\, r!}\sum_{\lambda=0}^{\alpha}(-1)^\lambda \binom{r+1}{\lambda} U^r_{\alpha-\lambda+r}(\zeta), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\sum\nolimits_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} W_{\rho_1}(\zeta)\cdot W_{\rho_2}(\zeta)\cdots W_{\rho_{r+1}}(\zeta) \\ &= \frac{1}{2^r\, r!}\sum_{\lambda=0}^{\alpha} \binom{r+1}{\lambda} U^r_{\alpha-\lambda+r}(\zeta), \end{aligned} \tag{2.5}$$

where sums run over all non-negative integers $\rho_1, \rho_2, \ldots, \rho_{r+1}$ such that $\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha$ with $\binom{r+1}{\lambda}=0$ for $\lambda > r+1$.

Proof. Please refer Kim T et.al.[14].

The main findings of this manuscript are as discussed under:

Theorem 1. *For $\alpha, r \geq 0$, we have*

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} V_{\rho_1}(\zeta)\cdot V_{\rho_2}(\zeta)\cdots V_{\rho_{r+1}}(\zeta)$$

$$= \tfrac{1}{2^r\ r!(1-\zeta^2)} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{r+1}{\lambda} [(2r-1)\zeta U_{\alpha-\lambda+r}^{r-1}(\zeta)$$

$$-[(\alpha-\lambda+2)(\alpha-\lambda+2r)] U_{\alpha-\lambda+r}^{r-2}(\zeta)],$$

where sum runs over all non-negative integers $\rho_1, \rho_2, \cdots, \rho_{r+1}$ *with* $\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha$, and $\binom{r+1}{\lambda}=0$ *for* $\lambda > r+1$.

Proof. Firstly, from [10], we will note that for any positive integer $\alpha \geq r > 0$,

$$U_{\alpha+r}^{r}(\zeta) = \frac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha+r}^{r-1}(\zeta) + \frac{(r-2)r-(\alpha+r)(\alpha+r+2)}{(1-\zeta^2)} U_{\alpha+r}^{r-2}(\zeta). \tag{2.6}$$

Thus,

$$U_{\alpha-\lambda+r}^{r}(\zeta) = \tfrac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-1}(\zeta) + \tfrac{(r-2)r-(\alpha-\lambda+r)(\alpha-\lambda+r+2)}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-2}(\zeta)$$

$$= \tfrac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-1}(\zeta) - \tfrac{[\alpha(\alpha-\lambda+r)-\lambda(\alpha-\lambda+r)+r(\alpha-\lambda+2)+2(\alpha-\lambda+r)]}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-2}(\zeta)$$

$$= \tfrac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-1}(\zeta) - \tfrac{[(\alpha-\lambda+2)(\alpha-\lambda+r)+r(\alpha-\lambda+2)]}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-2}(\zeta)$$

$$= \tfrac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-1}(\zeta) - \tfrac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-2}(\zeta),$$

$$\therefore U_{\alpha-\lambda+r}^{r}(\zeta) = \frac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-1}(\zeta) - \frac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-2}(\zeta). \tag{2.7}$$

Using Equation (2.7) in Equation (2.4), we have

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} V_{\rho_1}(\zeta)\cdot V_{\rho_2}(\zeta)\cdots V_{\rho_{r+1}}(\zeta)$$

$$= \tfrac{1}{2^r\ r!} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{r+1}{\lambda} U_{\alpha-i+\lambda}^{r}(\zeta)$$

$$= \tfrac{1}{2^r\ r!} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{r+1}{\lambda} \left[\tfrac{(2r-1)\zeta}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-1}(\zeta) - \tfrac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\zeta^2)} U_{\alpha-\lambda+r}^{r-2}(\zeta)\right]$$

$$= \tfrac{1}{2^r\ r!(1-\zeta^2)} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{r+1}{\lambda} [(2r-1)\zeta U_{\alpha-\lambda+r}^{r-1}(\zeta) - [(\alpha-\lambda+2)(\alpha-\lambda+2r)] U_{\alpha-\lambda+r}^{r-2}(\zeta)],$$

$$\therefore \sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} V_{\rho_1}(\zeta)\cdot V_{\rho_2}(\zeta)\cdots V_{\rho_{r+1}}(\zeta)$$

$$= \tfrac{1}{2^r\ r!(1-\zeta^2)} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{r+1}{\lambda} [(2r-1)\zeta U_{\alpha-\lambda+r}^{r-1}(\zeta) - [(\alpha-\lambda+2)(\alpha-\lambda+2r)] U_{\alpha-\lambda+r}^{r-2}(\zeta)].$$

This proves the Theorem 1.

Theorem 2. *For $\alpha, r \geq 0$, we have*

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} W_{\rho_1}(\zeta)\cdot W_{\rho_2}(\zeta)\cdots W_{\rho_{r+1}}(\zeta)$$

$$= \tfrac{1}{2^r\ r!(1-\zeta^2)} \sum_{\lambda=0}^{\alpha} \binom{r+1}{\lambda} [(2r-1)\zeta U_{\alpha-\lambda+r}^{r-1}(\zeta) - [(\alpha-\lambda+2)(\alpha-\lambda+2r)] U_{\alpha-\lambda+r}^{r-2}(\zeta)],$$

where sum runs over all non-negative integers $\rho_1, \rho_2, \cdots, \rho_{r+1}$ *with* $\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha$ *and* $\binom{r+1}{\lambda}=0$ *for* $\lambda > r+1$.

Proof. For proof of this theorem, we will proceed in a similar manner by using Equation (2.7) in Equation (2.5), to get the desired result.

Theorem 3. *For* $\alpha, r \geq 0$*, we have*

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} F_{2\rho_1+1}\cdot F_{2\rho_2+1}\cdots F_{2\rho_{r+1}+1}$$
$$=\frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{5}\binom{r+1}{\lambda}\left[2\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\tfrac{3}{2}\right)-3(2r-1)U_{\alpha-\lambda+r}^{r-1}\left(\tfrac{3}{2}\right)\right],$$

where sum runs over all non-negative integers $\rho_1, \rho_2, \cdots, \rho_{r+1}$ with $\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha$ and $\binom{r+1}{\lambda}=0$ for $\lambda > r+1$.

Proof. Using the fact that $U_\alpha\left(\frac{3}{2}\right)=F_{2\alpha+2}$ in Equation (2.1), we have

$$V_\alpha\left(\frac{3}{2}\right)=F_{2\alpha+1}. \tag{2.8}$$

Replacing ζ by $\frac{3}{2}$ in Theorem 1, we have

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} V_{\rho_1}\left(\frac{3}{2}\right)\cdot V_{\rho_2}\left(\frac{3}{2}\right)\cdots V_{\rho_{r+1}}\left(\frac{3}{2}\right)$$
$$=\frac{1}{2^r\ r!\left(1-\left(\frac{3}{2}\right)^2\right)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[(2r-1)\left(\frac{3}{2}\right)U_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)-\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)\right], \tag{2.9}$$

Using Equation (2.8) in Equation (2.9) with $\zeta=\frac{3}{2}$, it yields

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} F_{2\rho_1+1}\cdot F_{2\rho_2+1}\cdots F_{2\rho_{r+1}+1}$$
$$=\frac{1}{2^r\ r!}\left(-\tfrac{4}{5}\right)\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[(2r-1)\left(\tfrac{3}{2}\right)U_{\alpha-\lambda+r}^{r-1}\left(\tfrac{3}{2}\right)-\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\tfrac{3}{2}\right)\right]$$
$$=\frac{1}{2^{r+1}\ r!}\left(\tfrac{4}{5}\right)\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[2\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\tfrac{3}{2}\right)-3(2r-1)U_{\alpha-\lambda+r}^{r-1}\left(\tfrac{3}{2}\right)\right]$$
$$=\frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{5}\binom{r+1}{\lambda}\left[2\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\tfrac{3}{2}\right)-3(2r-1)U_{\alpha-\lambda+r}^{r-1}\left(\tfrac{3}{2}\right)\right],$$

That is,

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} F_{2\rho_1+1}\cdot F_{2\rho_2+1}\cdots F_{2\rho_{r+1}+1}$$
$$=\frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{5}\binom{r+1}{\lambda}\left[2\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\tfrac{3}{2}\right)-3(2r-1)U_{\alpha-\lambda+r}^{r-1}\left(\tfrac{3}{2}\right)\right],$$

which establishes the Theorem 3.

Theorem 4. *For* $\alpha, r \geq 0$*, we have*

$$\sum_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} L_{2\rho_1+1}\cdot L_{2\rho_2+1}\cdots L_{2\rho_{r+1}+1}$$
$$=\frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{1}{5}\cdot\binom{r+1}{\lambda}\left[2\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]U_{\alpha-\lambda+r}^{r-2}\left(\tfrac{3}{2}\right)-3(2r-1)U_{\alpha-\lambda+r}^{r-1}\left(\tfrac{3}{2}\right)\right],$$

where sum runs over all non-negative integers $\rho_1, \rho_2, \cdots, \rho_{r+1}$ with $\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha$ and $\binom{r+1}{\lambda}=0$ for $\lambda > r+1$.

Proof. For the proof of this Theorem, we will proceed in a similar manner as in Theorem 3. By using the fact $U_\alpha\left(\frac{3}{2}\right)=F_{2\alpha+2}$ in Equation (2.1), we get

$$W_\alpha\left(\frac{3}{2}\right)=L_{2\alpha+1}\,. \tag{2.10}$$

Taking $\zeta=\frac{3}{2}$ in Theorem 2 gives

$$\begin{aligned}&\sum\nolimits_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} W_{\rho_1}\left(\frac{3}{2}\right)\cdot W_{\rho_2}\left(\frac{3}{2}\right)\cdots W_{\rho_{r+1}}\left(\frac{3}{2}\right)\\ &=\frac{1}{2^r\ r!\left(1-\left(\frac{3}{2}\right)^2\right)}\sum_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\left[(2r-1)\left(\frac{3}{2}\right)U^{r-1}_{\alpha-\lambda+r}\left(\frac{3}{2}\right)-[(\alpha-\lambda+2)(\alpha-\lambda+2r)]U^{r-2}_{\alpha-\lambda+r}\left(\frac{3}{2}\right)\right],\end{aligned} \tag{2.11}$$

Using Equation (2.10) in Equation (2.11) yields,

$$\begin{aligned}&\sum\nolimits_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} L_{2\rho_1+1}\cdot L_{2\rho_2+1}\cdots L_{2\rho_{r+1}+1}\\ &=\tfrac{1}{2^r\ r!}\left(-\tfrac{4}{5}\right)\sum\nolimits_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\left[(2r-1)\left(\tfrac{3}{2}\right)U^{r-1}_{\alpha-\lambda+r}\left(\tfrac{3}{2}\right)-[(\alpha-\lambda+2)(\alpha-\lambda+2r)]U^{r-2}_{\alpha-\lambda+r}\left(\tfrac{3}{2}\right)\right]\\ &=\tfrac{1}{2^{r+1}\ r!}\left(\tfrac{4}{5}\right)\sum\nolimits_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+2r)]U^{r-2}_{\alpha-\lambda+r}\left(\tfrac{3}{2}\right)-3(2r-1)U^{r-1}_{\alpha-\lambda+r}\left(\tfrac{3}{2}\right)\right],\end{aligned}$$

That is,

$$\begin{aligned}&\sum\nolimits_{\rho_1+\rho_2+\cdots+\rho_{r+1}=\alpha} L_{2\rho_1+1}\cdot L_{2\rho_2+1}\cdots L_{2\rho_{r+1}+1}\\ &=\tfrac{1}{2^{r-1}\ r!}\sum\nolimits_{\lambda=0}^{\alpha}\tfrac{1}{5}\cdot\binom{r+1}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+2r)]U^{r-2}_{\alpha-\lambda+r}\left(\tfrac{3}{2}\right)-3(2r-1)U^{r-1}_{\alpha-\lambda+r}\left(\tfrac{3}{2}\right)\right].\end{aligned}$$

This proves the Theorem 4.

Corollary 1. *For $\alpha\geq 0$, we have*

$$\sum_{a+b+c=\alpha} V_a(\zeta)\cdot V_b(\zeta)\cdot V_c(\zeta)=\frac{1}{8}\sum_{\lambda=0}^{\alpha}(-1)^\lambda\binom{3}{\lambda}\left[P(\alpha,\lambda,\zeta)\,U_{\alpha-\lambda+1}(\zeta)-Q(\alpha,\lambda,\zeta)\,U_{\alpha-\lambda+2}(\zeta)\right],$$

where $P(\alpha,\lambda,\zeta)=\left(\frac{3\zeta(\alpha-\lambda+3)}{(1-\zeta^2)^2}\right)$ *and* $Q(\alpha,\lambda,\zeta)=\frac{(\alpha-\lambda+2)}{(1-\zeta^2)^2}\left((\alpha-\lambda+4)-(\alpha-\lambda-1)\zeta^2\right)$.

Proof. From [10], we see that

$$U'_\alpha(\zeta)=\frac{(\alpha+1)}{(1-\zeta^2)}\,U_{\alpha-1}(\zeta)-\frac{\alpha\zeta}{(1-\zeta^2)}\,U_\alpha(\zeta). \tag{2.12}$$

Taking $r=2$ in Theorem 2, we have

$$\begin{aligned}&\sum\nolimits_{a+b+c=\alpha} V_a(\zeta)\cdot V_b(\zeta)\cdot V_c(\zeta)\\ &=\frac{1}{2^2\ 2!\,(1-\zeta^2)}\sum_{\lambda=0}^{\alpha}(-1)^\lambda\binom{3}{\lambda}\left[3\zeta U'_{\alpha-\lambda+2}(\zeta)-[(\alpha-\lambda+2)(\alpha-\lambda+4)]\,U_{\alpha-\lambda+2}(\zeta)\right]\end{aligned} \tag{2.13}$$

Using Equation (2.12) in Equation (2.13), we have

$$\begin{aligned}&\sum\nolimits_{a+b+c=\alpha} V_a(\zeta)\cdot V_b(\zeta)\cdot V_c(\zeta)\\ &=\tfrac{1}{8\,(1-\zeta^2)}\sum\nolimits_{\lambda=0}^{\alpha}(-1)^\lambda\binom{3}{\lambda}\left\{\left[3\zeta\left(\tfrac{(\alpha-\lambda+3)}{(1-\zeta^2)}U_{\alpha-\lambda+1}(\zeta)-\tfrac{(\alpha-\lambda+2)\zeta}{(1-\zeta^2)}U_{\alpha-\lambda+2}(\zeta)\right)\right]-[(\alpha-\lambda+2)(\alpha-\lambda+4)]\,U_{\alpha-\lambda+2}(\zeta)\right\}\end{aligned}$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{3}{\lambda} \left[\left(\frac{3\zeta(\alpha-\lambda+3)}{8(1-\zeta^2)^2} \right) U_{\alpha-\lambda+1}(\zeta) - \left(\frac{3\zeta^2(\alpha-\lambda+2)}{8(1-\zeta^2)^2} + \frac{[(\alpha-\lambda+2)(\alpha-\lambda+4)]}{8\,(1-\zeta^2)} \right) U_{\alpha-\lambda+2}(\zeta) \right]$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{3}{\lambda} \left[\left(\frac{3\zeta(\alpha-\lambda+3)}{8(1-\zeta^2)^2} \right) U_{\alpha-\lambda+1}(\zeta) - \frac{(\alpha-\lambda+2)}{8\ (1-\zeta^2)} \left(\frac{3\zeta^2}{(1-\zeta^2)} + (\alpha-\lambda+4) \right) U_{\alpha-\lambda+2}(\zeta) \right]$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{3}{\lambda} \left[\left(\frac{3\zeta(\alpha-\lambda+3)}{8(1-\zeta^2)^2} \right) U_{\alpha-\lambda+1}(\zeta) - \frac{(\alpha-\lambda+2)}{8\ (1-\zeta^2)} \left(\frac{(\alpha-\lambda+4)-(\alpha-\lambda-1)\zeta^2}{(1-\zeta^2)} \right) U_{\alpha-\lambda+2}(\zeta) \right]$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{3}{\lambda} \left[\left(\frac{3\zeta(\alpha-\lambda+3)}{8(1-\zeta^2)^2} \right) U_{\alpha-\lambda+1}(\zeta) - \frac{(\alpha-\lambda+2)}{8\ (1-\zeta^2)^2} ((\alpha-\lambda+4)-(\alpha-\lambda-1)\zeta^2) U_{\alpha-\lambda+2}(\zeta) \right],$$

$$\therefore \sum_{a+b+c=\alpha} V_a(\zeta)\cdot V_b(\zeta)\cdot V_c(\zeta) = \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{3}{\lambda} [P(\alpha,\lambda,\zeta)\ U_{\alpha-\lambda+1}(\zeta) - Q(\alpha,\lambda,\zeta)\ U_{\alpha-\lambda+2}(\zeta)],$$

where $P(\alpha,\lambda,\zeta) = \left(\frac{3\zeta(\alpha-\lambda+3)}{8(1-\zeta^2)^2} \right)$ and $Q(\alpha,\lambda,\zeta) = \frac{(\alpha-\lambda+2)}{8\ (1-\zeta^2)^2} ((\alpha-\lambda+4)-(\alpha-\lambda-1)\zeta^2)$.

This proves the Corollary 1.

Corollary 2. *For* $\alpha \geq 0$*, we have*

$$\sum\nolimits_{a+b+c+d=\alpha} V_a(\zeta)\cdot V_b(\zeta)\cdot V_c(\zeta)\cdot V_d(\zeta)$$

$$= \sum\nolimits_{\lambda=0}^{\alpha} \tfrac{(-1)^{\lambda}}{48} \binom{4}{\lambda} [R(\alpha,\lambda,\zeta) U_{\alpha-\lambda+2}(\zeta) - S(\alpha.\lambda,\zeta) U_{\alpha-\lambda+3}(\zeta)],$$

where

$$R(\alpha,\lambda,\zeta) = \left[\frac{15\zeta^2-(\alpha-\lambda+2)(\alpha-\lambda+6)(1-\zeta^2)}{(1-\zeta^2)^3} \right] (\alpha-\lambda+4),$$

and

$$S(\alpha,\lambda,\zeta) = (\alpha-\lambda+3)\zeta \left(\frac{15\zeta^2-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)](1-\zeta^2)}{(1-\zeta^2)^3} \right).$$

Proof. Again from [10], we see that

$$(1-\zeta^2)\, U_{\alpha}''(\zeta) = 3\zeta\, U'_{\ \alpha}(\zeta) - \alpha(\alpha+2)\, U_{\alpha}(\zeta). \tag{2.14}$$

Taking r = 3 in Theorem 1 with Equations (2.12) and (2.14), we have

$$\sum\nolimits_{a+b+c+d=\alpha} V_a(\zeta)\cdot V_b(\zeta)\cdot V_c(\zeta)\cdot V_d(\zeta)$$

$$= \tfrac{1}{2^3\ 3!\,(1-\zeta^2)} \sum\nolimits_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{4}{\lambda} \left[5\zeta U''_{\alpha-\lambda+3}(\zeta) - ((\alpha-\lambda+2)(\alpha-\lambda+6)) U'_{\ \alpha-\lambda+3}(\zeta) \right]$$

$$= \frac{1}{48\ (1-\zeta^2)} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{4}{\lambda} \begin{pmatrix} 5\zeta \left(\frac{3\zeta}{(1-\zeta^2)} U'_{\ \alpha-\lambda+3}(\zeta) - \frac{(\alpha-\lambda+3)(\alpha-\lambda+5)}{(1-\zeta^2)} U_{\alpha-\lambda+3}(\zeta) \right) \\ -((\alpha-\lambda+2)(\alpha-\lambda+6)) U'_{\ \alpha-\lambda+3}(\zeta) \end{pmatrix}$$

$$=\frac{1}{48\left(1-\zeta^{2}\right)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\frac{15\zeta^{2}}{\left(1-\zeta^{2}\right)}-(\alpha-\lambda+2)(\alpha-\lambda+6)\right]U'_{\alpha-\lambda+3}(\zeta)-\frac{5\zeta(\alpha-\lambda+3)(\alpha-\lambda+5)}{\left(1-\zeta^{2}\right)}U_{\alpha-\lambda+3}(\zeta)$$

$$=\tfrac{1}{48\left(1-\zeta^{2}\right)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\tfrac{15\zeta^{2}-(\alpha-\lambda+2)(\alpha-\lambda+6)\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)}\right]$$
$$\left(\tfrac{(\alpha-\lambda+4)}{\left(1-\zeta^{2}\right)}U_{\alpha-\lambda+2}(\zeta)-\tfrac{(\alpha-\lambda+3)\zeta}{\left(1-\zeta^{2}\right)}U_{\alpha-\lambda+3}(\zeta)\right)-\tfrac{5\zeta(\alpha-\lambda+3)(\alpha-\lambda+5)}{\left(1-\zeta^{2}\right)}U_{\alpha-\lambda+3}(\zeta)$$

$$=\tfrac{1}{48\left(1-\zeta^{2}\right)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\tfrac{15\zeta^{2}-(\alpha-\lambda+2)(\alpha-\lambda+6)\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)}\right]$$
$$\tfrac{(\alpha-\lambda+4)}{\left(1-\zeta^{2}\right)}U_{\alpha-\lambda+2}(\zeta)-(\alpha-\lambda+3)\zeta\left(\tfrac{15\zeta^{2}-(\alpha-\lambda+2)(\alpha-\lambda+6)\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)^{2}}+\tfrac{5(\alpha-\lambda+5)}{\left(1-\zeta^{2}\right)}\right)U_{\alpha-\lambda+3}(\zeta)$$

$$=\sum_{\lambda=0}^{\alpha}\tfrac{(-1)^{\lambda}}{48}\binom{4}{\lambda}\left[\tfrac{15\zeta^{2}-(\alpha-\lambda+2)(\alpha-\lambda+6)\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)^{3}}\right]$$
$$(\alpha-\lambda+4)U_{\alpha-\lambda+2}(\zeta)-(\alpha-\lambda+3)\zeta\left(\tfrac{15\zeta^{2}-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)]\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)^{3}}\right)U_{\alpha-\lambda+3}(\zeta),$$

$$\therefore\sum_{a+b+c+d=\alpha}V_{a}(\zeta)\cdot V_{b}(\zeta)\cdot V_{c}(\zeta)\cdot V_{d}(\zeta)=\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{48}\binom{4}{\lambda}\left[R(\alpha,\lambda,\zeta)U_{\alpha-\lambda+2}(\zeta)-S(\alpha.\lambda,\zeta)U_{\alpha-\lambda+3}(\zeta)\right],$$

where

$$R(\alpha,\lambda,\zeta)=\left[\frac{15\zeta^{2}-(\alpha-\lambda+2)(\alpha-\lambda+6)\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)^{3}}\right](\alpha-\lambda+4),$$

a*nd*

$$S(\alpha.\lambda,\zeta)=(\alpha-\lambda+3)\zeta\left(\frac{15\zeta^{2}-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)]\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)^{3}}\right).$$

This proves Corollary 2.

Corollary 3. *For $\alpha\geq 0$, we have*

$$\sum_{a+b+c=\alpha}W_{a}(\zeta)\cdot W_{b}(\zeta)\cdot W_{c}(\zeta)=\sum_{\lambda=0}^{\alpha}\binom{3}{\lambda}\left[P(\alpha,\lambda,\zeta)\,U_{\alpha-\lambda+1}(\zeta)-Q(\alpha,\lambda,\zeta)\,U_{\alpha-\lambda+2}(\zeta)\right],$$

where

$P(\alpha,\lambda,\zeta)=\left(\frac{3\zeta(\alpha-\lambda+3)}{8\left(1-\zeta^{2}\right)^{2}}\right)$ *and* $Q(\alpha,\lambda,\zeta)=\frac{(\alpha-\lambda+2)}{8\left(1-\zeta^{2}\right)^{2}}\left((\alpha-\lambda+4)-(\alpha-\lambda-1)\zeta^{2}\right)$.

Proof. For the proof of Corollary, we will take r = 2 in Theorem 2 and proceed similarly as in the proof of Corollary 1 to achieve the desired results.

Corollary 4. For n $\geq$0, we have

$$\sum_{a+b+c+d=\alpha}W_{a}(\zeta)\cdot W_{b}(\zeta)\cdot W_{c}(\zeta)\cdot W_{d}(\zeta)=\sum_{\lambda=0}^{\alpha}\frac{1}{48}\binom{4}{\lambda}\left[R(\alpha,\lambda,\zeta)U_{\alpha-\lambda+2}(\zeta)-S(\alpha.\lambda,\zeta)U_{\alpha-\lambda+3}(\zeta)\right],$$

where

$$R(\alpha,\lambda,\zeta)=\left[\frac{15\zeta^{2}-(\alpha-\lambda+2)(\alpha-\lambda+6)\left(1-\zeta^{2}\right)}{\left(1-\zeta^{2}\right)^{3}}\right](\alpha-\lambda+4),$$

and

$$S(\alpha.\lambda,\zeta)=(\alpha-\lambda+3)\zeta\left(\frac{15\zeta^2-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)](1-\zeta^2)}{(1-\zeta^2)^3}\right).$$

Proof. For the proof of Corollary, we will take $r = 3$ in Theorem 2 and proceed similarly as in the proof of Corollary 2 to achieve the desired results.

Corollary 5. *For $\alpha \geq 0$, we have*

$$\sum_{a+b+c=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}=\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[A_{\alpha,\lambda}\,F_{2\alpha-2\lambda+4}+B_{\alpha,\lambda}F_{2\alpha-2\lambda+6}\right],$$

where $A_{\alpha,\lambda}=\frac{9}{25}(\alpha-\lambda+3)$ *and* $B_{\alpha,\lambda}=\frac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7)$.

Proof . Here, we will use Equation (2.8) together with the fact $U_\alpha\left(\frac{3}{2}\right)=F_{2\alpha+2}$ in Theorem 3 for $\zeta=\frac{3}{2}$ with r =2. Then, we have

$$\sum_{a+b+c=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}$$

$$=\frac{1}{2\cdot 2!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{5}\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]U_{\alpha-\lambda+2}\left(\tfrac{3}{2}\right)-3(3)U'_{\alpha-\lambda+2}\left(\tfrac{3}{2}\right)\right],$$

Using Equation (2.12) with $\zeta=\frac{3}{2}$, we have

$$=\frac{1}{4}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{5}\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]U_{\alpha-\lambda+2}\left(\frac{3}{2}\right)-9\left(\frac{(\alpha-\lambda+3)}{\left(1-\left(\frac{3}{2}\right)^2\right)}U_{\alpha-\lambda+1}\left(\frac{3}{2}\right)-\frac{(\alpha-\lambda+2)\left(\frac{3}{2}\right)}{\left(1-\left(\frac{3}{2}\right)^2\right)}U_{\alpha-\lambda+2}\left(\frac{3}{2}\right)\right)\right]$$

$$=\frac{1}{20}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]F_{2\alpha-2\lambda+6}-9\left(\frac{(\alpha-\lambda+3)}{\left(-\frac{5}{4}\right)}F_{2\alpha-2\lambda+4}-\frac{(\alpha-\lambda+2)\left(\frac{3}{2}\right)}{\left(-\frac{5}{4}\right)}F_{2\alpha-2\lambda+6}\right)\right]$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[[(\alpha-\lambda+2)(\alpha-\lambda+4)]F_{2\alpha-2\lambda+6}-9\left(\left(-\frac{2}{5}\right)(\alpha-\lambda+3)\,F_{2\alpha-2\lambda+4}-\left(-\frac{3}{5}\right)(\alpha-\lambda+2)F_{2\alpha-2\lambda+6}\right)\right]$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[[(\alpha-\lambda+2)(\alpha-\lambda+4)]F_{2\alpha-2\lambda+6}+9\left(\left(\frac{2}{5}\right)(\alpha-\lambda+3)\,F_{2\alpha-2\lambda+4}\left(\frac{3}{2}\right)-\left(\frac{3}{5}\right)(\alpha-\lambda+2)F_{2\alpha-2\lambda+6}\right)\right]$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[(\alpha-\lambda+2)\left((\alpha-\lambda+4)-\frac{27}{5}\right)F_{2\alpha-2\lambda+6}+\frac{18}{5}(\alpha-\lambda+3)\,F_{2\alpha-2\lambda+4}\right]$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[\frac{1}{5}(\alpha-\lambda+2)(5\alpha-5\lambda-7)F_{2\alpha-2\lambda+6}+\frac{18}{5}(\alpha-\lambda+3)\,F_{2\alpha-2\lambda+4}\right]$$

$$=\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[\frac{9}{25}(\alpha-\lambda+3)\,F_{2\alpha-2\lambda+4}+\frac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7)F_{2\alpha-2\lambda+6}\right],$$

$$\therefore\sum_{a+b+c=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}=\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[A_{\alpha,\lambda}\,F_{2\alpha-2\lambda+4}+B_{\alpha,\lambda}F_{2\alpha-2\lambda+6}\right],$$

where$A_{\alpha,\lambda} = \frac{9}{25}(\alpha-\lambda+3)$ *and* $B_{\alpha,\lambda} = \frac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7)$.
This proves Corollary 5.
Corollary 6. *For* $\alpha \geq 0$, *we have*

$$\sum_{a+b+c+d=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}\cdot F_{2d+1} = \frac{1}{150}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[C_{\alpha,\lambda}F_{2\alpha-2\lambda+8} - D_{\alpha,\lambda}F_{2\alpha-2\lambda+6}\right],$$

where

$$C_{\alpha,\lambda} = 3(\alpha-\lambda+3)(((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-5(\alpha-\lambda+5)),$$

and

$$D_{\alpha,\lambda} = 2[(\alpha-\lambda+2)(\alpha-\lambda+6)+27](\alpha-\lambda+4).$$

Proof. Here, we will follow the similar steps as in the proof of Corollary 6 and using Equation (2.8) in Theorem 3 for $\zeta = \frac{3}{2}$ with r =3 and $U_{\alpha}\left(\frac{3}{2}\right) = F_{2\alpha+2}$. We have

$$\begin{aligned}&\textstyle\sum_{a+b+c+d=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}\cdot F_{2d+1}\\ &= \tfrac{1}{5}\cdot\tfrac{1}{2^2 3!}\textstyle\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+6)]U'_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)-3\cdot 5\,U''_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)\right],\end{aligned}$$

Using Equations (2.12) and (2.14), with $\zeta = \frac{3}{2}$, we have

$$\textstyle\sum_{a+b+c+d=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}\cdot F_{2d+1}$$

$$= \tfrac{1}{120}\textstyle\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+6)]U'_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)-15\left(\frac{3\left(\frac{3}{2}\right)}{\left(1-\left(\frac{3}{2}\right)^2\right)}U'_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)-\frac{(\alpha-\lambda+3)(\alpha-\lambda+5)}{\left(1-\left(\frac{3}{2}\right)^2\right)}U_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)\right)\right]$$

$$= \frac{1}{120}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[[(\alpha-\lambda+2)(\alpha-\lambda+6)+54]U'_{\alpha-\lambda+3}\left(\frac{3}{2}\right)-12\left((\alpha-\lambda+3)(\alpha-\lambda+5)U_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\right)\right],$$

Again using Equation (2.12) with $= \frac{3}{2}$, we have

$$\textstyle\sum_{a+b+c+d=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}\cdot F_{2d+1} = \tfrac{1}{120}\textstyle\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}$$

$$\left[[(\alpha-\lambda+2)(\alpha-\lambda+6)+54]\left(\frac{(\alpha-\lambda+4)}{\left(1-\left(\frac{3}{2}\right)^2\right)}U_{\alpha-\lambda+2}\left(\tfrac{3}{2}\right)-\frac{(\alpha-\lambda+3)\left(\frac{3}{2}\right)}{\left(1-\left(\frac{3}{2}\right)^2\right)}U_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)\right)-12(\alpha-\lambda+3)(\alpha-\lambda+5)U_{\alpha-\lambda+3}\left(\tfrac{3}{2}\right)\right]$$

$$= \frac{1}{120}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\begin{array}{l}[(\alpha-\lambda+2)(\alpha-\lambda+6)+54]\left(\frac{(\alpha-\lambda+4)}{\left(-\frac{5}{4}\right)}U_{\alpha-\lambda+2}\left(\frac{3}{2}\right)-\frac{(\alpha-\lambda+3)\left(\frac{3}{2}\right)}{\left(-\frac{5}{4}\right)}U_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\right)\\ -12(\alpha-\lambda+3)(\alpha-\lambda+5)U_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\end{array}\right]$$

$$= \frac{1}{60}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\begin{array}{l}\left(-\frac{4}{5}\right)((\alpha-\lambda+2)(\alpha-\lambda+6)+27](\alpha-\lambda+4)U_{\alpha-\lambda+2}\left(\frac{3}{2}\right)+\left(\frac{6}{5}\right)\\ ((\alpha-\lambda+2)(\alpha-\lambda+6)+27)(\alpha-\lambda+3)U_{\alpha-\lambda+3}\left(\frac{3}{2}\right)-6(\alpha-\lambda+3)(\alpha-\lambda+5)U_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\end{array}\right],$$

$$\begin{aligned}&\textstyle\sum_{a+b+c+d=\alpha} F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}\cdot F_{2d+1}\\ &= \tfrac{1}{60}\textstyle\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\begin{array}{l}\left(-\frac{4}{5}\right)[(\alpha-\lambda+2)(\alpha-\lambda+6)+27](\alpha-\lambda+4)F_{2\alpha-2\lambda+6}\\ +(\alpha-\lambda+3)\left(\left(\frac{6}{5}\right)((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-6(\alpha-\lambda+5)\right)F_{2\alpha-2\lambda+8}\end{array}\right]\end{aligned}$$

$$= \frac{1}{60}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\begin{array}{l}\left(-\frac{4}{5}\right)\left[(\alpha-\lambda+2)(\alpha-\lambda+6)+27\right](\alpha-\lambda+4)F_{2\alpha-2\lambda+6}\\ +(\alpha-\lambda+3)\left(\left(\frac{6}{5}\right)((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-6(\alpha-\lambda+5)\right)F_{2\alpha-2\lambda+8}\end{array}\right]$$

$$= \frac{1}{30}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[\begin{array}{l}\left(-\frac{2}{5}\right)\left[(\alpha-\lambda+2)(\alpha-\lambda+6)+27\right](\alpha-\lambda+4)F_{2\alpha-2\lambda+6}\\ +\left(\frac{3}{5}\right)(\alpha-\lambda+3)\left(((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-5(\alpha-\lambda+5)\right)F_{2\alpha-2\lambda+8}\end{array}\right]$$

$$\begin{array}{l}= \frac{1}{150}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\left[3(\alpha-\lambda+3)\left(((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-5(\alpha-\lambda+5)\right)F_{2\alpha-2\lambda+8}\right]\\ -2\left[(\alpha-\lambda+2)(\alpha-\lambda+6)+27\right](\alpha-\lambda+4)F_{2\alpha-2\lambda+6},\end{array}$$

$$\therefore \sum_{a+b+c+d=\alpha}F_{2a+1}\cdot F_{2b+1}\cdot F_{2c+1}\cdot F_{2d+1} = \frac{1}{150}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}C_{\alpha,\lambda}F_{2\alpha-2\lambda+8}-D_{\alpha,\lambda}F_{2\alpha-2\lambda+6},$$

where

$$C_{\alpha,\lambda} = 3(\alpha-\lambda+3)\left(((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-5(\alpha-\lambda+5)\right)$$

and

$$D_{\alpha,\lambda} = 2\left[(\alpha-\lambda+2)(\alpha-\lambda+6)+27\right](\alpha-\lambda+4).$$

This proves the Corollary 6.

Corollary 7. *For $n, r \geq 0$, we have*

$$\sum_{a+b+c=\alpha}L_{2a+1}\cdot L_{2b+1}\cdot L_{2c+1} = \sum_{\lambda=0}^{\alpha}\binom{3}{\lambda}\left[A_{\alpha,\lambda}F_{2\alpha-2\lambda+4}+B_{\alpha,\lambda}F_{2\alpha-2\lambda+6}\right],$$

where $A_{\alpha,\lambda} = \frac{9}{25}(\alpha-\lambda+3)$ *and* $B_{\alpha,\lambda} = \frac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7)$.

Proof. For the proof of Corollary 7, we will proceed similarly as in the proof of Corollary 5 and use Equation (2.10) in Theorem 4 for $\zeta = \frac{3}{2}$ with r =2 and $U_{\alpha}\left(\frac{3}{2}\right) = F_{2\alpha+2}$ to achieve the desired results.

Corollary 8. *For $\alpha \geq 0$, we have*

$$\sum_{a+b+c+d=\alpha}L_{2a+1}\cdot L_{2b+1}\cdot L_{2c+1}\cdot l_{2d+1} = \frac{1}{150}\sum_{\lambda=0}^{\alpha}\binom{4}{\lambda}\left[C_{\alpha,\lambda}F_{2\alpha-2\lambda+8}-D_{\alpha,\lambda}F_{2\alpha-2\lambda+6}\right],$$

where

$$C_{\alpha,\lambda} = 3(\alpha-\lambda+3)\left[((\alpha-\lambda+2)(\alpha-\lambda+6)+27)-5(\alpha-\lambda+5)\right].$$

and

$$D_{\alpha,\lambda} = 2\left[(\alpha-\lambda+2)(\alpha-\lambda+6)+27\right](\alpha-\lambda+4).$$

Proof. For the proof of Corollary 8, we will proceed similarly as in the proof of Corollaries 6 and use Equation (2.10) in Theorem 4 for $\zeta = \frac{3}{2}$ with $r = 3$ and $U_{\alpha}\left(\frac{3}{2}\right) = F_{2\alpha+2}$ to achieve the desired results.

4 Conclusion

This study has considered the sums of finite products of Chebyshev polynomials of the 3rd and 4th kind as a linear combination of Chebyshev polynomials of 2nd kind and their derivatives. Similar results for Fibonacci and Lucas numbers are obtained. At the end, some particular cases of these results are discussed. These findings certainly provide prospective scholars with the possibility of expressing such sums in terms of other orthogonal polynomials. Additionally, it is feasible to examine the sums of finite products of different orthogonal polynomials as well. These findings are useful in handling summation problems arising in different fields of mathematics and the control theory is one such intriguing field.

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