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Optimal Mixed Strategy Selection for Two-Person Zero-Sum Games under Interval-Fuzzy Payoff Uncertainty

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Abstract

Objectives: To overcome the shortcoming of being limited-time interval derivative based game theory in dealing with the issue of uncertain payoffs and to suggest an optimization-based model of deriving plausible forms of mixed strategies in zero-sum games between two players. **Methods**: A hybrid interval-fuzzy payoff representation is proposed to describe uncertainty in more impressive way. The value of standard norm-based bounds is computed on approximate game values. There is a formed strategy selection process employed that is based on optimization where the derived bounds are used to derive the upper and lower thresholds of mixed strategies (psup and pinf). These are realistic interval matrices and synthetic simulations. **Findings**: The suggested approach does not require solving linear programming problems and it has a competitive set of game values as compared to conventional approaches. The strategy on the 3×3 matrix games provided consistent approximate values $[2.666, 5.000]$. The algorithm is scalable and analyzable like bigger games for 10×10 matrix. **Novelty**: The paper presents the first interval-fuzzy payoff approach which maintains both the interval width and the fuzzy confidence at the same time in the zero-sum games. A bounding mechanism that is norm based is created to estimate game values without the use of linear programming, allowing analysis of large matrices on a scale. Theoretical results are now determined new to interval fuzzy saddle-point conditions, mixed-strategy interval convexity and mixed-strategy interval duality-gap control.

Keywords: Two-Person Zero-Sum Game; Interval-Fuzzy Matrix; Mixed Strategy; Norm-based Estimation; Saddle Point; Game Theory

1 Introduction

Two-player zero sum allows a basic structure upon which to model the adversarial decision-making in economics, defense and competitive markets[1]. In the classical method, payoff is known, and the players should be able to compute the saddle points or mixed strategies in the case of exact payoff[2]. Nevertheless, everyday situations seldom meet this assumption, as uncertainties include measurement errors, environmental variations as well as cognitive evaluation. This led to game theory increasingly generalizing above deterministic models to interval, fuzzy, probabilistic and hybrid models across research.

The approaches based on intervals form an initial outflow of a work on the processing of payoff uncertainty. In the case of interval-valued matrix games, Karmakar[3] suggested the use of linear programming methods and Dey and Zaman[4] suggested the use of robust optimization models when uncertainty is of an interval uncertainty type. In more recent times, Afreen and Bhurjee[5] presented a norm-based bounding approach to approximate the value of interval games without resolution equations. The contributions made created theoretical limits on uncertain games, but were limited to pure interval models, nearly invariably with no mechanism to extract strategic information readable off-book or to scale them to play very large games.

Fuzzy and probabilistic extensions have been added to accommodate imprecision in the payoffs. The model of fuzzy interval and matrix game solutions who combined the theory of fuzzy sets and the uncertainty[6]. Evidence base Consistency[7] generalized the concept to fuzzy rough matrix games, whereas Wu and Lisser[8] looked at uncertain parameter game-theoretic models with probabilistic formulations. It is these studies that contributed to the modeling space yet failed to come up with unified frameworks with intersecting ability at integrating interval and fuzzy semantics and could not systematically approach mixed-strategy interpretability.

The developments in optimization and algorithms have played an important role in computational practices of zero-sum games. The authors developed a formulation as a mixed-integer linear programming (MILP) model, under which one party makes action decisions on both sides given a budget constraint, which is more expensive than the heuristic strategies[9]. In recent studies, the force of optimizing two-person zero-sum games to a linear programming problem and obtaining optimal solution with simplex optimization was emphasized with higher results[10],[11]. In addition, Fey[12] applied set theory and ergodic theory to illustrate that certain statements in paradox games cannot be proven and use the technical limitations of the field underpinning standard research challenges.

Theoretical boundaries have also been pushed further by dynamic and hybrid formulations. Jimenez Leudo and Sanfelice[13] used hybrid dynamical equations and Hamilton-Jacoby-Bellman-Isaacs while assessing the values of a game without requiring solving the system. This research further implemented hybrid inclusion frameworks and obtained stability guarantees in the most optimal feedback schemes. In their study, Musthofa and Engwerda[14] derived conditions on feedback saddle points of dynamic games in the discrete time based on Riccati recursions and extended their findings to multi-player games.

Learning-based and risk sensitive frameworks have gained attention recently. Rajab and Shamma[15] suggested an optimization model of minimax discoveries of scenarios, which gives guaranteed probabilities of tolerance and convergence. A Markov game two-step minimax Q-learning algorithm formulated by Shreyas and Vijesh[16] established the convergence of algorithmic almost sure even in the conditions of the model free. These papers enhanced the strength of stochastic models though they never factored in integrating hybrid fuzzy-interval uncertainty and the fuzzy focus on understandable strategies.

Technical balances that include scalability and computability have been examined as well. Systematic exploration of code algorithms on extensive-form poker games was conducted by Navidi and Keshavarzi[17], which identified trade-offs of exploitability, utility and convergence. Wang and Chizat[18] proposed a finite particle method to compute Nash equilibria in high-dimensional nonmodular multistage strategy spaces expressed in terms of Wasserstein-Fisher-Rao gradient flows. The input of these people highlighted the computational complexity of the scaling of the zero-sum game models.

Three big gaps are created out of this wealth of literature. The first is that interval-only models fail to represent the complex uncertainty structure emerged by fuzzy semantics. Second, the current implementations are highly dependent on linear programming solvers, so that scaling and interpretability cannot be achieved in broader games. Third, attributable mixed strategies are not often presented by fuzzy and probabilistic extensions[19], thus reducing the degree to which they might be useful as a decision-support tool.

Rough-interval game models have been investigated to reflect either of two types of uncertainty, in which there is both interval variability and boundary vagueness. Research on rough interval payoffs offered valuable extensions to models that can be applied in solving telecommunication market share problems but have not been found to produce sensible mixed strategies because they are based on rough approximations of boundary models[20]. Picture fuzzy models go a step further and introduce positive, neutral and negative degrees at the same time. A non-linear mathematical framework of picture fuzzy payoff matrix games showed better representational richness, particularly in a cyberterrorism risk analysis context, but the multi-dimensional membership structure raises the cost of computation and makes strategy derivations more complex[21].

Neutrosophic game models, such as those based on single-valued trapezoidal neutrosophic numbers, have elements of truth-membership, indeterminacy and falsity[22]. These approaches permit much richer characterization of imprecise payoffs but involve complicated reduction or score functions which reduce the uncertainty to a single representative value, leading to loss of interval width. Hesitant fuzzy matrix games are another approach that can be used by providing a number of possible membership values of each payoff, which could represent the phenomenon of hesitation. The necessary aggregation or score-based selection, however, may have a high cost of computation. The difficulty in scaling the method to large payoff matrices [23].

The dense fuzzy environment models are generalizations of fuzzy sets, which take into account continuous dense membership distributions. Scenario domains in which matrix games are developed based on dense fuzzy payoffs or dense fuzzy bimatrix structures have been used in natural disaster management [24]. These models are able to reflect small nuances of uncertainty but generally demand complex integration steps and do not produce closed strategy limits. Type-2 intuitionistic fuzzy matrix games add a more significant uncertainty framework. It includes embedded membership functions and distance-based assessments. The expressiveness of these models is exhibited by applications like biogas plant implementation, which require large amounts of time and minimize interpretability. Equally, more recent defuzzification schemes of type-2 fuzzy variables, such as the one used to assess plastic-ban policy, provide alternative reduction schemes, but reduce multidimensional uncertainty to a scalar result, with no strategic limit [25].

In all these approaches, a unifying drawback arises: most approaches either (i) attempt to focus on linguistic or higher order uncertainty, to the detriment of their computational tractability (ii) employ defuzzification and reduction functions that drive out interval width and provide a single crisp value. These models do not offer a unified mechanism of both interval bounds and confidence semantics, as well as providing interpretable and norm-based mixed strategy bounds, without the use of linear programming. The main reason why these limitations are the basis of the interval-fuzzy norm-based framework is developed in this research.

Although interval-valued, fuzzy-valued, probabilistic and hybrid game-theoretic models have been developed extensively, there are several inadequacies in the current literature in the field. First, interval-based formulations can represent limited uncertainty but are not able to reflect semantic or belief-oriented uncertainty which commonly occurs in expert-based or partially observed environment. On the other hand, fuzzy-valued equations use linguistic confidence but eliminate the interval width. It generates solutions that decay uncertainty to unique defuzzified numbers. A fundamental modeling gap to the real-world decision environment is due to the absence of a single structure that can represent two modeling properties at the same time: interval boundaries and confidence semantics.

Second, most of the currently existing interval or fuzzy matrix-game models depend on either the use of linear programming or nonlinear optimization to calculate the values and optimal strategies of games. This type of solver-based method has scalability issues in large games. It is also poor at giving an insight into the effect of uncertainty on mixed-strategy distributions. Their indecipherability and complexity of computation make them less appropriate to high-dimensional structures of payoff.

Third, very few of the existing extensions of zero-sum games under uncertainty generate explainable mixed-strategy bounds. Defuzzification, ranking and hesitant fuzzy aggregation methods usually give one representative value of payoff without the derivation of transparent upper-lower probability distributions of strategies. This limits empirical application in economic, military and risk sensitive applications where limited strategic advice is necessary.

These gaps are filled in this paper with the introduction of an interval-fuzzy combination of payoffs representations with normal bounding methods to keep values in games approximated for non-solver linear programming. It introduces a new algorithm that is referring to Fuzzy Interval Weighted Average Estimate (FIWAE) so that it provides single-point estimates of the game value, which are highly explainable. The theoretical support of the framework is the new theorems, corollaries and convexity results and the empirical examples of the new framework provide examples of scalability and interpretation (synthetic games, benchmark games). This way, the given study contributes to the state of the art in the modeling and solution of uncertain two-player zero-sum games.

The main contribution of the research is that a general interval-fuzzy strategy selection framework of two-person zero-sum games is developed. The model combines interval uncertainty with fuzzy confidence semantics and presents a norm-based approach to calculating strategy bounds with no linear programming model solution. This allows scaling and analysis of uncertain games. It is also shown in the study that new theoretical outcomes exist, such as interval-fuzzy saddle-point existence, convexity of the feasible mixed-strategy region and a duality-gap bound, which measures the accuracy of the approximation. Moreover, a mechanism that is suggested in the study, the FIWAE, generates a risk-conscious, single-point estimation of the game value, but is consistent with the uncertainty envelope. These contributions offer a big leap forward in comparison with the current interval only, fuzzy only, type 2 and hesitant fuzzy game models.

2 Methodology

Let $\mathcal{G} = (S_1, S_2, \mathcal{M})$ denote a two-person zero-sum game, where $S_1 = \{\alpha_1, \dots, \alpha_m\}$ and $S_2 = \{\beta_1, \dots, \beta_n\}$ denote the pure strategy sets for Player I and Player II respectively and $\mathcal{M} \in \mathcal{IF}^{m \times n}$ denotes the interval-fuzzy payoff matrix. This section constructs the analytical foundation for such games, defines novel optimality bounds and develops closed-form theorems for saddle-point existence and optimal mixed strategy estimation.

2.1 Preliminaries and Notation

Let $I \subset R \times R$ be the space of closed real intervals. Each element $[a,b] \in I$ satisfies $a \leq b$, representing imprecise but bounded quantities. Further, let $\widetilde{m_{ij}} \in \mathcal{IF}$ denote an interval-fuzzy number, i.e.,

$$\widetilde{m_{ij}} = \left(\left[a_{ij}, b_{ij}\right], \mu_{ij}\right)$$

where $\mu_{ij} \in [0,1]$ is a possibility membership function attached to the interval $\left[a_{ij}, b_{ij}\right]$. The set of all such numbers is denoted $\mathcal{IF} = I \times [0,1]$. The matrix $\mathcal{M} = \left[\widetilde{m_{ij}}\right] \in \mathcal{IF}^{m \times n}$ generalizes the classical crisp payoff matrix.

2.2 Payoff Evaluation Under Interval-Fuzzy Uncertainty

The evaluation of expected payoffs under interval-fuzzy uncertainty requires extending scalar products to handle intervals. Let $p = (p_1, \ldots, p_m) \in \mathcal{P}_m$ and $q = (q_1, \ldots, q_n) \in \mathcal{P}_n$ be probability vectors (mixed strategies) in the standard simplex:

$\mathcal{P}_{k} = \left\{x \in R^k \mid \sum_{i=1}^{k} x_i = 1,\ x_i \geq 0\right\}$.

We define the interval expected payoff $E_{IF}(p,q)$ as:

$$E_{IF}(p,q) = \left(\left[\sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j a_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j b_{ij}\right], \mu(p,q)\right)$$

where,

$$\mu(p,q) = \min_{i,j} \mu_{ij} \text{ such that } p_i q_j > 0$$

This captures the least-confidence among all activated payoffs.

2.3 Norm-Based Bounding of Game Value

To avoid solving linear or nonlinear systems, we employ a norm-based bounding scheme. Let $|\cdot|_{I1}\ and\ |\cdot|_{I\infty}$ be matrix norms adapted for interval matrices.

Let $\mathcal{M}_{\mathcal{L}} = \left[a_{ij}\right]$ and $\mathcal{M}_{\mathcal{U}} = \left[b_{ij}\right]$ be the lower and upper bound matrices of $\mathcal{M}$. Define:

$$|\mathcal{M}|_{I1} := \max_{1 \leq j \leq n} \sum_{i=1}^{m} \max\left\{\left|a_{ij}\right|, \left|b_{ij}\right|\right\}$$

$$|\mathcal{M}|_{I\infty} := \max_{1 \leq i\ \leq m} \sum_{j=1}^{n} \max\left|a_{ij}\right|, \left|b_{ij}\right|\}$$

These yield a rough over-approximation of the scalar value function. Using this, we define the estimated game value interval:

$$\nu(\mathcal{M}) \in \left[\frac{k}{|\mathcal{M}|_{I\infty}}, |\mathcal{M}|_{I1}\right]$$

where

$$k = \inf_{p,q \in \mathcal{P}_m \times \mathcal{P}_n} \left\{\sum_{i,j} p_i q_j \left|a_{ij}\right|\right\}$$

The bounds refine existing works by adding fuzzy-adjusted weights.

2.4 Saddle Point Characterization under Interval-Fuzzy Conditions

We now derive conditions under which a pure strategy saddle point exists for the interval-fuzzy matrix game.

Definition 2.1 (Interval-Fuzzy Saddle Point):

Let $(\alpha_i, \beta_j) \in S_1 \times S_2$. The strategy pair is an interval-fuzzy saddle point if:

$$\forall i' \in [1,m],\ [a_{i'j}, b_{i'j}] \succeq [a_{ij}, b_{ij}], \quad \text{and} \quad \forall j' \in [1,n],\ [a_{ij'}, b_{ij'}] \preceq [a_{ij}, b_{ij}]$$

where the partial ordering on intervals is defined as:

$$[a,b] \preceq [c,d] \Longleftrightarrow a \leq c \text{ and } b \leq d$$

This generalizes the classical von Neumann saddle point condition.

Theorem 2.1 (Existence Condition for Saddle Point)

Let $\mathcal{M} \in \mathcal{IF}^{m\times n}$. If there exists (i^*, j^*) such that:

$$\forall i,\ a_{ij^*} \geq a_{i^*j^*}, \quad \forall j,\ b_{i^*j} \leq b_{i^*j^*}$$

then $(\alpha_{i^*}, \beta_{j^*})$ is a saddle point in the sense of minimal interval uncertainty and

$$\nu = ([a_{i^*j^*}, b_{i^*j^*}], \mu_{i^*j^*})$$

is the game value.

Proof:

Follows directly from bounding the row-minimums and column-maximums under the defined partial ordering. Since all elements in the same column are bounded below by $a_{i^*j^*}$ and all in the same row bounded above by $b_{i^*j^*}$, no unilateral deviation can improve payoff under worst-case logic.

2.5 Mixed Strategy Optimization Bounds

Let $\mathcal{G} = (S_1, S_2, \mathcal{M})$ be the interval-fuzzy zero-sum game as defined. When a pure-strategy saddle point does not exist (as characterized in Theorem 2.1), a mixed strategy pair $(p^*, q^*) \in \mathcal{P}_m \times \mathcal{P}_n$ must be used. This section presents a new framework to bound such strategies using matrix norms.

2.5.1. Strategy Space Definitions

Let the payoff matrix $\mathcal{M} = [\widetilde{m_{ij}}] = [([a_{ij}, b_{ij}], \mu_{ij})]$ be split into two crisp matrices:

- $M_L = [a_{ij}] \in R^{m\times n}$ – the lower matrix
- $M_U = [b_{ij}] \in R^{m\times n}$ – the upper matrix

Define:

- $\nu_L := \min_{p\in\mathcal{P}_m, q\in\mathcal{P}_n} p^T M_L q$
- $\nu_U := \max_{p\in\mathcal{P}_m, q\in\mathcal{P}_n} p^T M_U q$

Thus, the actual game value lies within $[\nu_L, \nu_U]$.

2.5.2. Theorem: Norm-Based Bounds for Mixed Strategies

Theorem 2.2 (Interval Norm Bounds for Strategy Set):

Let $M_L, M_U \in R^{m\times n}$ be the lower and upper payoff matrices (refer section 2.3). Let $\nu \in [\nu_L, \nu_U]$ be the interval game value. Then the upper and lower bounds for mixed strategies of Player I satisfy:

$$p_{\text{sup}} \geq \sup\left\{1 - \frac{\nu_U}{|M|_{I1}}, \frac{\nu_L}{|M'|_{I1}}\right\}$$

$$p_{\text{inf}} \leq \inf\left\{1 - \frac{\nu_L}{|M''|_{I1}}, \frac{\nu_U}{|M|_{I1}}\right\}$$

where:

- M' is any column-wise generated matrix (i.e., M with one column removed),
- M'' is any row-wise generated matrix (i.e., M with one row removed).

Proof:

The idea is to evaluate the payoff distribution over the support of the strategies by comparing the marginal contributions. Let $\nu = p^T Mq$. Suppose Player I allocates weight p_i on row i, the worst-case contribution is:

$$\sum_j p_i q_j \cdot a_{ij} \leq \nu_L, \quad \text{for all } i$$

Since ν_L is a linear combination over all rows, any p_i should not exceed the ratio $\frac{\nu_L}{\text{row sum}}$. Hence:

$$p_i \leq \frac{\nu_L}{\sum_j |a_{ij}|} \leq \frac{\nu_L}{|M|_{I1}}$$

Similarly, since $\sum_i p_i = 1$, removing any row forces:

$$p_{\text{sup}} \geq 1 - \frac{\nu_U}{|M|_{I1}}$$

$$p_{\text{inf}} \leq \frac{\nu_U}{|M|_{I1}}$$

This gives the bounds above.

2.5.3. Corollary: Fuzzy-Aware Strategy Bounds

Let $\mu_{ij} \in [0,1]$ be the fuzzy confidence for each m_{ij}. Define the fuzzy-lowered norm:

$$|M|_{\mu,I1} := \max_j \sum_i \mu_{ij} \cdot |a_{ij}|$$

$$|M|_{\mu,I\infty} := \max_i \sum_j \mu_{ij} \cdot |a_{ij}|$$

Then the bounds improve to:

$$p_{\text{sup}} \geq 1 - \frac{\nu_U}{|M|_{\mu,I1}}$$

$$p_{\text{inf}} \leq \frac{\nu_U}{|M|_{\mu,I1}}$$

This incorporates semantic trustworthiness of the entries.

2.5.4. Theorem: Convexity of Strategy Bounds

Theorem 2.3 (Convexity of Strategy Interval):

The set $\mathcal{S}_p := \{p \in \mathcal{P}_m \mid p_{\text{inf}} \leq p_i \leq p_{\text{sup}}, \forall i\}$ forms a convex polytope within the simplex $\mathcal{P}_m$.

Proof:

The constraints are linear inequalities on each p_i, intersected with the linear equality $\sum_i p_i = 1$ and non-negativity $p_i \geq 0$. Hence, $\mathcal{S}_p$ is convex.

This allows applying Lagrangian-based constrained optimization within $\mathcal{S}_p$ to refine strategies further.

2.6 Game Value Approximation Without Solving LP

In classical game theory, the value v of a two-person zero-sum game is obtained by solving a pair of linear programs or a saddle-point formulation. Under interval-fuzzy uncertainty, such approaches become ambiguous or intractable due to undefined scalar arithmetic on fuzzy quantities. This section proposes a non-LP-based approach to approximate v by directly using interval norms and support estimates derived from payoff uncertainty.

2.6.1. Definition: Interval-Fuzzy Game Value Envelope

Let $\mathcal{M} = [\widetilde{m_{ij}}] = [([a_{ij}, b_{ij}], \mu_{ij})] \in \mathcal{IF}^{m\times n}$. Define the following scalar matrices:

- $M_L = [a_{ij}]$
- $M_U = [b_{ij}]$
- $M_C = \left[\frac{a_{ij}+b_{ij}}{2}\right]$ – Center matrix

Let $\sigma_{ij} = \frac{b_{ij}-a_{ij}}{2}$ be the radius of each interval cell.

Then the game value envelope is defined as:

$$\mathcal{V}(\mathcal{M}) := [\underline{v},\, \overline{v}] = \left[\min_{p,q} p^T M_L q,\ \max_{p,q} p^T M_U q\right]$$

This envelope bounds the unknown true game value $v \in \mathcal{V}(\mathcal{M})$.

2.6.2. Theorem: Norm-Based Envelope Bounds

Theorem 2.4 (Interval Game Value Bounds):

Let $\mathcal{M} \in \mathcal{IF}^{m\times n}$. Define:

- $|M_L|_1 = \max_j \sum_i |a_{ij}|$
- $|M_U|_\infty = \max_i \sum_j |b_{ij}|$

Then the value of the game $v \in [\underline{v}, \overline{v}]$ satisfies:

$\underline{v} \geq \frac{1}{n} \min_i \sum_j a_{ij}, \quad \overline{v} \leq \frac{1}{m} \max_j \sum_i b_{ij}.$

Proof:

Since $p, q \in \mathcal{P}_m, \mathcal{P}_n$, their components are non-negative and sum to 1. The minimum value over all i-th rows of M_L (under uniform q) bounds the lowest expected payoff:

$$p^T M_L q \geq \min_i \sum_j a_{ij} \cdot \frac{1}{n} = \frac{1}{n} \min_i \sum_j a_{ij}$$

Similarly, the maximum occurs at the most favorable row-column pair under M_U, bounded by:

$$p^T M_U q \leq \frac{1}{m} \max_j \sum_i b_{ij}$$

Thus, the bounds hold.

2.6.3. Definition: Approximate Game Value via Center Matrix

Let $M_C = \left[\frac{a_{ij}+b_{ij}}{2}\right]$. Then define the center-mean value:

$$v_C := \max_{p\in\mathcal{P}_m} \min_{q\in\mathcal{P}_n} p^T M_C q = \min_q \max_p p^T M_C q$$

which approximates the most likely value of the game under symmetric uncertainty.

2.6.4. Theorem: Game Value under Possibility Confidence Threshold

Theorem 2.5 (Confidence-Weighted Game Value):

For a fixed confidence level $\lambda \in [0,1]$, define the restricted matrix:

$$\mathcal{M}^{(\lambda)} = \left\{ \widetilde{m_{ij}} \in \mathcal{M} \mid \mu_{ij} \geq \lambda \right\}$$

Let $M_C^{(\lambda)}$ be its corresponding center matrix. Then the restricted game value $v^{(\lambda)}$ satisfies:

$$v^{(\lambda)} = \text{Value of game with matrix } M_C^{(\lambda)} \in [\underline{v}, \overline{v}]$$

with $v^{(\lambda)} \to v_C \; as \; \lambda \to \min_{i,j} \mu_{ij}$

Proof Sketch:

The confidence filter eliminates payoff entries with poor possibility scores, causing the resulting strategy space to narrow. The center-mean value on this reduced matrix defines a lower-confidence version of v_C, bounded within the interval game value. As λ reduces to include all entries, $v^{(\lambda)} \to v_C$.

2.6.5. Fuzzy Interval Weighted Average Estimate (FIWAE)

We now define a novel approximation function using all fuzzy-confidence weights:

$$v_{FIWAE} := \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \cdot \frac{a_{ij} + b_{ij}}{2}$$

where $w_{ij} = \frac{\mu_{ij}}{\sum_{i,j} \mu_{ij}}$

This represents a normalized fuzzy-weighted average and serves as a practical single-point estimate of the interval-fuzzy game value. It is easy to compute and aligned with expected value logic.

2.6.6. Corollary: Envelope Tightening Criterion

Let $\delta_{ij} := b_{ij} - a_{ij}$ and define:

$\Delta = \max_{i,j} \delta_{ij}, \quad \delta_{\text{mean}} = \frac{1}{mn} \sum_{i,j} \delta_{ij}$.

Then if $\Delta \to 0$, the following holds:

$$v_C \to v, \quad v_{FIWAE} \to v, \quad \mathcal{V}(\mathcal{M}) \to [v, v]$$

i.e., all approximations converge to the exact scalar value as fuzziness diminishes.

2.6.7. Algorithm for Game Value Approximation

Given an interval-fuzzy matrix $\mathcal{M}$, the steps for approximating game value without solving LP are:

1. Construct M_L, M_U, M_C
2. Compute bounds $\underline{v}, \overline{v}$ from Theorem 2.4
3. Compute $v_C =$ value of M_C using LP-free maximin/minimax expressions
4. Filter $\mathcal{M}^{(\lambda)}$ for desired fuzzy-confidence level λ
5. Compute fuzzy-weighted average v_{FIWAE}
6. Report envelope $[\underline{v}, \overline{v}]$ and approximations $v_C, v_{FIWAE}, v^{(\lambda)}$

Corollary 2.1 (Existence of Fuzzy-Bounded Game Value Interval)

Let $\mathcal{G} = (S_1, S_2, \mathcal{M})$ be a two-person zero-sum game where $\mathcal{M} = \left[\left(\left[a_{ij}, b_{ij} \right], \mu_{ij} \right) \right]$ is the interval-fuzzy payoff matrix. Let $M_L = \left[a_{ij} \right], M_U = \left[b_{ij} \right]$ and define:

$$V_L := \min_{p \in \mathcal{P}_m} \max_{q \in \mathcal{P}_n} p^T M_L q, \quad V_U := \max_{p \in \mathcal{P}_m} \min_{q \in \mathcal{P}_n} p^T M_U q$$

Then the actual game value V^* under the fuzzy-interval model satisfies:

$$V^* \in \left[\int_{\lambda}^{1} V_L \, d\lambda, \int_{\lambda}^{1} V_U \, d\lambda \right] \subseteq [V_L, V_U]$$

where $\lambda \in [0,1]$ is the fuzzy confidence threshold. The bounds narrow as $\lambda \to 1$ and converge to a crisp saddle value when all $\mu_{ij} = 1$.

Proof Sketch:

As fuzzy confidence $\mu_{ij} \to 1$, the payoff matrix reduces to a standard interval matrix. The fuzzy integrals shrink the uncertainty envelope due to exclusion of low-confidence terms. Since V_L and V_U already envelope the possible outcomes, integrating only the credible segment results in a narrower bound.

Claim 2.1 (FIWAE Minimizes Fuzzy-Aware Risk)

Let $\mathcal{G} = (S_1, S_2, \mathcal{M})$ be an interval-fuzzy zero-sum game. Let $E_\mu[M]$ denote the expected fuzzy-interval payoff matrix using:

$$E_\mu\left[m_{ij}\right] = \mu_{ij} \cdot \frac{a_{ij} + b_{ij}}{2}$$

Let $V_{\text{FIWAE}} := \min_p \max_q p^T E_\mu[M] q$. Then, among all weighted averaging schemes of the interval matrix, FIWAE (Fuzzy Integrated Weighted Average Estimate) minimizes the following fuzzy risk functional:

$$\mathcal{R}(V) := \sum_{i,j} \mu_{ij} \cdot \left| V - \frac{a_{ij} + b_{ij}}{2} \right|^2$$

subject to $V \in [V_L, V_U]$ and $\sum_i p_i = 1, \sum_j q_j = 1$.

Justification:

This statement holds that by applying an estimate of the payoff with a weighted center based on the weight-value, namely, as the payoff matrix, the fuzzy deviation mean square with respect to weighted center would reach its minimal values, which are penalized by confidence. Basically, FIWAE is optimal when the loss is quadratic with weighted belief strength and, therefore, it is very appropriate with robust minimax approximation in the fuzzy uncertainty.

The following outcome is another in the form of a lemma and a proposition which is aimed at enhancing the theoretical base of the suggested model of fuzzy-interval games. These dwell on saddle-point characterization and duality gap of the uncertain game setting.

Lemma 2.1 (Fuzzy Saddle Point Preservation under Interval Core Reduction)

Let $\mathcal{G} = (S_1, S_2, \mathcal{M})$ be a two-person zero-sum game with interval-fuzzy payoffs $\mathcal{M} = \left[\left(\left[a_{ij}, b_{ij}\right], \mu_{ij}\right)\right]$. Let $\mathcal{M}_{\text{cr}}$ be the fuzzy-credibility core submatrix, defined as:

$$\mathcal{M}_{\text{cr}} = \left\{(i,j) \in [1,m] \times [1,n] \,\middle|\, \mu_{ij} \geq \lambda_0\right\}$$

for some threshold $\lambda_0 \in (0,1]$. If the fuzzy-interval matrix $\mathcal{M}$ admits a saddle point in the classical sense within $\mathcal{M}_{\text{cr}}$, then that saddle point is preserved in the full game $\mathcal{G}$ under FIWAE reduction, provided that the remaining payoff cells contribute zero gradient to the risk functional.

Proof Outline:

- The saddle point condition requires that the chosen row and column indices form a max-min and min-max equality in the payoff space.
- If $\mu_{ij} < \lambda_0$, then under FIWAE, these elements are scaled down and contribute negligibly to the expected payoff.
- Hence, their impact on optimal strategy selection diminishes.
- When the remaining gradient of those cells is zero or negligible, the solution derived from $\mathcal{M}_{\text{cr}}$ remains valid globally.

Proposition 2.1 (Upper Bound on Duality Gap under FIWAE Model)

Let $V_{\text{minimax}} := \min_{p \in \mathcal{P}_m} \max_{q \in \mathcal{P}_n} p^T E_\mu[M] q \; and \; V_{\text{maximin}} := \max_{q \in \mathcal{P}_n} \min_{p \in \mathcal{P}_m} p^T E_\mu[M] q$, where $E_\mu[M]$ is the FIWAE-aggregated payoff matrix.

Then the duality gap $\delta := V_{\text{minimax}} - V_{\text{maximin}}$ under fuzzy-integrated approximation satisfies:

$$\delta \leq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - \mu_{ij}) \cdot |b_{ij} - a_{ij}|$$

That is, the gap is upper bounded by the average uncertainty-weighted spread of the interval payoffs.

Implication:

This proposition demonstrates that when belief degrees μ_{ij} approach 1, the duality gap collapses to zero, restoring classical LP-based zero-sum game equilibrium. Conversely, when uncertainty is high, approximation introduces bounded deviation but quantifiable and explainable using confidence scores and interval widths.

2.6.7.1. Optimal solution and game value under interval–fuzzy payoffs. Consider a two-person zero-sum game with interval–fuzzy payoff matrix $M \in R^{m\times n}$. The mixed-strategy sets for Player I and Player II are given by the simplices

$$\Delta_m = \left\{ x \in R^m \mid x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\}, \quad \Delta_n = \left\{ y \in R^n \mid y_j \geq 0, \sum_{j=1}^{n} y_j = 1 \right\}$$

Let M_L and M_U denote the lower and upper payoff matrices associated with (M). For any mixed-strategy pair $(x,y) \in \Delta_m \times \Delta_n$, define the lower and upper expected payoffs

$$\underline{v}(x,y) = x^\top M_L y, \quad \overline{v}(x,y) = x^\top M_U y$$

The classical lower and upper values of the game are then defined as

$$v_L = \sup_{x\in\Delta_m} \inf_{y\in\Delta_n} \underline{v}(x,y), v_U = \inf_{y\in\Delta_n} \sup_{x\in\Delta_m} \overline{v}(x,y),$$

which produce the interval game value

$$\mathcal{V} = [v_L, v_U].$$

An optimal mixed-strategy pair in the interval–fuzzy sense is any pair $(x^\star, y^\star) \in \Delta_m \times \Delta_n$ satisfying

$$\underline{v}(x^\star, y) \geq v_L \quad \text{for all } y \in \Delta_n, \ \overline{v}(x, y^\star) \leq v_U \quad \text{for all } x \in \Delta_m.$$

Such a pair ensures that Player I cannot guarantee a lower payoff than v_L and Player II cannot enforce a higher payoff than v_U under the worst-case interpretation of interval payoffs.

In the interval–fuzzy method, the FIWAE generates an aggregated crisp payoff matrix M^{FIWAE} by combining midpoint information and fuzzy confidence weights. The corresponding FIWAE game value is given by

$$v_{\text{FIWAE}} = \max_{x\in\Delta_m} \min_{y\in\Delta_n} x^\top M^{\text{FIWAE}} y,$$

subject to the classical minimax formulation. The scalar value v_{FIWAE} lies inside the envelope $[v_L, v_U]$ and acts as an interpretable, risk-aware representative of the interval–fuzzy game value.

2.6.7.2. Nash equilibrium considerations in interval–fuzzy games. For finite two-person zero-sum matrix games with crisp payoffs, the minimax theorem guarantees the existence of a game value (v) and at least one pair of optimal mixed strategies $(x^\star, y^\star)$, which forms a Nash equilibrium in mixed strategies. The interval–fuzzy formulation considered in the present study generalizes the payoff space from scalars to interval–fuzzy quantities. Hence, standard equilibrium concepts require careful adaptation.

The interval game value $\mathcal{V} = [v_L, v_U]$ defined above describes a band of admissible payoffs rather than a single scalar. Under this setting, a natural equilibrium notion involves an interval–fuzzy saddle point: a pair $(x^\star, y^\star)$ such that no unilateral deviation can improve the payoff under the partial order on intervals and under the associated fuzzy-confidence structure. Existence of such saddle points can be established under the sufficient conditions provided by the norm-based bounds and the ordering assumptions on M_L and M_U. These conditions guarantee stability of the game value within the interval $\mathcal{V}$. It does not imply a universal Nash existence theorem for all possible interval–fuzzy matrices.

A complementary perspective arises from the FIWAE-aggregated game with payoff matrix M^{FIWAE}. The aggregated game constitutes a standard finite zero-sum matrix game. The minimax theorem therefore ensures existence of at least one mixed-strategy Nash equilibrium $(x^{\text{FIWAE}}, y^{\text{FIWAE}})$ and a well-defined scalar value v_{FIWAE}. The proposed method thus guarantees Nash existence for the FIWAE-aggregated representation. It provides convex strategy bounds around such equilibrium strategies through the norm-based inequalities.

It would need more topological and ordering conditions and would be beyond the scope of the present contribution, to prove a general Nash existence theorem on the entire interval fuzzy payoff space. This current work, rather, concentrates on (i) strict definition of optimal solutions, and game values in terms of interval and fuzzy, (ii) generating of adequate conditions of interval and fuzzy saddle-point existence and (iii) giving of an LP-free, FIWAE-based aggregated game where classical Nash equilibrium existence may be ensured.

2.6.7.3. Crisp Conversion in Interval–Fuzzy Payoff Processing. The interval-fuzzy payoff matrix has two types of uncertainty: (i) interval width which is a limited numerical imprecision, and (ii) fuzzy membership which is subjective confidence. Any solution process of mixed strategies involves operations of scalar values, e.g. minimax comparisons, row and column dominance checks, norm calculations, which are unable to be directly computed on interval-fuzzy objects without going into contradictions in ranking or breaking fundamental algebraic rules. This is the reason why a controlled transformation into crisp quantities is required so that optimization can proceed along with the necessary form of uncertainty.

The proposed conversion does this conversion in a very limited and information preserving way. The method defuzzifies not the individual payoffs like in a single point but derives lower and upper interval matrices $(M_L M_U)$ and then builds norm-based envelopes and fuzzy membership is used solely as a weighting element in FIWAE. This makes sure that propagating the payoff interval as well as the confidence attached to it affects the strategy boundaries that are obtained. The conversion does not, however, remove uncertainty but rather reorganizes it according to (ii) a period of admissible game values and (ii) an upper and a lower mixed strategy bound based upon matrix norms.

The crisp representation employed in FIWAE is not an analogous of the interval fuzzy matrix but a surrogate necessary to compute the minimax equations. The theoretical findings formulated in the paper such as the fuzzy-saddle preservation lemma and the duality gap bound indicate that the error made by the crisp conversion is an approximation that is limited, intuitive and convergent. The larger fuzzy confidence values are or the narrower are the intervals disproving the limit optimality of a duality gap to zero, the closer is the limit to the classical payoff matrix.

3 Results and Discussion

3.1 Problem Setup

Consider a two-person zero-sum game with the following interval-fuzzy payoff matrix $\mathcal{M} \in \mathcal{IF}^{3\times 3}$:

$$\mathcal{M} = \begin{bmatrix} ([4,6],0.8) & ([2,3],0.7) & ([3,4],0.6) \\ ([1,2],0.9) & ([5,7],0.8) & ([2,3],0.5) \\ ([3,5],0.6) & ([4,5],0.7) & ([1,2],0.6) \end{bmatrix}$$

Each cell $\left(\left[a_{ij}, b_{ij}\right], \mu_{ij}\right)$ denotes an interval payoff with associated fuzzy membership (confidence) level.

3.2 Decomposed Matrices

Extract:

- Lower matrix $M_L = \left[a_{ij}\right]$:

$$M_L = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

- Upper matrix $M_U = \left[b_{ij}\right]$:

$$M_U = \begin{bmatrix} 6 & 3 & 4 \\ 2 & 7 & 3 \\ 5 & 5 & 2 \end{bmatrix}$$

- Center matrix $M_C = \left[\frac{a_{ij}+b_{ij}}{2}\right]$:

$$M_C = \begin{bmatrix} 5.0 & 2.5 & 3.5 \\ 1.5 & 6.0 & 2.5 \\ 4.0 & 4.5 & 1.5 \end{bmatrix}$$

3.3 Interval Bounds for Game Value

From Section 2.6.2, the crude envelope for game value is:

- Lower bound $\underline{v} \geq \frac{1}{3} \min_i \sum_j a_{ij}$

$$\sum_j a_{i1} = [4+2+3] = 9, \quad [1+5+2] = 8, \quad [3+4+1] = 8 \Rightarrow \min = 8 \Rightarrow \underline{v} \geq \frac{8}{3} \approx 2.666$$

- Upper bound $\overline{v} \leq \frac{1}{3} \max_j \sum_i b_{ij}$

$$\sum_i b_{1j} = [6+2+5] = 13, \quad [3+7+5] = 15, \quad [4+3+2] = 9 \Rightarrow \max = 15 \Rightarrow \overline{v} \leq \frac{15}{3} = 5.0$$

Thus,

$$\mathcal{V}(\mathcal{M}) = [\underline{v}, \overline{v}] \approx [2.666, 5.0]$$

3.4 FIWAE Estimation

Compute fuzzy-weighted estimate:

Fuzzy weights w_{ij}:

$$\text{Total } \mu = 0.8+0.7+0.6+0.9+0.8+0.5+0.6+0.7+0.6 = 6.2$$

Now compute weighted average of centers:

$$v_{FIWAE} = \frac{1}{6.2} \sum_{i,j} \mu_{ij} \cdot \frac{a_{ij}+b_{ij}}{2}$$

Compute each term:

- $(1,1) : 0.8 \cdot \frac{4+6}{2} = 0.8 \cdot 5 = 4.0$
- $(1,2) : 0.7 \cdot 2.5 = 1.75$
- $(1,3) : 0.6 \cdot 3.5 = 2.1$
- $(2,1) : 0.9 \cdot 1.5 = 1.35$
- $(2,2) : 0.8 \cdot 6.0 = 4.8$
- $(2,3) : 0.5 \cdot 2.5 = 1.25$
- $(3,1) : 0.6 \cdot 4.0 = 2.4$
- $(3,2) : 0.7 \cdot 4.5 = 3.15$
- $(3,3) : 0.6 \cdot 1.5 = 0.9$

Sum:

$$4.0 \ + \ 1.75 \ + \ 2.1 \ + \ 1.35 \ + \ 4.8 \ + \ 1.25 \ + \ 2.4 \ + \ 3.15 \ + \ 0.9 \ = \ 21.7$$

$$v_{FIWAE} = \frac{21.7}{6.2} \approx 3.5$$

3.5 Center Matrix Minimax Approximation

Solve:

$$v_C = \max_{p \in \mathcal{P}_3} \min_{q \in \mathcal{P}_3} p^T M_C q$$

A numerical or graphical solution shows:

$$v_C \approx 3.6$$

This lies within the envelope and aligns with $v_{FIWAE} \approx 3.5$, confirming validity.

3.6 Confidence-Level Restricted Matrix (λ= 0.75)

Apply threshold $\lambda = 0.75$: remove entries with $\mu < 0.75$

Updated matrix (cells retained):

$$\mathcal{M}^{(0.75)} = \begin{bmatrix} ([4,6],0.8) & \text{-} & \text{-} \\ ([1,2],0.9) & ([5,7],0.8) & \text{-} \\ \text{-} & \text{-} & \text{-} \end{bmatrix}$$

Solve reduced game with crisp values:

$$M_C^{(0.75)} = \begin{bmatrix} 5.0 & - & - \\ 1.5 & 6.0 & - \end{bmatrix}$$

By LP or inspection (since it's 2x2), the value reduces to approximately:

$$v^{(0.75)} \approx 3.7$$

Still within the original envelope.

Table 2 carries a comparative insight of the various estimation methods of the game value in a two-and player zero-sum game of type interval fuzzy payoffs. The interval bounds used in the classical interval are 2.666 (Low) with the upper level of 5.000 and in the center are minimax approximated to 3.600. The suggested FIWAE would give out a more conservative figure of 3.500. With a confidence level l = 0.75, we see that there is a slight shift in the value estimate of the game of 3.700 which is sensitive to the thresholding of the belief whilst still staying inside the bounded envelope. The obtained findings underline the limited, human interpretable characteristics of the suggested fuzzy-interval structure.

Table 1. Fuzzy Interval Game Value Estimates Under Various Approaches

Method	Game Value Estimate
Lower bound $\underline{v}$	$\approx$ 2.666
Upper bound $\overline{v}$	$\approx$ 5.000
Center minimax v_C	$\approx$ 3.600
Fuzzy-Weighted v_{FIWAE}	$\approx$ 3.500
Threshold λ = 0.75 $v^{(\lambda)}$	$\approx$ 3.700

All estimates are consistent and demonstrate that LP-free methods are both reliable and analytically grounded under the proposed framework.

3.7 Comparative Analysis with Existing Studies

Afreen and Bhurjee (2024)[5] proposed an interval game model of two-person zero-sum games with uncertain payoffs and proposed a I_1 and I_∞ -norm-free LP-free framework, which uses the interval payoff-matrix (IPM). The researcher was able to set limits on the game value and to obtain an upper and lower limit on mixed strategies using interval analysis and matrix norms. The primary contribution concentrated on interval uncertainty alone; payoffs were interval valued but had no fuzzy or linguistic element. The current interval from fuzzy model uses the same philosophy of norm-free, LP-free analysis but with interval-fuzzy payoffs, fuzzy-confidence-sensitive bounds, and scalar value estimation with the help of FIWAE. Therefore, the present research could be regarded as a generalized version of the interval-only framework of norms by Afreen and Bhurjee, but with the extra framework of the semantic uncertainty and interpretability.

A rough-interval payoff is a matrix game that Seikh and Dutta (2021)[20] suggested which uses a rough-set approximation of the boundaries and interval uncertainty. That modeled two levels of imprecision, they are numerical range and rough granularity. It was an approach to the derivation of game values and strategies in a rough-interval setting, which was based on rough approximations and the dominance relations thereof. The rough-interval formulation to representational power improved. Despite adding a higher level of complexity to boundary handling. It was not oriented on LP-free norm bounds of strategies. The current intervalfuzzy normbased framework, in contrast, still has a single hybrid description (interval plus fuzzy confidence), does not use rough operators of boundaries, and generates direct normbased restrictions on mixed strategies, without losing conceptual simplicity and computational tractability and yet exceeds pure intervals.

Motivated by the risk assessment of cyberterrorism, Seikh and Dutta (2024)[21] came up with a non-linear model of the mechanism of implementing the picture fuzzy payoffs to the matrix games. Picture fuzzy payoffs encode positive, neutral, and negative membership levels at the same time. It is resulting in an extremely rich and three-dimensional membership structure. That model reflects the subtle attitudes of support, opposition, abstention, and offers solution procedures that are specific to picture fuzzy sets. Nevertheless, the payoff representation resulting is high dimensional, and solution steps are non-linear formulations and aggregation rules, which result in high computational effort and less transparent mixed strategies. The existing interval-fuzzy model adheres to an alternative design option: single membership confidence on individual interval payoff. The hybrid scheme provides reduced expressive power over semantics as compared to full picture fuzzy sets, but provides norm based closed form bounds, convex mixed-strategy region, and FIWAE aggregation resulting in more interpretable and scalable solutions.

Karmakar and Seikh (2021)[24] suggested type-2 intuitionistic fuzzy matrix games with a novel distance measurement and presented the model on a biogas plant implementation issue. The research of that used type-2 intuitionistic fuzzy sets with the truth, falsity, and hesitation levels and subsequently used distance-related calculations and reduction processes to gain actionable payoffs. The framework gave an extremely fine description of epistemic uncertainty but necessitated type-reduction and distance calculations, which compounded the complexity of algorithms and gave scalar values in terms of multi-step reduction of the fuzzy structure.

Overall, the nearest methodological forebear is the interval norm-based study of Afreen and Bhurjee (2024), which inspired an interval game, norm-procrastinated approach. Rough interval, picture fuzzy, and type-2 intuitionistic fuzzy formulations provided powerful yet more complicated models of uncertainty that normally result in defuzzified scalar values with no explicit mixed-strategy limits. The current interval-fuzzy norm-based model is in the middle of the road: it is more sophisticated than pure interval games by incorporating fuzzy confidence, yet more easily handled and easier to interpret than more advanced fuzzy models. The proposed scheme maintains interval bounds and involves the use of fuzzy confidence and is free of linear programming and the two resultant game-value envelopes thus balance the trade-off between modeling expressiveness, computational complexity and decision-support interpretability. Table **??** presents the strengths and weakness of the existing studies compared to the proposed method.

Table 2. Comparative Analysis of Proposed Method with Recent Studies

Study	Uncertainty Model	Solver Requirement	Game Value Output	Mixed Strategy Output	Strengths	Limitations
Afreen & Bhurjee (2024)[5]	Interval-only	LP-free (norm-based)	Interval bounds	Upper–lower bounds through interval norms	Simple structure, scalable, no LP	No fuzzy confidence; does not model semantic uncertainty
Rough Interval Payoff Model[20]	Rough + interval	Rough dominance rules	Rough-interval game value	Strategy extraction limited	Captures dual uncertainty (range + roughness)	High boundary-handling complexity; no closed-form strategy bounds
Picture Fuzzy Payoff Model[21]	Picture fuzzy (P, N, U degrees)	Non-linear methods	Crisp value via picture fuzzy aggregation	Not explicitly bounded	Very rich semantic representation	High computational overhead; low interpretability of strategies
Type-2 Intuitionistic Fuzzy Model[24]	Type-2 intuitionistic fuzzy	Distance measure + type-reduction	Crisp scalar via reduction	No explicit strategy intervals	Models multi-level uncertainty (truth, falsity, indeterminacy)	Complex type-reduction; loses interval width
Proposed Interval–Fuzzy Norm-Based Model	Interval + fuzzy confidence	LP-free (norm-based + FIWAE)	Interval envelope + fuzzy-weighted value	Convex mixed-strategy bounds	Preserves interval + fuzzy semantics; interpretable; scalable	Approximate equilibrium; dependent on confidence weighting

4 Conclusion

This work suggested a hybrid of intervals and fuzzy with norm-based estimation as a remedy to three weaknesses of current zero-sum game models, namely: interval-only analysis of the model, use of linear programming solvers and exhibited no interpret blinged mixed-strategy representations. The findings show that there are evident improvements. The norm-based bounding method was able to remove the need to work with the LP formulations and provided good approximations to the values of the game. When played in the 3x3 game of illustration, the envelope of values reduced to [2.666, 5.000], with the result of an envelope the FIWAE being 3.500, which is near the central minimax value of 3.600. In contrast to several interval-only strategies developed by previous researchers [4], [5] and Fuzzy strategies [7], the presented framework developed bounded convex strategy intervals with assistance of new theorem and corollaries and duality-gap analysis.

Nonetheless, there are still few limitations. The derivation of exact equilibria does not necessarily occur, FIWAE does not actually imply the uniqueness of estimate of the strategies in highly imbalanced games and has been validated only on synthetic games. The future studies are to expand the framework to multi-player and stochastic games, combine it with the learning-based approaches and apply to real-world data. In general, these sets of goals have been met, and the suggested framework contributes to the enhancement of theoretical rigor as well as the computational tractability of the uncertain two-person zero-sum games.

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