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RESEARCH ARTICLE

Ordered Pseudo-Ideals of An Ordered Ternary Semigroup

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Abstract

Objectives: This research explores the concept of ordered pseudo-ideals in an ordered ternary semigroup G, which builds on the existing idea of pseudo-ideals in a semigroup. **Methods:** The investigation delves into some unique properties of ordered pseudo-ideals in an ordered ternary semigroup G. **Findings:** This study has established many sufficient conditions for $(LMN]$ to be an ordered pseudo-ideal of G where L, M and N are non-empty subsets of G. Additionally, the study provides conditions for subsets $(LBB]$ and $(BBL]$ to be recognized as ordered left and right pseudo-ideals, respectively, as well as ordered bi-ideals within G. **Novelty:** This contributes to a deeper understanding of ideal theory in structured algebraic systems and offers insights that could be beneficial to abstract algebra and theoretical computer science. This advancement in understanding how algebraic structures can be ordered and organized opens up new possibilities for applying these concepts in various mathematical and computational theories.

Keywords: Ordered Ternary Semigroup; Commutative Ordered Ternary Semigroup; Ordered Ideal; Ordered Pseudo-ideal; Ordered Bi-ideal

1 Introduction

The concept of semigroups has played a vital role in abstract algebra, laying a foundation for numerous extensions and applications. Initially explored as basic algebraic structures, semigroups were later extended into ordered and ternary forms to address increasingly complex algebraic systems. Ordered semigroups, which impose a partial order, offer richer structure and have been instrumental in various fields, including logic, computer science, and data structures. Expanding further, ternary semigroups incorporate ternary operations, yielding ordered ternary semigroups—an area where significant advancements have been made in recent years.

The concept of ternary operations has gained traction due to its unique associative properties and relevance in mathematical physics and cryptographic systems. The study of ordered ternary semigroups, specifically, became formalized through the work of A. Iampan, who provided essential definitions and theoretical frameworks. His contributions led to a systematic understanding of ordered ternary substructures and ideal theory in n-ary semigroups.

D. N. Shinde and M. T. Gophane[1] explored several fascinating characteristics of ideals within the structure of a partially ordered ternary semigroup. Their research focused on three types of ideals: standard ideals, bi-ideals, and pseudo-symmetric ideals. Specifically, they examined the conditions under which the combination of three non-empty subsets U, V, and W of ordered ternary semigroup, denoted as $(UVW]$, forms these different types of ideals. In[2] , M. T. Gophane and D. N. Shinde explored the characteristics of various pseudo-ideals of a partially ordered ternary semigroup. Their investigations not only expanded understanding of these mathematical structures but also led to the significant discovery that the collection of all strongly irreducible pseudo-ideals in such semigroups forms a compact space.

In 1965, Zadeh introduced the innovative idea of a fuzzy set, which has since found applications across various disciplines. Many authors have studied fuzzy ideals and fuzzy bi-ideals in ternary semigroups. In[3] , A. Phukhaengsi, J. Suradach and T. Gaketem introduced the notion of ordered almost bi-ideals and fuzzy ordered almost bi-ideals of ordered ternary semigroups and studied their properties. In[4], the concepts of almost bi-ideal and fuzzy almost bi-ideal in ternary semigroups are introduced.

The present study aims to address these gaps by studying the notion of ordered pseudo-ideals in an ordered ternary semigroup G. In an ordered ternary semigroup, establish many sufficient conditions for $(LMN]$ to be an ordered pseudo-ideal of G where L, M, N are non-empty subsets of G. This advancement not only contributes to algebraic theory but also opens pathways for applications in computational structures and cryptographic frameworks where ternary operations are beneficial.

2 Methodology

In the development of this article, we have compiled some essential definitions and outcomes as follows.

A set $G(\neq \emptyset)$ furnished with a ternary operation $[\ \ ]:\ G \times G \times G \to\ G$. We say G forms a ternary semigroup if the ternary operation is associative in the following sense: for any $\lambda, \mu, \nu,\ \xi,\ \sigma$ in G, the equality $[\ \lambda\ \mu\ \nu\ \xi\ \sigma\ ] = [\ [\ \lambda\ \mu\ \nu\ ]\ \ \xi\ \sigma\ \ ] = [\ \lambda\ \ [\ \mu\ \nu\ \xi\ ]\ \sigma\ ] = [\ \lambda\ \mu\ \ [\nu\ \xi\ \sigma\ ]\ ]$ holds. This associative property must be valid across all combinations of elements from G. For three non-empty subsets X_1, X_2 and X_3 of a ternary semigroup G, let $[X_1X_2X_3] = \{[xyz]\ :\ x \in X_1, y \in X_2\ and\ z \in X_3\}$. For simplicity, we often denote this set as $X_1X_2X_3$, and similarly, we write $[xyz]$ as xyz and $[X_1X_1X_1]$ as ${X_1}^3$.

A ternary semigroup, denoted as G, becomes an ordered ternary semigroup when it incorporates a partial order, usually expressed as $\leq$. This ordering relationship must fulfill certain criteria: if $a \leq\ b$, then for any x, y in G, the relationships $xya \leq\ xyb,\ xay \leq\ xby,\ and\ axy \leq\ bxy$ should also hold. This condition ensures that the order is preserved no matter how the elements are combined. For $L \subseteq G$, let $(L] = \{s \in\ G\ \ :\ \ s \leq\ u,\ for\ some\ \ u \in\ L\ \}$. If $L\ =\ \{l\}$ then we write, $(L]$ as $(l]$. In this article, we will refer to G as an ordered ternary semigroup with respect to ternary operation $[\ ]$ unless otherwise stated. For non-empty subsets $L,\ M$ and N of G, we have, 1) $N \subseteq (N]$, 2) $((N]]\ =\ (N]$, 3) $(L](M](N]\ \subseteq\ (L\ MN]$, 4) $(M]\ \subseteq\ (\ N]$ if $M\ \subseteq\ \ N$. An ordered ternary semigroup G is considered commutative if all possible combinations of three elements from G result in the same outcome, no matter the order of these elements. A subset $M\ (\neq \emptyset)$of G is an ordered ternary subsemigroup of G if $M^3 = MMM \subseteq\ M$ and $(M]\ =\ M$. A subset $N\ (\neq \emptyset)$ of G is an ordered left (right, middle) ideal of G if $GGN \subseteq\ N$ ($NGG \subseteq\ N,\ GNG \subseteq\ N$) and $(N]\ = N$. If subset $L\ (\neq \emptyset)$ of G is an ordered left, ordered right and ordered middle ideal of G, then it is called an ordered ideal of G.

By an ordered left (right, middle) pseudo-ideal of G we mean an ordered ternary subsemigroup L of G such that $(ssssL] \subseteq\ L$ ($(Lssss] \subseteq\ L,\ (ssLss] \subseteq L$) for all $s \in\ G$. By a two-sided ordered pseudo-ideal, we mean a subset of G which is both an ordered left and an ordered right pseudo-ideal of G. If a subset $L\ (\neq \emptyset)$ of G is an ordered left, right and middle pseudo-ideal of G, then it is called an ordered pseudo-ideal of G. An ordered pseudo-ideal L of G is a properly ordered pseudo-ideal of G if it differs from G.

3 Result and Discussion

Theorem 3.1 .

The non-empty intersection of an arbitrary collection of ordered pseudo-ideals of G is an ordered pseudo-ideal of G.

Proof. Let $\{I_\alpha\}_{\alpha \in \Delta}$ be a collection of ordered pseudo-ideals of G . Let $x, y, z \in\ \bigcap_{\alpha \in \Delta} I_\alpha \Rightarrow\ \ x, y, z \in\ I_\alpha$ for all $\alpha \in \Delta$. Here $x, y, z \in\ \ I_\alpha$ and I_α is an ordered pseudo-ideal of $G \Rightarrow\ \ xyz \in\ I_\alpha$ and $(\ I_\alpha] =\ I_\alpha\ \Rightarrow\ xyz \in\ I_\alpha$ for all $\alpha \in \Delta \Rightarrow\ \ xyz \in \bigcap_{\alpha \in \Delta} I_\alpha$ and $(\bigcap_{\alpha \in \Delta} I_\alpha]$ = $\bigcap_{\alpha \in \Delta} I_\alpha \Rightarrow$ I is an ordered ternary subsemigroup of G. Now, let $t \in\ G$ and $a \in \bigcap_{\alpha \in \Delta} I_\alpha$. We have I_α is an ordered pseudo-ideal of $G \Rightarrow\ [tttta] \in\ I_\alpha, [atttt] \in\ I_\alpha$ and $[ttatt] \in\ I_\alpha$ for all $\alpha \in \Delta$. This shows that $[tttta]\ , [atttt]\ ,\ [ttatt] \in\ \bigcap_{\alpha \in \Delta} I_\alpha$. This shows that $\bigcap_{\alpha \in \Delta} I_\alpha$ is an ordered pseudo-ideal of G.

Example 3.2.

Let $P = \{0,\ 1,\ 2\}$ and $G = \{\emptyset, \{0\},\ \{1\},\ \{2\},\ \{0,1\},\ \{0,2\},\ \{1,2\},\ \{0,\ 1,\ 2\}\}$. Define ternary operation [ ] on G by $[L\ MN] = \{\ s \in\ P\ :\ s \in\ L\ \ or\ s \in\ M\ or\ s \in\ N\ or\ s \in\ L,\ M, N\ \}$ for all L , M, $N \subseteq G$. Then $G = <G,\ [\ ]\ , \subseteq>$ is an ordered ternary semigroup together with partial ordering relation $\subseteq$ on G. Let $G = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ and $H = \{\emptyset, \{0\}, \{2\}, \{0,2\}\}$. Then G and H are ordered pseudo-ideals of G. For $\{0\}, \{1\}, \{2\} \in G \cup H$, we have $[\{0\}\ \{1\}\ \{2\}] = \{0,\ 1,\ 2\}$. This implies that $[\{0\}\{1\}\{2\}] \notin \mathcal{G} \cup \mathcal{H}$. Therefore $G \cup H$ is not an ordered pseudo-ideal of G.

Theorem 3.3 .

If $\{I_\alpha\}_{\alpha \in \Delta}$ forms a chain of ordered pseudo-ideals of G then $\bigcup_{\alpha \in \Delta} I_\alpha$ is an ordered pseudo-ideal of G.

Remark 3.4 .

Every ordered ideal of an ordered ternary semigroup G is an ordered pseudo-ideal of G.

Theorem 3. 5.

If I is an ordered ideal of G and J is an ordered ideal of I satisfying the condition $J = J^3$, then J is an ordered pseudo-ideal of G.

Proof. Consider, $JJJ \subseteq\ IIJ \subseteq\ J$ (since J is an ordered ideal of I). Let $s \in\ J$ and $t \in\ G$ such that $t \leq\ s.\ s \in\ J \subseteq I,\ t \in\ G,\ t \leq\ s$ and I is an ordered ideal of $G \Rightarrow\ t \in\ I$. Now, we have $J \subseteq\ I$ and J is an ordered ideal of I, $s \in\ J$ and $t \in\ I \subseteq\ G$ such that $t \leq\ s \Rightarrow\ t \in\ J$. This shows that J is an ordered ternary subsemigroup of G. For $x \in\ G$, we have $[xxxxJ] = [xxxxJ^3] = [xxxxJJJ] \subseteq [xxxxIIJ] \subseteq [[GGGGI]\,IJ] \subseteq\ [[GGI]\,IJ] \subseteq\ [IIJ] \subseteq\ J$ (since J is an ordered ideal of I). Similarly, $[Jxxxx] = [J^3xxxx] = [JJJxxxx] \subseteq [JIIxxxx] \subseteq\ [JI\,[IGGGG]] \subseteq\ [JI\,[IGG]] \subseteq\ [JII] \subseteq\ J$ and $[xxJxx] =\ [xxJ^3xx] = [xxJJJxx] \subseteq [xxIJIxx] \subseteq [GGIJIGG] = [[GGI]\,J\,[IGG]] \subseteq [IJI] \subseteq\ J$. This proves that J is an ordered pseudo-ideal of G.

We establish conditions for $(LMN]$ to be an ordered pseudo-ideal of G, where $L, M,\ N$ are non-empty subsets of G.

Theorem 3. 6.

Let L be any non-empty subset of G. Then $(GLG]$ is an ordered pseudo-ideal of G.

Proof. Consider, $([G\mathcal{L}G])^3 = (G\mathcal{L}G](G\mathcal{L}G](G\mathcal{L}G] \subseteq (G\mathcal{L}GG\mathcal{L}GG\mathcal{L}G] = (G[\mathcal{L}GG][\mathcal{L}GG]\mathcal{L}G] \subseteq (G[GGG][GGG]\mathcal{L}G] \subseteq ([GGG]\mathcal{L}G] \subseteq\ (G\mathcal{L}G]$. Now, let $s \in\ (G\mathcal{L}G]$ and $t \in\ G$ such that $t \leq\ s.\ \ s \in\ (GLG) \Rightarrow s \leq\ x\ for\ some\ x \in\ GLG$. Therefore $t \leq\ s,\ s \leq\ x \Rightarrow\ t \leq\ x\ \Rightarrow\ t \leq\ x,\ t \in\ G,\ x \in\ GLG \Rightarrow\ t \in\ (GLG]$. This shows that $(GLG]$ is an ordered ternary subsemigroup of G. For $x \in\ G$, we have $[xxxx\,(GLG]] \subseteq \lceil (x]\,(x]\,(x]\,(x]\,(GLG] \rceil \subseteq (xxxxG\mathcal{L}G] = ([xxxxG]\,\mathcal{L}G) \subseteq (G\mathcal{L}G]$ (since $xxxxG \subseteq\ G$). Similarly, we can show that, $[(GLG]\,xxxx] \subseteq\ (GLG]$ and $[xx\,(GLG]\,xx] \subseteq\ (GLG]$. Hence $(GLG]$ is an ordered pseudo-ideal of G.

Theorem 3.7 .

Let L and M be any two non-empty subsets of G. Then $(LMG]$ is an ordered right pseudo-ideal of G.

Proof. Consider, $((\mathcal{LM}G])^3 = (\mathcal{LM}G]\,(\mathcal{LM}G]\,(\mathcal{LM}G] \subseteq ([\mathcal{LM}G]\,[\mathcal{LM}G]\,[\mathcal{LM}G]] = (\mathcal{LM}\,[G\mathcal{LM}]\,[G\mathcal{LM}]\,G] \subseteq (\mathcal{LM}\,[GGG]\,[GGG]\,G] \subseteq (\mathcal{LM}\,[GGG]] \subseteq\ (\mathcal{LM}G]$. Now, let $s \in\ (\mathcal{LM}G]$ and $t \in\ G$ such that $t \leq\ s.\ \ s \in\ (LMG) \Rightarrow s \leq\ x\ for\ some\ x \in\ LMG$. Therefore $t \leq\ s,\ s \leq\ x \Rightarrow\ t \leq\ x\ \Rightarrow\ t \leq\ x,\ t \in G,\ \ x \in LMG \Rightarrow\ t \in\ (LMG]$. This shows that $(LMG]$ is an ordered ternary subsemigroup of G. For $x \in\ G$, we have $[(LMG)\,xxxx] \subseteq \lceil (LMG]\,(x]\,(x]\,(x]\,(x] \rceil \subseteq (LMGxxxx) = (LM\,(Gxxxx] \rceil \subseteq (LMG]\ \ (since\ Gxxxx \subseteq\ G)$. Hence $(LMG]$ is an ordered right pseudo-ideal of G.

Corollary 3. 8.

Let L be a non-empty subset of G. Then $(LGG]$ is an ordered right pseudo-ideal of G.

Theorem 3. 9.

Let L be an ordered left pseudo-ideal of G and M be a non-empty subset of G. Then $(LMG]$ is a two sided ordered pseudo-ideal of G.

Proof. From **Theorem 3.7**, $(LMG]$ is an ordered right pseudo-ideal of G. Now let $x \in G$, we have $[xxxx(LMG]] \subseteq [(x](x](x](x](LMG]] \subseteq (xxxxLMG] = ([xxxxL]MG] \subseteq (GLG]$. This prove that $(LMG]$ is an ordered left pseudo-ideal of G. Hence $(LMG]$ is a two sided ordered pseudo-ideal of G.

Corollary 3.10.

Let L be an ordered left pseudo-ideal of G. Then $(LGG]$ is a two sided ordered pseudo-ideal of G.

Based on **Theorems 3.7** and **3.9**, we can state the following simple results.

1) Let L and M be any two non-empty subsets of G. Then $(GLM]$ is an ordered left pseudo-ideal of G.

2) Let L be a non-empty subset of G and M be an ordered right pseudo-ideal of G. Then $(GLM]$ is a two sided ordered pseudo-ideal of G.

3) Let L be a non-empty subset of G then $(GGL]$ is an ordered left pseudo-ideal of G.

4) Let L be an ordered right pseudo-ideal of G. Then $(GGL]$ is a two sided ordered pseudo-ideal of G.

Theorem 3.11 .

Let L be an ordered left pseudo-ideal of G and M be an ordered right pseudo-ideal of G. Then $(LGM]$ is a two sided ordered pseudo-ideal of G.

Proof. Consider, $((LGM])^3 = (LGM](LGM](LGM] \subseteq ([LGM][LGM][LGM]] = (L[GML][GML]GM] \subseteq (L[GGG][GGG]GM] \subseteq (LGGGM] \subseteq (L(GGG]M] \subseteq (LGM]$. Now, let $s \in (LGM]$ and $t \in G$ such that $t \leq s$. $s \in (LGM] \Rightarrow s \leq x$ for some $x \in LGM$. Therefore $t \leq s$, $s \leq x \Rightarrow t \leq x \Rightarrow t \leq x, t \in G, x \in LGM \Rightarrow t \in (LGM]$. This shows that $(LGM]$ is an ordered ternary subsemigroup of G. For $x \in G$, we have $[xxxx(LGM]] \subseteq [(x](x](x](x](LGM]] \subseteq (xxxxLGM] = ([xxxxL]GM] \subseteq (LGM]$ (since $xxxxL \subseteq L$). Similarly, we can show that, $[(LGM]xxxx] \subseteq (LGM]$. Hence $(LGM]$ is a two sided ordered pseudo-ideal of G.

Theorem 3.1 2.

Let L and M be non-empty subsets of a commutative ordered ternary semigroup G. Then $(LGM]$ is an ordered pseudo-ideal of G.

Proof. As L, M are non-empty subsets of G, $(LGM]$ is an ordered ternary subsemigroup of G (see the proof of **Theorem 3.11.**). For $x \in G$, we have $[xxxx(LGM]] \subseteq [(x](x](x](x](LGM]] \subseteq (xxxxLGM] = (xx[xxL]GM] = (xx[Lxx]GM] = ([xx\mathcal{L}][xxG]\mathcal{M}] \subseteq [(\mathcal{L}xx]GM] = (\mathcal{L}[xxG]\mathcal{M}] \subseteq (\mathcal{L}G\mathcal{M}]$. Hence $(LGM]$ is an ordered left pseudo-ideal of G. As G is a commutative ordered ternary semigroup, $(LGM]$ is an ordered right and an ordered pseudo-ideal of G. This shows that $(LGM]$ is an ordered pseudo-ideal of G.

A non-empty subset $(L\,MN]$ of G satisfies the sufficient condition to be an ordered pseudo-ideal, as stated in the following theorems.

Theorem 3.1 3.

Let L, M, N be three non-empty subsets of a commutative ordered ternary semigroup G. Then the set $(L\,MN]$ is an ordered pseudo-ideal of G if at least one of the following conditions is satisfied in G.

1) $L\,, M \subseteq N$ and N is an ordered left (middle or right) pseudo-ideal of G.

2) $L, N \subseteq M$ and M is an ordered left (middle or right) pseudo-ideal of G.

3) $M, N \subseteq L$ and L is an ordered left (middle or right) pseudo-ideal of G.

Proof. We present the proof for statement (1) only, as the proof for statements (2) and (3) follow a similar approach. Consider, $((LMN])^3 = (LMN](LMN](LMN] \subseteq (LMNLMNLMN] = (\mathcal{L}\mathcal{M}[\mathcal{N}\mathcal{L}\mathcal{M}][\mathcal{N}\mathcal{L}\mathcal{M}]\mathcal{N}) \subseteq (LMNNN]$ (since $L, M \subseteq N$, $[NLM] \subseteq [NNN] \subseteq N) \subseteq (LMN]$. Now, let $s \in (LMN]$ and $t \in G$ such that $t \leq s$. $s \in (LMN] \Rightarrow s \leq x$ for some $x \in LMN$. Therefore $t \leq s$, $s \leq x \Rightarrow t \leq x \Rightarrow t \leq x, t \in G, x \in LMN \Rightarrow t \in (LMN]$. This shows that $(LMN]$ is an ordered ternary subsemigroup of G. For $x \in G$, we have $\lceil xxxx(LMN]\rceil \subseteq \lceil (x](x](x](x](LMN]\rceil \subseteq (xxxxLMN] = (L\,MNxxxx]$ (since G is commutative) $= (L\,M[Nxxxx]] \subseteq (L\,MN]$ (since N is an ordered right pseudo-ideal of G, $(xxxxL] \subseteq L$). Hence $(LMN]$ is an ordered left pseudo-ideal of G. As G is a commutative, $(L\,MN]$ is an ordered right and an ordered middle pseudo-ideal of G. This proves that $(L\,MN]$ is an ordered pseudo-ideal of G.

Corollary 3.14.

Let L be a non-empty subset of a commutative ordered ternary semigroup G. Then $(GLG]$, $(GGL]$ and $(LGG]$ are ordered pseudo-ideals of G.

Corollary 3.15.

Let L and M be two non-empty subsets of a commutative ordered ternary semigroup G. Then $(LMG]$, $(LGM]$ and $(GLM]$ are ordered pseudo-ideals of G.

Theorem 3.1 6.

If $L,\ M,\ N$ are three ordered pseudo-ideals of a commutative ordered ternary semigroup G then $(LMN]$ is an ordered pseudo-ideal of G.

Proof. Consider, $((LMN])^3 = (LMN](L\,MN](L\,MN] \subseteq (L\,MNLMNL\,MN] = ([LLL][MMM][NNN]]$ (since G is commutative) $\subseteq (LMN]$. Now, let $s \in (LMN]$ and $t \in G$ such that $t \leq s$. $s \in (LMN] \Rightarrow s \leq x$ for some $x \in LMN$. Therefore $t \leq s,\ s \leq x \Rightarrow t \leq x \Rightarrow t \leq x,\ t \in G,\ x \in LMN \Rightarrow t \in (LMN]$. This shows that $(LMN]$ is an ordered ternary subsemigroup of G. For $s \in G$, we have $[ssss(\mathcal{LMN}]] \subseteq [(s](s](s](s](\mathcal{LMN}]] \subseteq (ssss\mathcal{LMN}] = ([ssss\mathcal{L}]\mathcal{MN}] \subseteq (\mathcal{LMN}]$. Similarly, $[(\mathcal{LMN}]ssss] \subseteq [(\mathcal{LMN}](s](s](x](s]] \subseteq (\mathcal{LMN}ssss] = (\mathcal{LM}[\mathcal{N}ssss]] \subseteq (\mathcal{LMN}]$ and $[ss(\mathcal{LMN}]ss] \subseteq [(s](s](\mathcal{LMN}](s](s]] \subseteq (ss\mathcal{LMN}ss] = (\mathcal{L}ss\mathcal{M}ss\mathcal{N}] = (\mathcal{L}[ss\mathcal{M}ss]\mathcal{N}] \subseteq (\mathcal{LMN}]$. This proves that $(LMN]$ is an ordered pseudo-ideal of G.

Theorem 3.1 7.

If $L,\ M,\ N$ are three ordered ideals of G then $(LMN]$ is an ordered pseudo-ideal of G.

Proof. Consider, $((LMN])^3 = (LMN](LMN](LMN] \subseteq ((LMN](LMN](LMN]] \subseteq ((LGG](GMG](GGN]] \subseteq (LMN]$ (since $L,\ M,\ N$ are ideals of G). Now, let $s \in (LMN]$ and $t \in G$ such that $t \leq s$. $s \in (LMN] \Rightarrow s \leq x$ for some $x \in LMN$. Therefore $t \leq s,\ s \leq x \Rightarrow t \leq x \Rightarrow t \leq x,\ t \in G,\ x \in LMN \Rightarrow t \in (LMN]$. This shows that $(LMN]$ is an ordered ternary subsemigroup of G. For $x \in G$, we have $\lceil xxxx(LMN]\rceil \subseteq \lceil (x](x](x](x](LMN]\rceil \subseteq (xxxxLMN] \subseteq ([GGGGL]MN] \subseteq ([GGL]MN] \subseteq (LMN]$. Similarly, we have $[(LMN]xxxx] \subseteq [(LMN](x](x](x](x]] \subseteq (LMNxxxx] = (LM[NGGGG]] \subseteq (LM[NGG]] \subseteq (LMN]$ and $[xx(\mathcal{LMN}]xx] \subseteq [(x](x](\mathcal{LMN}](x](x]] \subseteq (xx\mathcal{LMN}xx] \subseteq (GG\mathcal{LMN}GG] = ([GGX]\mathcal{M}[\mathcal{N}GG]] \subseteq (\mathcal{LMN}]$. This proves that $(LMN]$ is an ordered pseudo-ideal of G.

Theorem 3.18 .

Let $L,\ M,\ N$ be three non-empty subsets of a commutative ordered ternary semigroup G. Then $(LMN]$ is an ordered pseudo-ideal of G, if one of the following conditions is true for G.

1) L is an ordered left (middle or right) ideal of G.
2) M is an ordered left (middle or right) ideal of G.
3) N is an ordered left (middle or right) ideal of G.

Proof. We present the proof for statement (1) only, as the proofs for statements (2) and (3) follow a similar approach. Consider, $((LMN])^3 = (LMN](LMN](LMN] \subseteq (LMNLMNLMN] \subseteq ([GGG][LMN][GGG]] \subseteq (G[LMN]G] \subseteq (GG[LMN]]$ (since G is commutative) $= ([GGL]MN] \subseteq (LMN]$. Now, let $s \in (LMN]$ and $t \in G$ such that $t \leq s$. $s \in (LMN) \Rightarrow s \leq x$ *for some* $x \in LMN$. Therefore $t \leq s,\ s \leq x \Rightarrow t \leq x \Rightarrow t \leq x,\ t \in G,\ x \in LMN \Rightarrow t \in (LMN]$. This shows that $(LMN]$ is an ordered ternary subsemigroup of G. For $x \in G$, we have $\lceil xxxx(LMN]\rceil \subseteq \lceil (x](x](x](x](LMN]\rceil \subseteq (xxxxLMN] \subseteq ([GGGGL]MN] \subseteq ([GGL]MN] \subseteq (LMN]$. Similarly, we have $[(LMN]xxxx] \subseteq \lceil (LMN](x](x](x](x]\rceil \subseteq (LMNxxxx] = (LM[NGGGG]] \subseteq ([GGGGL]MN] \subseteq ([GGL]MN] \subseteq (LMN]$ and $[xx(LMN]xx] \subseteq [(x](x](LMN](x](x]] \subseteq (xxLMNxx] \subseteq (GGLMNGG] = ([GGL]MNGG] \subseteq (LMNGG] \subseteq (GGLMN] \subseteq (LMN]$. This proves that $(LMN]$ is an ordered pseudo-ideal of G.

A sufficient condition for $(LBB]$ (respectively, $(BBL]$) to be an ordered left (respectively, right) pseudo-ideal and an ordered bi-ideal of G is given in the following theorems.

Theorem 3. 19.

If L is an ordered left pseudo-ideal and B is an ordered bi-ideal of G then $(LBB]$ is an ordered left pseudo-ideal and an ordered bi-ideal of G.

Proof. Consider, $((LBB])^3 = (LBB](LBB](LBB] \subseteq ((LBB](LBB](LBB]] = (LBBLBBLBB] = (\mathcal{L}[\mathcal{B}[\mathcal{BLB}]\mathcal{BLB}]\mathcal{B}] \subseteq (\mathcal{L}[\mathcal{BGBLB}]\mathcal{B}$ (since $[\mathcal{BLB}] \subseteq [GGG] \subseteq G) \subseteq (\mathcal{L}[\mathcal{BGBGB}]\mathcal{B}] \subseteq (\mathcal{LBB}]$ $(since\ [BGBGB] \subseteq B)$. Now, let $s \in (LBB]$ and $t \in G$ such that $t \leq s$. $s \in (LBB] \Rightarrow s \leq x$ for some $x \in LBB$. Therefore $t \leq s$, $s \leq x \Rightarrow t \leq x \Rightarrow t \leq x$, $t \in G$, $x \in LBB \Rightarrow t \in (LBB]$. This shows that $(LBB]$ is an ordered ternary subsemigroup of G. For $x \in G$, we have $[xxxx(LBB]] \subseteq [(x](x](x](x](LBB]] \subseteq (xxxxLBB] \subseteq ([xxxxL]BB] \subseteq (LBB] \subseteq (LBB]$. This shows that $(LBB]$ is an ordered left pseudo-ideal of G. Now, consider $(LBB]G(LBB]G(LBB] = (LBB](G](LBB](G](LBB] = (LBBGLBBGLBB] = (L[B[BGL]B[BGL]B]B] \subseteq (L[BGBGB]B]$ (since $[BGL] \subseteq [GGG] \subseteq G) \subseteq (LBB]$ $(since\ [BGBGB] \subseteq B)$. Let $s \in (LBB]$ and $t \in G$ such that $t \leq s$. $s \in (LBB] \Rightarrow s \leq x$ for some $x \in LBB$. Therefore $t \leq s$, $s \leq x \Rightarrow t \leq x \Rightarrow t \leq x$, $t \in G$, $x \in LBB \Rightarrow t \in (LBB]$. This shows that $(LBB]$ is an ordered bi-ideal of G.

Theorem 3.2 0.

If L is an ordered right pseudo-ideal and B is an ordered bi-ideal of G then $(BBL]$ is an ordered right pseudo-ideal and an ordered bi-ideal of G.

Proof. Following the proof of **Theorem 3.19**

4 Conclusion

This study introduces ordered pseudo-ideals in ordered ternary semigroups, filling a key gap by extending pseudo-ideal properties from binary to ternary structures. We establish conditions for subsets $(LMN]$ to act as ordered pseudo-ideals within an ordered ternary semigroup G. While these findings offer valuable theoretical advancements, limitations remain, particularly in addressing ordered pseudo-ideal behaviors in non-commutative ternary semigroups and in extending these properties to more complex algebraic structures. Future research can build on this by exploring these structures computationally and in applied fields. Ultimately, ordered pseudo-ideals present a novel tool in ternary semigroup theory, broadening the scope for further study.

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