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Mathematical Model for Optimal Control of Water Pollutant Discharges into Aquatic Ecosystem

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Abstract

Objectives: To develop an optimal control model that can be used to control organic pollutants, inorganic pollutants and invasive aquatic species within the ecosystem to enhance the level of dissolved oxygen and fish populations. **Methods**: The study formulated a nonlinear differential equation model that involves the interaction between pollutants, aquatic plants, bacteria, oxygen and fish. Optimal controls for inorganic pollutant treatment, organic pollutant treatment, and the removal of invasive aquatic plants were devised using Pontryagin's Maximum Principle. Four different combinations of control strategies were applied, and numerical simulations were performed using a fourth-order Runge–Kutta algorithm. **Findings**: Combining all control measures of treating both inorganic and organic pollutants and removal of invasive aquatic plants resulted in a higher ecological recovery. Strategy I has improved the level of concentration of dissolved oxygen up to 60% and also increased the population of fish up to 62% which is better compared to all other strategies. Strategy II failed to improve the oxygen level, although it increased the fish population by 42% due to managing invasive plants well, which would induce eutrophication. Strategy III managed to improve oxygen concentration by 60% and fish population by 57% but untreated organic pollutants still impaired water clarity. Strategy IV produced the highest oxygen improvement of 68% but resulted in no fish survival because of untreated inorganic toxicity. Compared with earlier models that examined either pollutants or invasive plants in isolation, the model presented herein integrates all major ecological drivers into a unified optimal control framework for the first time. Thereby, the findings are more realistic, holistic, and quantitatively superior for long-term ecosystem restoration. **Novelty**: A combined optimal control model that incorporates organic pollutants, inorganic pollutants, and invasive aquatic plants simultaneously to restore the ecological ecosystem.

Keywords: Aquatic species; Dissolved oxygen; Numerical simulation; Optimal control; Pollutants; Wastewater

1 Introduction

High quantities of wastewater from urban and industrial sources are generated and discharged into rivers, lakes and oceans [1]. This wastewater usually contains pollutants that can be categorized into two types, namely organic and inorganic pollutants [2]. Organic pollutants, including oils, fats, proteins and sugar, can undergo biodegradation and can also be decomposed by microorganisms. When these organic pollutants are in higher amounts, they reduce dissolved oxygen levels in water, hence becoming danger to aquatic life [3]. Inorganic pollutants include heavy metals, phosphorus, nitrogen, and other compounds. These can be accumulated in the ecosystem, leading to negative effects on the quality of water as well as to biodiversity [4],[5].

It is challenging to manage these pollutants, particularly when multiple sources of wastewater discharge into the same water body. This is because the interaction of different types of pollutants with aquatic ecosystems is complicated and changes over time [6],[7]. Water hyacinths and algae are among of invasive aquatic plants that also make it difficult to manage the ecosystem [8],[9]. These aquatic plants grow well when there is a higher amount of nutrients, brought by pollutants from wastewater [10]. Algae blooms caused by organic pollutants cause killing of aquatic species like fish by taking away dissolved oxygen [11]. Water hyacinths grow very fast, cover the water surface and limit sunlight and oxygen from penetrating the water [12]. Therefore, invasive species affect the ecosystems, and if they are not well controlled, they can cause problems to aquatic life and the ecosystem in general [13].

Mathematical models are widely used to assess the effects of water pollutants on aquatic ecosystems. The study presented in [14] analyzed the effects of organic and inorganic pollutants on the survival of fish without considering aquatic plants as contributors to oxygen production. The study [15] developed a reaction–diffusion–advection model to examine how exposure to toxicants in a flowing river influences the spatial distribution and long-term survival of aquatic populations. However, the study did not address or propose any control strategy for managing or mitigating the effects of these toxic pollutants. The model proposed in [16] considered the combined effects of water pollution and eutrophication on the dissolved oxygen levels. This model demonstrated that the impact on oxygen levels is higher in both cases. However, it did not take into consideration the effects of toxicants on the fish population over time. Later on, research presented in [17] focused on the effect of toxicants on biological populations, indicating the need for toxic emission regulation, but did not take into consideration the broader ecological and human health effects. In addition to that, the study in [18] applied the nonlinear differential equation model to analyze the nutrient-driven species growth but failed to consider the interaction between pollutants and other components of the ecosystem.

More importantly, earlier studies did not look at the best way to combine control strategies. Optimal control theory offers a robust framework for dealing with the complex problems associated with managing wastewater discharges while preserving the ecological balance of aquatic organisms such as fish, bacteria, and aquatic plants [19]. By constructing mathematical models that consider organic and inorganic pollutants, coupled with the infestation of invasive aquatic plants, it is possible to derive the optimum methods of controlling the pollutants and invasive plants while still maintaining the health of the ecosystem. These methods offer an effective means of formulating interventions that comply with water quality standards while safeguarding the biodiversity of aquatic organisms [20].

This study addresses a significant gap, as aquatic plants play a dual role by supplying oxygen to the water and serving as a food source for aquatic organisms, but they can also negatively impact the survival and functioning of aquatic species. It presents a complete model for the optimum methods of controlling the release of water pollutants, taking into consideration aquatic plants as an integral part of how oxygen moves around.

2 Methodology

2.1 Model Formulation

In a model of [21], sensitivity analysis showed that some of the parameters had a great influence on the dynamics of the system. These findings motivated further research using optimal control in order to determine the best combination of strategies for managing ecosystems. This research seeks to find the best control strategy for wastewater treatment and invasive aquatic species management. For this purpose, we employ three time-dependent control parameters, anchored to the sensitive parameters in the basic model. This will allow us to modify crucial parts of the system in real time, as is required for the sensitive variables. The parameter $u_1(t)$ is used for controlling the inorganic pollutants in the water system. It is intended to enhance the inorganic pollutant treatment processes to be as effective as possible, perhaps by altering the rates or methods of treatment. Similarly, $u_2(t)$ is the control parameter associated with organic pollutants. It facilitates the application of strategies aimed at reducing organic pollutants, such as chemical treatments or deploying biological agents. Finally, $u_3(t)$ is applied to control invasive aquatic plants such as water hyacinths and algae. The parameter addresses the eradication or control of these species through various means, which include chemical treatments, mechanical harvesting, or biological control measures. In optimizing the control parameters, the aim is to derive a compromise between treatment efficiency and ecological effect with the view of ensuring the sustainability of the aquatic ecosystem.

This model analyzes the movement of pollutants within aquatic ecosystems. It does this by applying a system of differential equations in considering organic pollutants (P_o) and the inorganic pollutants (P_i). Other state variables include the density of bacteria (B), the density of aquatic plants such as algae and water hyacinth (A), the concentration of dissolved oxygen (C) and the density of fish (F).

Pollutants enter the water at rates of Q_1 (inorganic) and Q_2(organic), and they break down over time at rates of μ_1 and μ_2 respectively. Organic pollutants help bacteria grow, but fish grow more slowly when they eat inorganic pollutants. Bacterial populations experience natural mortality at a specific rate μ_1 and engage in intra-species competition defined by λ_{20}. Aquatic plants also die naturally at a rate of μ_6, and they are limited by competition within their own species, which is shown by λ_{30}. The rate at which dissolved oxygen (C) is added back is Λ, and the rate at which it naturally goes down is μ_4. Oxygen also helps bacteria break down into organic matter at a rate of β_{02}. Fish growth is critically dependent on dissolved oxygen levels and is adversely affected by the direct ingestion of inorganic pollutants as well as the oxygen depletion caused by organic pollutants. Fish also die naturally at a rate of, compete with other fish of the same species at a rate of λ_{10}, and die from toxins in inorganic pollutants at a rate of θ.

Species growth and nutrient dynamics are modeled using Monod-type functions. The consumption of organic pollutants P_o by bacteria, fish, and aquatic plants occurs at rates of $\frac{\beta_{20}P_oB}{\beta_{21}+\beta_{22}P_o}$, $\frac{k_1P_oF}{k_{12}+k_{11}P_o}$ and $\frac{\beta_{01}P_oA}{\beta_{12}+\beta_{11}P_o}$ respectively. Oxygen is depleted during the decomposition of organic matter by bacteria at a rate $\frac{\beta_{02}P_oB}{\beta_{21}+\beta_{22}P_o}$. Additionally, fish benefit from the consumption of organic pollutants at a rate of $\frac{\lambda_4k_1P_oF}{k_{12}+k_{11}P_o}$. Hence, the optimal control model is formalized as a system of nonlinear differential equations, as presented mathematically in **Eq. (1)**.

$$\begin{cases} \dfrac{dP_i}{dt} = Q_1(1-u_1) - \alpha(1-u_1)P_iF - \mu_1P_i, \\ \dfrac{dP_o}{dt} = Q_2(1-u_2) - \dfrac{\beta_{20}(1-u_2)P_oB}{\beta_{21}+\beta_{22}P_o} - \dfrac{k_1(1-u_2)P_oF}{k_{12}+k_{11}P_o} - \dfrac{\beta_{01}(1-u_2)P_oA}{\beta_{12}+\beta_{11}P_o} - \mu_2P_o, \\ \dfrac{dA}{dt} = \Lambda_2(1-u_3)A + \dfrac{\lambda_3(1-u_3)\beta_{01}P_oA}{\beta_{12}+\beta_{11}P_o} - \mu_3A - \lambda_{30}A^2, \\ \dfrac{dB}{dt} = \Lambda_3B + \dfrac{\lambda_2(1-u_2)\beta_{20}P_oB}{\beta_{21}+\beta_{22}P_o} - \mu_6B - \lambda_{20}B^2, \\ \dfrac{dC}{dt} = \Lambda - \dfrac{\beta_{02}(1-u_2)P_oB}{\beta_{21}+\beta_{22}P_o} - \gamma_1CF - \gamma_2(1-u_3)A - \gamma_3B - \mu_4C, \\ \dfrac{dF}{dt} = \Lambda_1F + \dfrac{\lambda_4(1-u_2)k_1P_oF}{k_{12}+k_{11}P_o} + \gamma_1CF - \theta(1-u_1)P_iF - \mu_5F - \lambda_{10}F^2. \end{cases} \tag{1}$$

With initial conditions $P_i(0) > 0, P_o(0) > 0, B(0) > 0, A(0) > 0, F(0) > 0, C(0) > 0$.

To determine the optimal control measure from the given set of control measures (u_1, u_2, u_3) we formulate the objective function J.

$$J(u_1,u_2,u_3) = \int_0^{t_f} \left(A_1P_i + A_2P_o + A_3A + \frac{C_1u_1^2}{2} + \frac{C_2u_2^2}{2} + \frac{C_3u_3^2}{2} \right) dt \tag{2}$$

where, A_1, A_2 and A_3 are positive constant weights representing cost for minimizing inorganic pollutants, organic pollutants and aquatic plants in water body, respectively. Additionally, C_1, C_2 and C_3 are positive constant weights describing costs incurred in implementing the control measures u_1, u_2 and u_3, respectively. It should be noted that, costs associated to any control measure takes a non-linear form. This fact can be observed from **equation (2)**, where $\frac{C_1u_1^2}{2}$ stands for the cost associated with inorganic pollutants treatment system to minimize inorganic pollution. Furthermore, $\frac{C_2u_2^2}{2}$ stands for the cost associated with organic pollutants treatment system to minimize inorganic pollution and $\frac{C_3u_3^2}{2}$ stands for the cost associated with removing invasive species of aquatic plants. Using the objective function $J(u_1,u_2,u_3)$, we aim at reducing cost of water treatment and invasive aquatic plants removal. The optimal control solutions $u_1^*(t), u_1^*(t)$ and $u_1^*(t)$ are searched in such a way that:

$$J(u_1^*,u_2^*,u_3^*) = min\{J(u_1,u_2,u_3)\,|u_1,u_2,u_3 \in \mathbf{u}\}\,dt$$

where, $\mathbf{u} = u_1, u_2, u_3$ in which u_1, u_2 and u_3 are measurable within the closed interval $0 \leq u_1, u_2, u_3 \leq 1$ under the control set $t \in \left[0, t_f\right]$.

2.2 Optimal control characterization

In this subsection, the conditions required for an optimal control are derived. Furthermore, an optimality system that uses lower and upper bound approaches is formulated in order to describe the formulated optimal control problem. The necessary condition is that the optimal control problem should satisfy Pontryagin's maximum principle[(19)],[(20)],[(21)]. This principle transposes system (1) and **equation (2)** to a point-wise minimization problem of a Hamiltonian function $\mathcal{H}$ with respect to u_1, u_2 and u_3 in the following manner:

$$
\begin{aligned}
\mathcal{H} \;=\;& A_1 P_i + A_2 P_o + A_3 A + \frac{C_1 {u_1}^2}{2} + \frac{C_2 {u_2}^2}{2} + \frac{C_3 {u_3}^2}{2} \\
&+\lambda_{P_i}\left[Q_1(1-u_1) - \alpha(1-u_1)P_iF - \mu_1 P_i\right] \\
&+\lambda_{P_O}\left[Q_2(1-u_2) - \frac{\beta_{20}(1-u_2)P_oB}{\beta_{21}+\beta_{22}P_o} - \frac{k_1(1-u_2)P_oF}{k_{12}+k_{11}P_o} - \frac{\beta_{01}(1-u_2)P_oA}{\beta_{12}+\beta_{11}P_o} - \mu_2 P_o\right] \\
&+\lambda_A\left[\Lambda_2(1-u_3)A + \frac{\lambda_3(1-u_3)\beta_{01}P_oA}{\beta_{12}+\beta_{11}P_o} - \mu_3 A - \lambda_{30}A^2\right] \\
&+\lambda_B\left[\Lambda_3 B + \frac{\lambda_2(1-u_2)\beta_{20}P_oB}{\beta_{21}+\beta_{22}P_o} - \mu_6 B - \lambda_{20}B^2\right] \\
&+\lambda_C\left[\Lambda - \frac{\beta_{02}(1-u_2)P_oB}{\beta_{21}+\beta_{22}P_o} - \gamma_1 CF - \gamma_2(1-u_3)A - \gamma_3 B - \mu_4 C\right] \\
&+\lambda_F\left[\Lambda_1 F + \frac{\lambda_4(1-u_2)k_1P_oF}{k_{12}+k_{11}P_o} + \gamma_1 CF - \theta(1-u_1)P_iF - \mu_5 F - \lambda_{10}F^2\right]
\end{aligned}
\tag{3}
$$

$\lambda_{P_i},\ \lambda_{P_O}, \lambda_A\ ,\ \lambda_B, \lambda_C\ and\ \ \lambda_F$ are the co-state variables. By using Pontryagin's maximum principle described by[(20)],[(21)],[(22)], we get Lemma 1:

Lemma 1: *For optimal controls$u_1^*, u_2^* and u_3^*$that minimizes$J(u_1, u_2, u_2)$over$\mathbf{u}$, there exist co-state variables $\lambda_{P_i},\ \lambda_{P_O}, \lambda_A\ ,\ \lambda_B, \lambda_C\ and\ \ \lambda_F$satisfying:*

$$
\begin{aligned}
\frac{\partial \lambda_{P_i}}{\partial t} &= \lambda_{P_i}\left(\alpha(1-u_1)F + \mu_1\right) + \lambda_F \theta(1-u_1)P_iF - A_1, \\
\frac{\partial \lambda_{P_O}}{\partial t} &= \lambda_{P_O}\left(\frac{\beta_{20}(1-u_2)\beta_{21}B}{\left(\beta_{21}+\beta_{22}P_o\right)^2} + \frac{k_1(1-u_2)k_{12}F}{\left(k_{12}+k_{11}P_o\right)^2} + \frac{(1-u_2)\beta_{12}\beta_{01}A}{\left(\beta_{12}+\beta_{11}P_o\right)^2} + \mu_2\right) \\
&\quad + \frac{\lambda_C\beta_{02}(1-u_2)\beta_{21}B}{\left(\beta_{21}+\beta_{22}P_o\right)^2} \\
&\quad - \left(\frac{\lambda_A\lambda_3(1-u_3)\beta_{01}\beta_{12}A}{\left(\beta_{21}+\beta_{22}P_o\right)^2} + \frac{\lambda_B\lambda_2(1-u_2)\beta_{20}\beta_{21}B}{\left(\beta_{21}+\beta_{22}P_o\right)^2} + \frac{\lambda_F\lambda_4(1-u_2)k_1k_{12}F}{\left(k_{12}+k_{11}P_o\right)^2} + A_2\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \lambda_A}{\partial t} &= \frac{\lambda_{P_O}(1-u_2)\beta_{01}P_o}{\beta_{12}+\beta_{11}P_o} + \lambda_A\left(\mu_3 + 2\lambda_{30}A\right) + \lambda_F\gamma_2(1-u_3) \\
&\quad - \left(\lambda_A\left(\Lambda_2(1-u_3) + \frac{\lambda_3(1-u_2)\beta_{01}P_o}{\beta_{12}+\beta_{22}P_o}\right) + A_3\right), \\
\frac{\partial \lambda_B}{\partial t} &= \frac{\lambda_{P_O}\beta_{20}(1-u_2)P_oB}{\beta_{21}+\beta_{22}P_o} - \lambda_A\left(\Lambda_3 + \frac{\lambda_3\beta_{20}(1-u_2)P_o}{\beta_{21}+\beta_{22}P_o} - \left(\mu_6 + 2\lambda_{20}B\right)\right), \\
\frac{\partial \lambda_C}{\partial t} &= \lambda_C\left(\gamma_1 F + \mu_4\right) - \lambda_F\gamma_1 F, \\
\frac{\partial \lambda_F}{\partial t} &= \frac{\lambda_{P_O}(1-u_2)k_1P_o}{k_{12}+k_{11}P_o} + \lambda_{P_i}\alpha P_i + \lambda_F\gamma_2\left(\theta(1-u_1)P_i + \mu_5 + 2\lambda_{20}B\right) \\
&\quad - \lambda_F\left(\Lambda_1 + \frac{\lambda_4(1-u_2)k_{11}P_oF}{k_{12}+k_{11}P_o} + \gamma_1 C\right).
\end{aligned}
$$

with transversality conditions

$$
\lambda_{P_i}\left(t_f\right) = \lambda_{P_0}\ \left(t_f\right) = \lambda_A\left(t_f\right) = \lambda_B\left(t_f\right) = \lambda_C\left(t_f\right) = \lambda_F(t_f) = 0. \tag{5}
$$

2.3 Existence of optimal controls

Lemma 2 *Given$J(\mathbf{u})$subject to system (1) with non-negative initial conditions of the state variables, that is$P_i(0) \geq 0, P_o(0) \geq 0, A(0) \geq 0, B(0) \geq 0, C(0) \geq 0,andF(0) \geq 0,$there exists an optimal control$\mathbf{u} = (u_1^*, u_2^*, u_3^*)$that minimizes$J(\mathbf{u})$over$\mathbf{u}$, and the* *corresponding adjoint variables* $P_i, P_o,\ A,\ B,\ C and F$ *satisfying the following equations.*

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial u_1} & = & C_1 u_1 + \left(\lambda_{P_i}\alpha + \lambda_F \theta\right) P_i F - \lambda_{P_i} Q_1, \\ \dfrac{\partial \mathcal{H}}{\partial u_2} & = & C_2 u_2 - \mathcal{V}, \\ \dfrac{\partial \mathcal{H}}{\partial u_3} & = & C_3 u_3 - A\left(\lambda_C \gamma_2 + \lambda_A \left(\Lambda_2 + \dfrac{\lambda_3 \beta_{01} P_o}{\beta_{12} + \beta_{11} P_o}\right)\right) \end{cases} \tag{6}$$

where

$$\mathcal{V} = \left(\frac{\left(\lambda_{P_0} + \lambda_B\right)\beta_{20}\ P_o B}{\beta_{21} + \beta_{22} P_o} + \frac{\lambda_C \beta_{02} P_o\ B}{\beta_{21} + \beta_{22} P_o} + \frac{\left(\lambda_{P_0} + \lambda_F\right) k_1\ F P_o}{k_{12} + k_{11} P_o} + \frac{\left(\lambda_{P_0} + \lambda_A\right)\lambda_3 \beta_{01}\ P_o A}{\beta_{21} + \beta_{22} P_o}\right)$$

The control set $\mathbf{u} = (u_1^*, u_2^*, u_3^*)$ gives $\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0$, in particular, $\frac{\partial \mathcal{H}}{\partial u_1^*} = 0,\quad \frac{\partial \mathcal{H}}{\partial u_2^*} = 0$ and $\frac{\partial \mathcal{H}}{\partial u_3^*} = 0$. Thus, solving the system (6) for u_1^*, u_2^* and u_3^* gives

$$\begin{aligned} u_1^* &= \tfrac{1}{c_1}\left(\lambda_{P_i} Q_1 - \left(\lambda_{P_i}\alpha + _F \theta\right) P_i F\right) \\ u_2^* &= \tfrac{1}{c_2}\left(\lambda_{P_0} Q_2 - \mathcal{V}\right) \\ u_3^* &= \tfrac{1}{c_3}\left(A\left(\lambda_C \gamma_2 + \lambda_A \left(\Lambda_2 + \tfrac{\lambda_3 \beta_{01} P_o}{\beta_{12} + \beta_{11} P_o}\right)\right)\right) \end{aligned}$$

The characterization (8) holds on the interior of the control set $\mathbf{u}^*$.

$$\begin{aligned} u_1^* &= \max\left\{0, \min\left(1, \frac{1}{c_1}\left(\lambda_{P_i} Q_1 - (\lambda_{P_i}\alpha + \lambda_F \theta) P_i F\right)\right)\right\}, \\ u_2^* &= \max\left\{0, \min\left(1, \frac{1}{c_2}\left(\lambda_{P_o} Q_2 - \mathcal{V}\right)\right)\right\}, \\ u_3^* &= \max\left\{0, \min\left(1, \frac{1}{c_3}\left(A\left(\lambda_C \gamma_2 + \lambda_A \left(\Lambda_2 + \frac{\lambda_3 \beta_{01} P_o}{\beta_{12} + \beta_{11} P_o}\right)\right)\right)\right)\right\} \end{aligned} \tag{7}$$

Proof. The form of the adjoint (or costate) system, given by **Eq. (4),** and the transversality conditions, as specified in **Eq. (5)**, are standard outcomes derived from [22]. To derive the costate system described in **Eq. (4)**, we compute the partial derivatives of the Hamiltonian function $\mathcal{H}$ (**Eq. (3)**) with respect to each state variable as follows:

$$\begin{cases} \dfrac{d\lambda_{P_i}}{dt} = -\dfrac{\partial \mathcal{H}}{\partial P_i}; \quad \lambda_{P_i}(t_f) = 0, \\ \quad\vdots \\ \dfrac{d\lambda_F}{dt} = -\dfrac{\partial \mathcal{H}}{\partial F}; \quad \lambda_F(t_f) = 0. \end{cases} \tag{9}$$

The optimality system, represented by **Eq. (7)**, is obtained by taking the partial derivatives of the Hamiltonian function **Eq. (3)** with respect to each control variable and solving for the optimal values u_i^* where these derivatives are zero; specifically, $\frac{\partial \mathcal{H}}{\partial u_i} = 0$ for $i = 1, 2, 3$. Solving for u_i^* under the given constraints yields the characterization **equation (8)**.

3 Results and Discussion

In this section, the study present a detailed interpretation of the numerical results obtained from the optimal control model and situate our findings within the broader context of existing literature. To find the best control solution, you need to solve the optimality system, which has both the adjoint and state systems. Using the Fourth-order Runge–Kutta iterative method and an initial guess of the control variables over the simulation period, the state equations were solved. This method is known for its fourth-order accuracy and has a truncation error of $\mathcal{O}(h^5)$, where h is the step size. This means that the error at each step is proportional to h^5, which makes it very accurate. The Fourth-order Runge–Kutta method is also known for being stable across many problems, which makes it useful for many different things. The convergence rate is $\mathcal{O}(h^4)$, which means that the error goes down quickly as the step size gets smaller.

The adjoint equations were solved using the backward Fourth-order Runge–Kutta scheme, with the solutions to the state equations as inputs because of the transversality conditions. Additionally, the controls were revised by utilizing a convex combination of the preceding controls and the values derived from the characterizations[7]. The iterative process continued until the difference between the values of the unknowns from successive iterations was sufficiently minimal, indicating convergence. Table 1 shows the numbers that were used for the numerical simulation. The initial values used in this study are $[P_i(0),\ P_o(0),\ A(0),\ B(0),\ C(0),\ F(0)] = [50.1,\ 0.15,\ 1.5,\ 0.01,\ 4.0,\ 3.0]$. Due to the lack of enough data to figure out the weights, theoretical values were used as examples in this paper, with the weights set as $A_1 = 25.00$, $A_2 = 5.00$, $A_3 = 5.00$, and the costs as $C_1 = 12.00$, $C_2 = 6.00$, and $C_3 = 1.50$. A larger weight was assigned to inorganic pollutants due to their higher toxicity and persistence, while organic pollutants and invasive plants were assigned smaller but equal weights. The cost coefficients reflect the relative implementation difficulty, with inorganic pollutant control being more expensive than the other. Although three control variables theoretically yield seven possible combinations, existing studies have demonstrated that combined control strategies are more effective than the application of single controls[23]. Accordingly, this study considers only four strategies involving combinations of controls, while single-control scenarios are omitted as they provide limited additional insight and reduced management effectiveness. Figures 1– 8 show the simulation results, which show how different control strategies affect the high-risk classes. The red dashed lines show how the population would change without any control measures, and the blue solid lines show how the population would change with certain control strategies. Additionally, in the graphs that display the control profiles during the implementation of particular strategies, the blue, black, and green lines correspond to the control options $u_1(t)$, $u_2(t)$, and $u_3(t)$, respectively.

Table 1. Description of parameters used in numerical simulations

Parameters	Value	Source	Parameters	Value	Source
Q_1	18.05019×10^6	[24]	β_{11}	0.051	[25]
Q_2	18.056	[24]	β_{12}	1	[25]
Λ	16.913	[24]	β_{21}	7.81	[13]
Λ_1	0.5	Assumed	β_{22}	1.48	[13]
Λ_2	0.5	Assumed	k_1	1	Assumed
Λ_3	0.5	Assumed	k_{11}	1	Assumed
α	0.05	Assumed	k_{12}	1	Assumed
θ	0.02	Assumed	μ_1	0.2	Assumed
λ_2	0.33	[24]	μ_2	1.804	[24]
λ_3	0.2	Assumed	μ_3	0.031	Assumed
λ_4	0.13	Assumed	μ_4	0.3	[24]
λ_{10}	0.446	[24]	μ_5	1.5	[24]
λ_{20}	8.278	[24]	μ_6	0.28	[24]
λ_{30}	0.5	Assumed	γ_1	0.1	[24]
β_{01}	0.029	[25]	γ_2	0.5	Assumed
β_{02}	0.112	[24]	γ_3	0.05	Assumed
β_{20}	4.38	[12]			

3.1 Strategy I: Treatment of both Organic and Inorganic Pollutants and Controlling Aquatic Plants

Figure 1 illustrates that the treatment of both organic and inorganic pollutants, coupled with controlling excessive growth of aquatic plants, results in significant ecological improvements in the aquatic environment. Lowered pollutant levels reduce water toxicity, lowering the mortality rate among fish and promoting the recovery of their populations. In Figure 2, we can see how this affects oxygen levels and fish populations. Furthermore, invasive plant species, such as algae, do not take over and cause eutrophication, which reduces oxygen levels when the plants die and decay. In effect, dissolved oxygen levels normalize, allowing a much healthier environment for aquatic organisms, especially fish, without hypoxia. Overall, these measures contribute to enhanced ecosystem stability and biodiversity conservation.

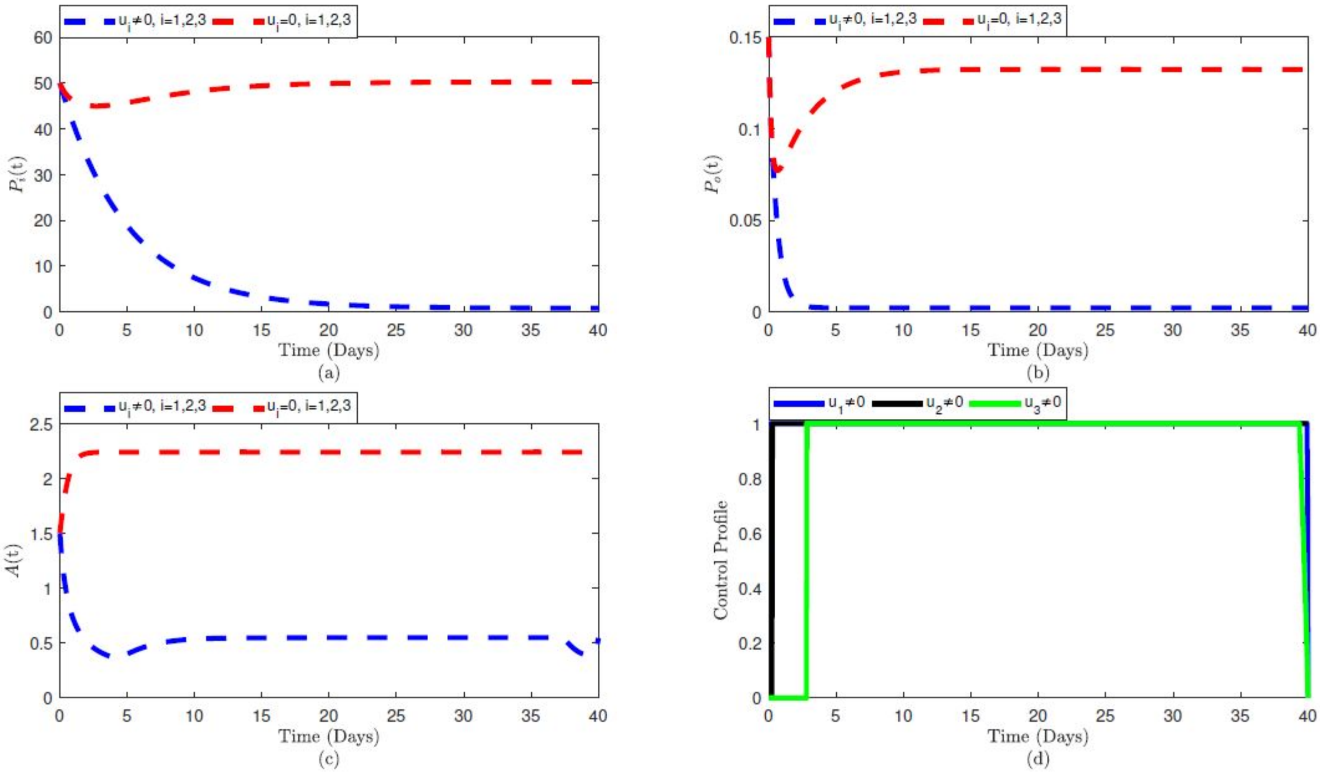

Fig 1. Treatment of both organic and inorganic pollutants and controlling aquatic plants

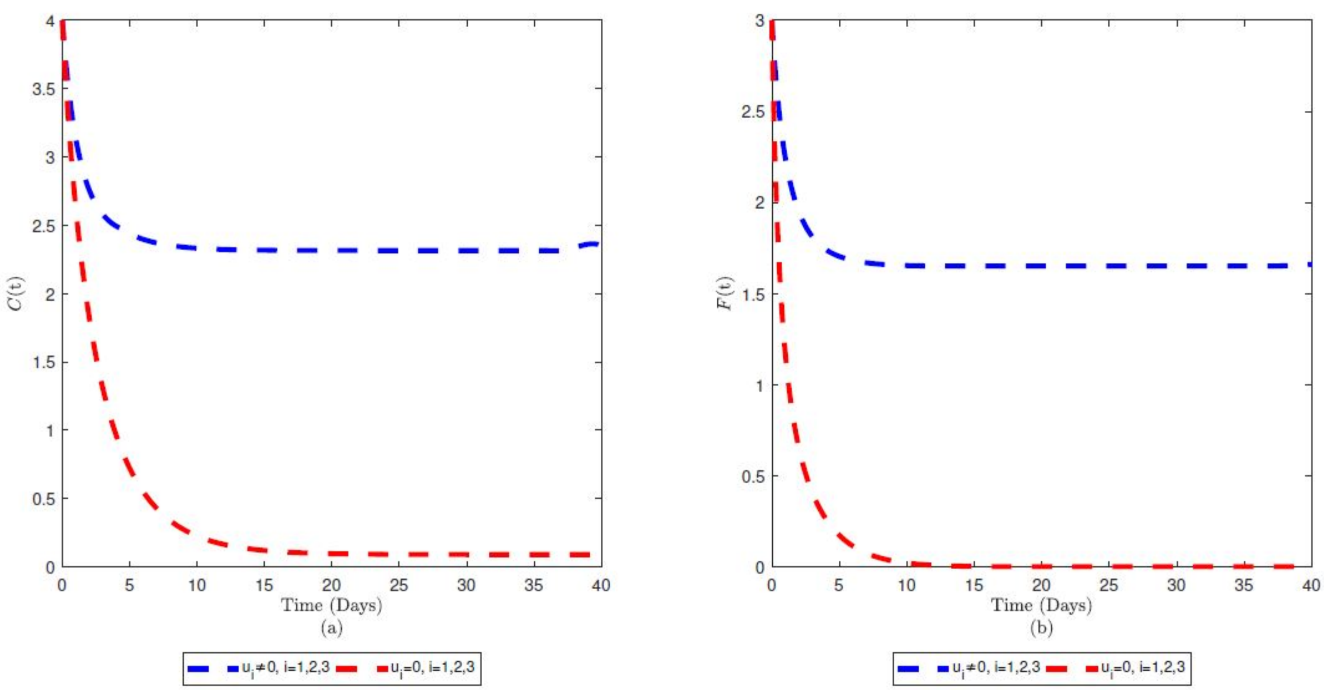

Fig 2. Impact of treatment of both organic and inorganic pollutants and controlling aquatic plants to level of oxygen concentration and fish population

3.2 Strategy II: Treatment of both Organic and Inorganic Pollutants

Fig. 3: Keeping both organic and inorganic pollutants within the limits but not managing aquatic plants strongly favors ecological benefits. A decrease in the level of pollutants reduces the toxicity of water, which improves the recovery of fish populations. Organic pollutants may help in the growth of fish by providing nutrition. However, uncontrolled growth of aquatic plants-especially algae-may result in eutrophication followed by oxygen depletion. The outcome could be lower levels of dissolved oxygen over some time, which would affect the health of fish. **Fig. 4:** The above effect on oxygen levels and fish populations. Thus, the population of fish may improve initially but the negative impacts on long-term water quality due to unmanaged plant growth will actually reduce the sustainability of the fish over a longer period.

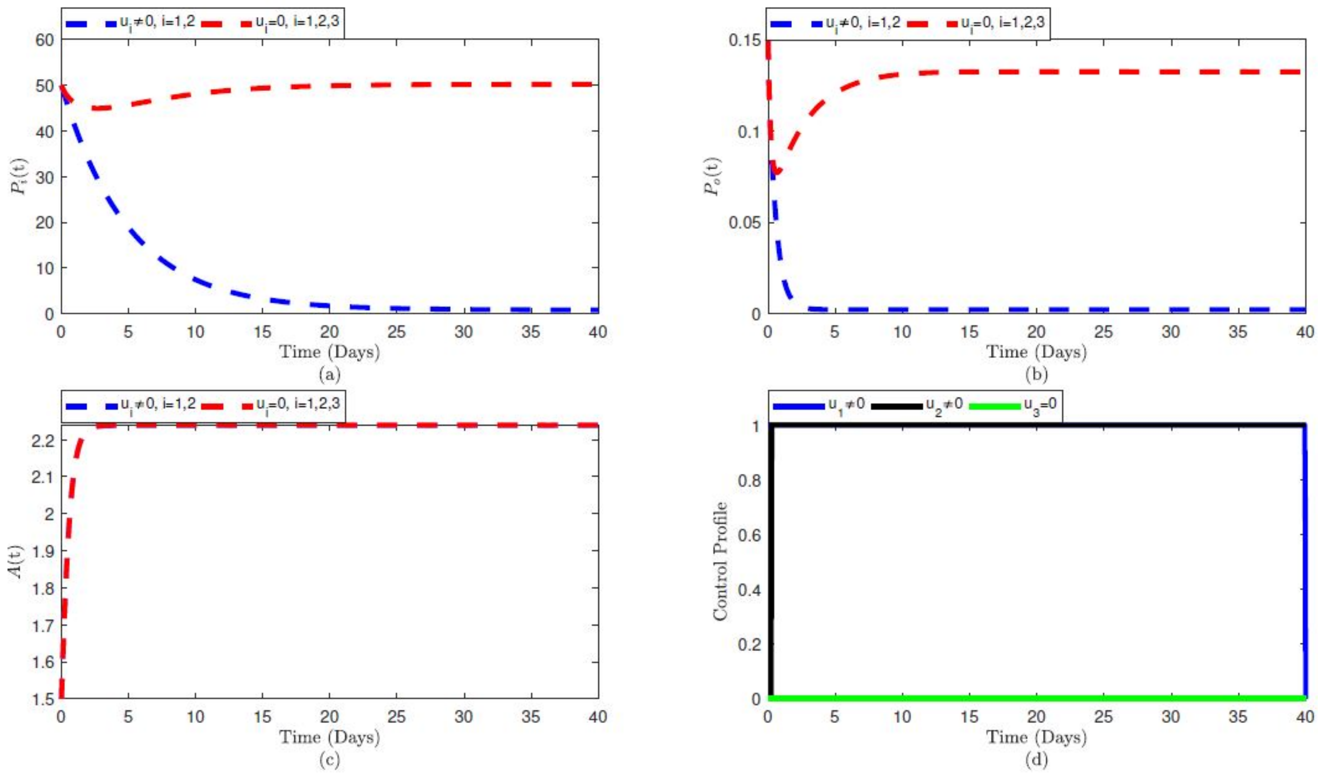

Fig 3. Treatment of both organic and inorganic pollutants without controlling aquatic plants

3.3 Strategy III: Treatment of Inorganic Pollutants and Controlling Aquatic Plants

Figure 5 shows that the removal of inorganic pollutants and aquatic plant growth control creates partial improvement in the ecosystem. The decrease in inorganic pollutants reduces the toxicity of the water, which favors fish populations. However, uncontrolled organic pollutants can provide a food source and further increase fish populations. This would still provide an effect in terms of the growth of fish, but water quality is compromised because organic pollutants may still affect turbidity and dissolve oxygen. As such, even though the population of fish can be increased, the stabilization of clarity and dissolved oxygen could not be guaranteed. The effect on dissolved oxygen and fish population is presented by Figure 6.

3.4 Strategy IV: Treatment of Organic Pollutants and Controlling Aquatic Plants

Referring to Figure 7, we can see that while controlling the organic pollutants and managing the aquatic plants, leaving the inorganic pollutants untreated would have dire consequences on the aquatic ecosystem. This can be explained by noting that even though a reduction in organic pollutants would increase oxygen concentration because there is no eutrophication, the untreated inorganic pollutants being toxic to fish and other aquatic species will end up killing all the fish in the ecosystem. Even though the oxygen concentration increases due to reduced organic matter and managed plant growth, the toxicity created due to inorganic pollutants would outweigh the benefits related to increased oxygen concentration, resulting in collapse of all the fish within the ecosystem and a severe alteration in ecosystem dynamics, as represented in Figure 8.

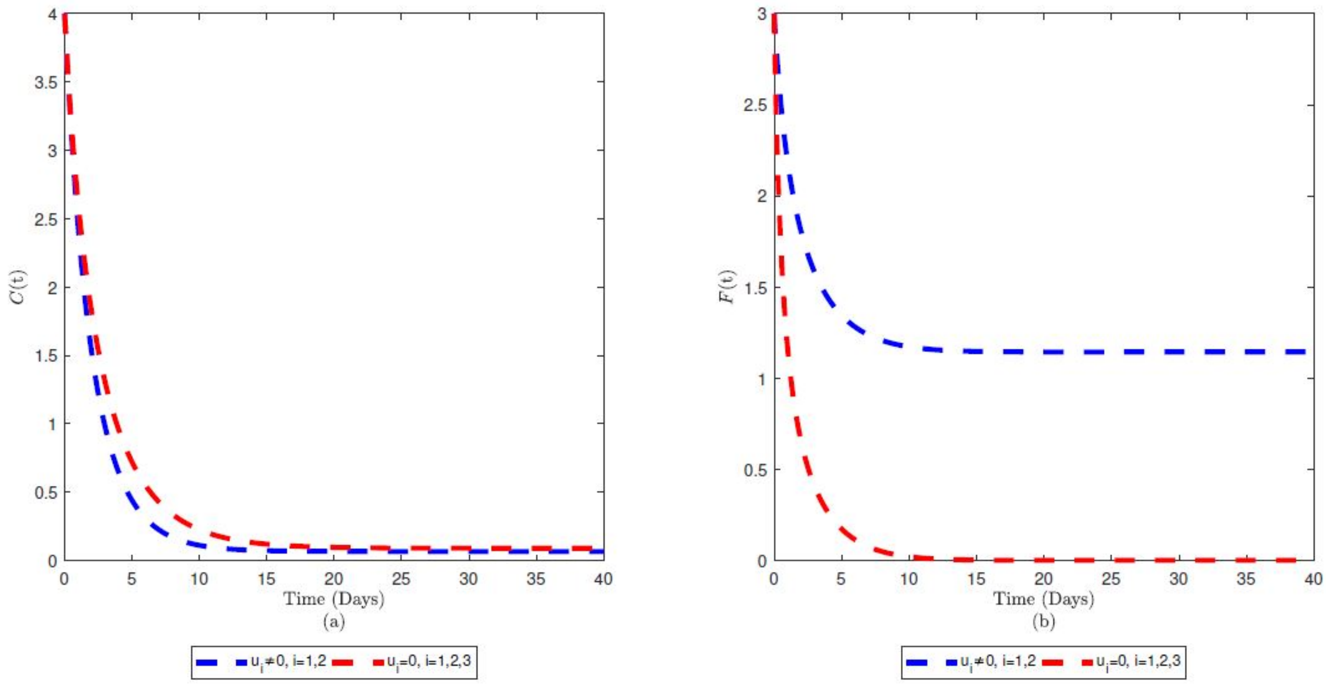

Fig 4. Impact of treatment of both organic and inorganic pollutants without controlling aquatic plants to level of oxygen concentration and fish population

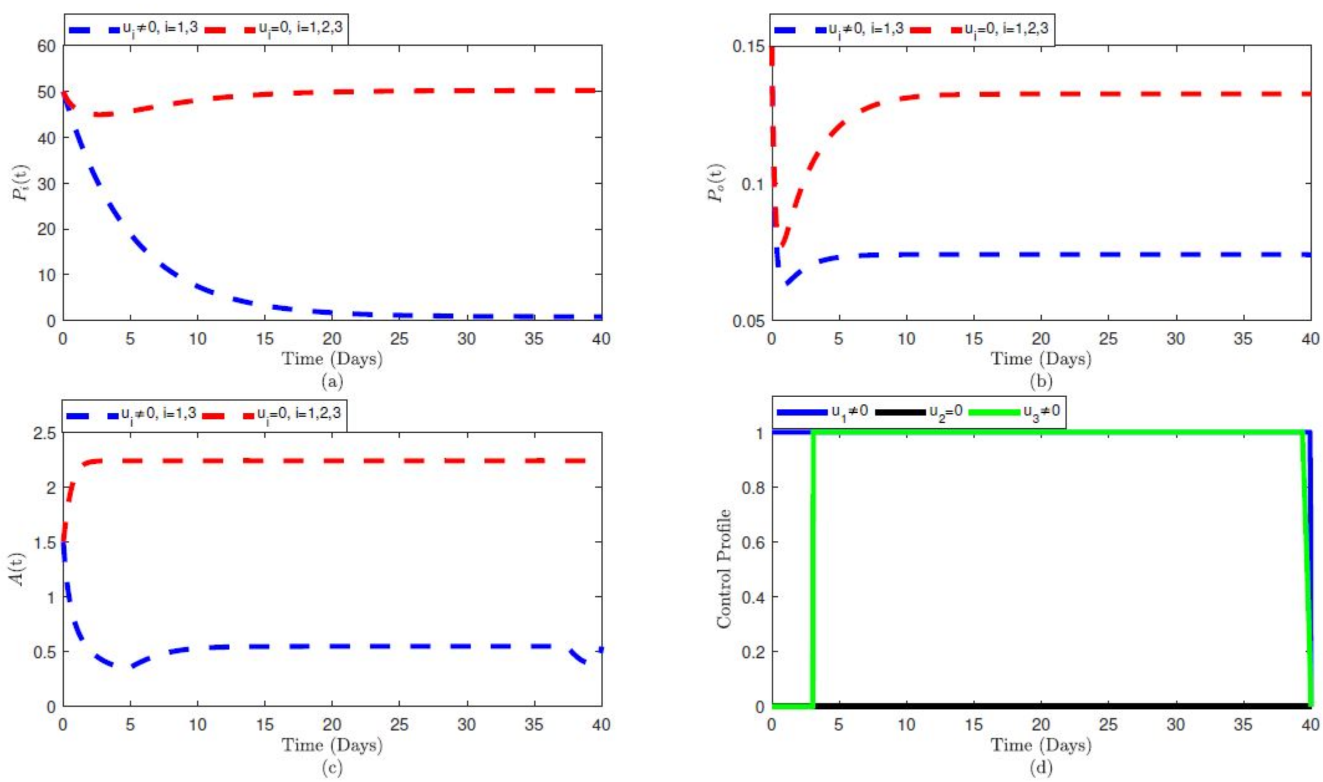

Fig 5. Treatment of inorganic pollutants and controlling aquatic plants

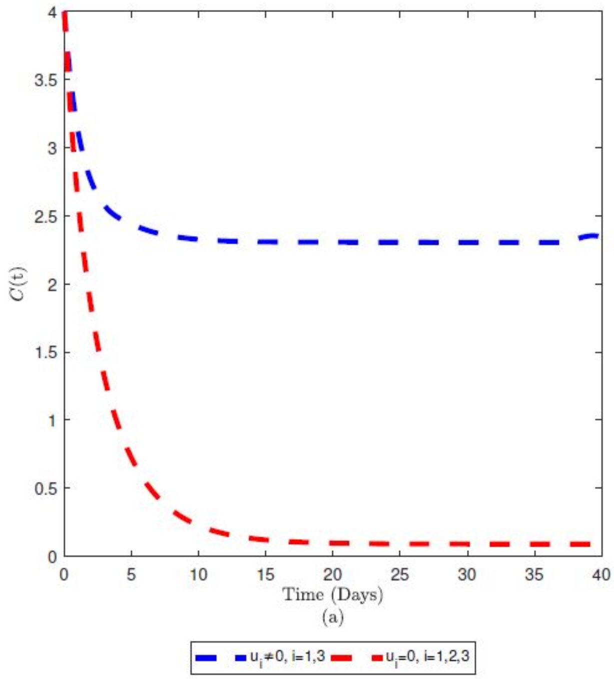

Fig 6. Impact of treatment of inorganic pollutants and controlling aquatic plants to level of oxygen concentration and fish population

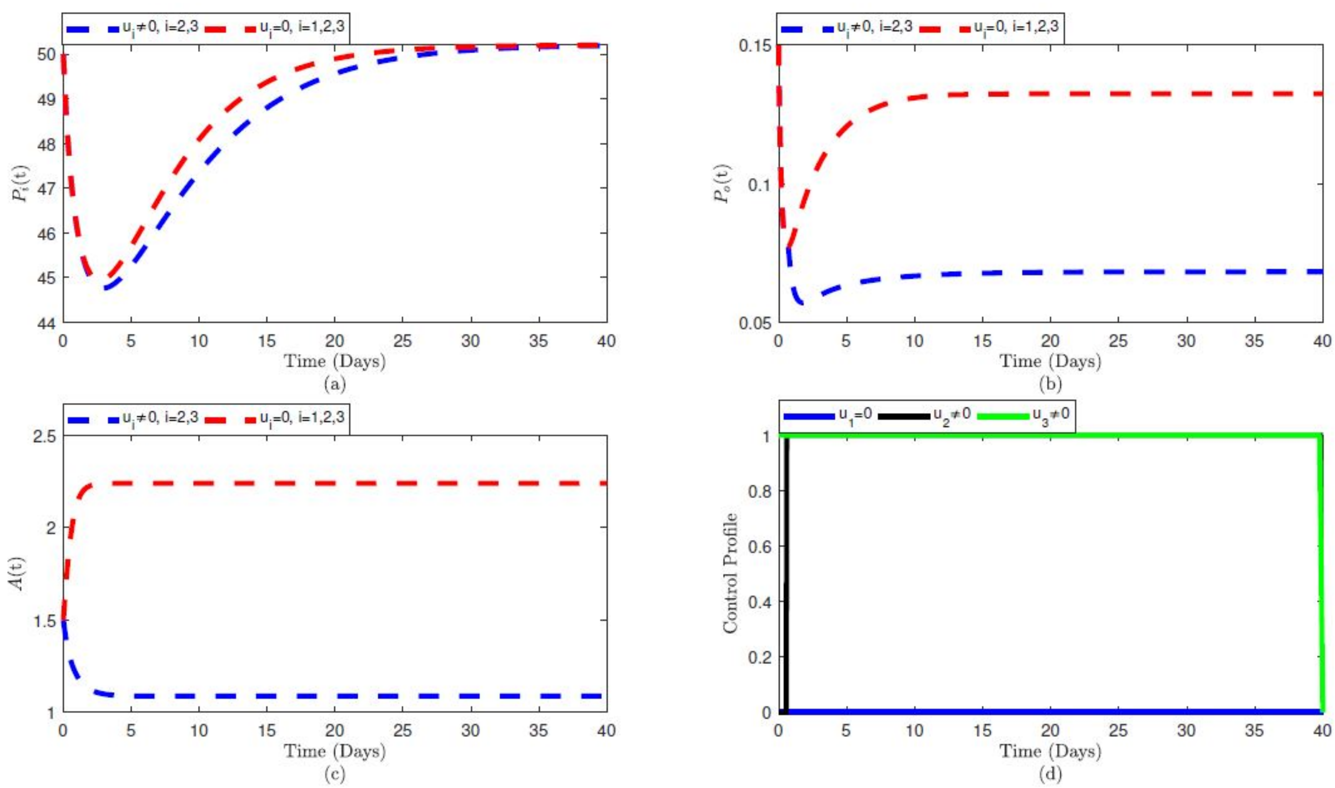

Fig 7. Treatment organic pollutants and controlling aquatic plants

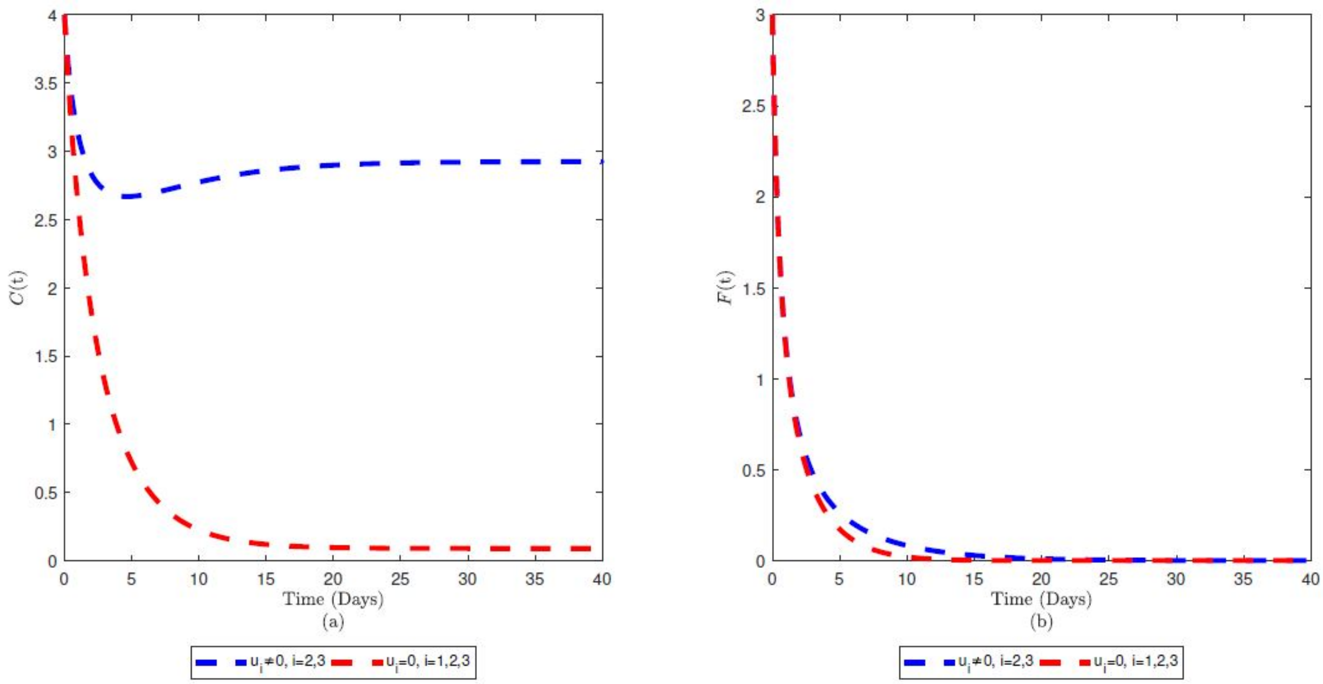

Fig 8. Impact of treatment organic pollutants and controlling aquatic plants to level of oxygen concentration and fish population

The results of this work show a significantly improved ecological outcome if it is applied compared to previous mathematical models on aquatic pollution dynamics. For example, most of the earlier works study the dynamics of either the impact of organic pollutants on oxygen depletion [15], toxicant impacts on fish survival [12], [15] or algal growth-induced eutrophication [14]. Although these provided some satisfactory insights, they didn't consider a common framework within which the joint and interactive pressures of organic pollutants, inorganic pollutants, and invasive aquatic plants can be dealt with. The model presented in this paper goes beyond these earlier works by embedding all three stressors into an optimal control framework and by calculating the ecological benefits of various combined intervention strategies.

In the model formulated by [14], organic pollutants were seen to drive oxygen depletion, but the study did not incorporate inorganic pollutants or plant-mediated feedback. Similarly, [13] analyzed algal bloom–oxygen interactions using Holling-Type III kinetics, but the work did not account for toxic inorganic pollutants or fish mortality through pollutant ingestion. These limitations resulted in restricted ecological prediction power of earlier models, particularly in environments where multiple pollutant types co-exist. The present study demonstrates that the exclusion of any pollutant class may lead to misleading outcomes. As an example, Strategy IV in our analysis depicts that while a 68% increase in dissolved oxygen is achieved, untreated inorganic pollutants yield 0% improvement in fish survival, thereby suggesting the inadequacy of models that consider only organic pollutants. Likewise, [13] and [15] formulated works related to the toxic effects of inorganic pollutants on fish populations. Yet, these models lacked any explicit representation of organic pollutant dynamics and also did not include aquatic plant interactions. These omissions prevented any synergistic or compensatory dynamics between pollutant types to be detected. In contrast, our model identifies the fact that organic pollutants, while harmful if in excess, are also a food source that could support fish recovery when controlled (Strategies I and III demonstrate an improvement of 62% and 57%, respectively, in fish populations). This dual role of organic pollutants is poorly highlighted in previous modelling studies.

A further novelty of this work consists of capturing the ecological consequences of plant-mediated eutrophication. Though several works such as [8] and [9] have discussed qualitatively how nutrient enrichment stimulates algal overgrowth, most of them do not embed the plant dynamics within the pollutant–oxygen–fish mathematical system. In fact, this study shows by explicit modelling of aquatic plants and their optimal control that aquatic plants will significantly affect the outcome: under Strategy II, where invasive plants are left unmanaged, there is 0% recovery in dissolved oxygen levels, despite the treatment of both kinds of pollutants. This quantitative insight underlines the importance of plant control, an aspect hardly explored mathematically in pollutant–ecosystem models.

Overall, the comparative performance of all four strategies underlines how superior ecological outcomes have been realized only when all three stressors are tackled simultaneously. Strategy I always outperforms others in oxygen recovery (60% increase) and fish population enhancement (62% increase), which shows that multi-component interventions create synergistic benefits impossible to achieve with partial strategies. This corresponds to ecological evidence from studies on eutrophication and invasive species management[7], [8], [11]; however, the current work is the first to mathematically optimize such combined interventions.

4 Conclusion

This shows that the application of optimal control theory provides a powerful, quantifiable, and verifiable framework for the management of organic and inorganic pollutants, including invasive aquatic plants. Among all the simulated strategies, Strategy I, involving integrated control, produced the most significant ecological improvements, with dissolved oxygen levels increasing by 60% and fish populations recovering by 62%, clearly demonstrating a strong synergistic effect when pollutants and invasive plants are managed simultaneously. These quantitative results reinforce the effectiveness of optimal control as a scientifically sound approach for designing balanced interventions that address both pollutant dynamics and biological imbalances caused by invasive species such as algae and water hyacinths. By jointly managing pollutants and invasive plants, the proposed control strategies help mitigate low oxygen levels, excessive algal growth, and toxic accumulation issues that are often addressed separately in previous studies. Consequently, this work bridges an important gap in aquatic ecosystem modeling by providing an integrated, multi-pollutant, and multi-species perspective based on quantifiable ecological gains. However, it is important to note that the quantitative outcomes and optimal strategy rankings may vary with different choices of the weighting parameters and cost coefficients. Nevertheless, the overall qualitative conclusions regarding the superiority of integrated control and the relevance of optimal control–based management remain robust, indicating that such approaches can offer substantial benefits for improving water quality and enhancing long-term ecological resilience in complex and dynamically evolving aquatic environments.

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