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Common Fixed Point Theorems Involving Cubic Terms of d(x, y) in b-Metric Spaces

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Abstract

Objective/Aim: To establish the existence and uniqueness of fixed points for self maps in b-metric spaces. **Methods:** We have used generalized ϕ-weak contractive condition involving cubic terms of d(x, y) and weak compatibility of two maps in the setting of b-metric spaces. **Findings:** Some fixed point theorems for a self map and common fixed point theorems for two maps have been proved and some suitable examples are also given to justify the proven results. **Novelty:** In b-metric spaces, the existence of fixed points for mappings satisfying generalized ϕ-weak contractive conditions involving cubic terms of d(x, y) has not been proved yet by others. Furthermore, our results extend and generalize the results of Dutta and Choudhury, Murthy and Prasad, Hao and Guan, Jain and Kaur, Petru, sel, A. and Petru, sel, J., Peng et. al. etc.
Mathematics Subject Classification: 54H25, 47H10.

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1 Introduction

Fixed point Theory is one of the most important and useful tool in many branches of Science, Economics, Engineering and Computer Science, etc. The majority of applied mathematics problems reduce to fixed point problems. The famous Banach contraction principle proves the existence and uniqueness of fixed points for a contraction mapping in complete metric space. It has been generalized by many researchers by finding fixed point of single map or common fixed points of two or more mapping in various spaces satisfying various contractive conditions which are either commuting or weakly commuting or compatible or weakly compatible etc.

In the development of various metric spaces, one of the significant generalizations of metric space is b- metric space introduced by Bakhtin[(1)] in 1989. Afterwards, Czerwik[(2)] proved some fixed point theorems in b- metric spaces in 1993. Aydi et al.[(3)] proved common fixed point results for single valued and multi-valued mapping satisfying φ-weak contraction in b-metric spaces. After that, several interesting results about the existence of fixed points and common fixed points using different contractive conditions in b-metric spaces have been obtained by Akkouchi[(4)], Kir et.al[(5)], Agraw l et. al.[6], Hao and Guan[7], Jain and Kaur[8], Guan and Li[9], Petrusel, A. and Petrusel, J.[10], Peng et. al.[11] etc.

On the other hand, contraction mapping was generalized by many researchers. Boyd and Wong replaced the contraction mapping by φ -contraction mapping and proved the following result in 1969.

Theorem 1.1.

"Let T be a φ - **contraction** mapping defined on a complete metric space X, i.e.,

d(Tx, Ty) $\leq \varphi$(d(x, y)) for all x, y $\in$ X where φ: $[0, \infty) \to [0, \infty)$ is an upper semi-continuous function from right such that $0 \leq \varphi(t) < t$ for all t > 0. Then, T has a unique fixed point."

In 1997, Alber and Guerre-Delabriere introduced φ **– weak contraction** and proved the following.

Theorem 1.2.

"T be a φ **– weak contraction** mapping defined on a complete metric space X, i.e.,

d(Tx, Ty) $\leq$ d(x, y) - φ(d(x, y)) for all x, y $\in$ X where φ: $[0, \infty) \to [0, \infty)$ is an upper semi-continuous function from right such that $\varphi(t) > 0$ for all t > 0 and $\varphi(0) = 0$. Then, T has a unique fixed point."

In 2013, Murthy and Prasad[12] introduced a new type of inequality with cubic terms of d(x, y) involving p $\geq$ 0 called a "generalized ϕ-weak contractive condition with cubic terms involving p" and established the following result:

Theorem 1.3.

"Let (X, d) be a complete metric space and T be a self-map of X satisfying the following condition:

$[1 + pd(x, y)]d^2(Tx, Ty) \leq$ pmax $\{\frac{1}{2}[d^2(x, Tx)d(y, Ty) + d(x, Tx)\, d^2(y, Ty)],$

d(x, Tx)d(x, Ty)d(y, Tx), d(x, Ty)d(y, Tx)d(y, Ty)}

where

m(x, y) = max$\{d^2(x, y)$, d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx),

$\frac{1}{2}$[d(x, Tx)d(x, Ty)+ d(y, Tx)d(y, Ty)]},

In the present paper, we generalize the above result in the setting of b- metric space and extend our result for a pair of weakly compatible self mappings in b- metric space. We also obtain some suitable examples to justify the proven results. Our results extend and generalize the results of Dutta and Choudhury[13], Murthy and Prasad[12], Hao and Guan[7], Jain and Kaur[8], Petrusel, A. and Petrusel, J.[10], Peng et. al.[11] etc.

2 Preliminaries

We now give some known definitions and standard notations which will be needed in the sequel:

Definition 2.1[1]

Let X be a nonempty set and let s $\geq$ 1be a given real number. A function d: X$\times$X $\to [0, \infty)$ is said to be b-metric if the following conditions hold good:

(i) d(x, y) = 0 if and only if x = y.

(ii) d(x, y) = d(y, x).

(iii) d(x, y) $\leq$ s[d(x, z) + d(z, y)], for all x, y and z $\in$ X.

Then, the triplet (X, d, s) is called a **b-metric** or metric type space.

Remark 2.2.

The class of b-metric spaces is larger than that of metric spaces, since a b-metric is a metric when s = 1. All the metric spaces are b-metric spaces but the converse is not true. We can illustrate this with the help of the following examples:

Example 2.3.

"Let X = {0, 1, 3}, d(x, y) = $(x-y)^2$ and s = 2.

Then, d(x, y) $\leq$ s[d(x, z) + d(z, y)], for all x, y and z $\in$ X but the triangle inequality d(x, y) $\leq$ d(x, z) + d(z, y) does not hold for x = 0, y = 3 as d(0, 3) > d(0, 1) + d(1, 3). Therefore, (X, d, s) is a b- metric space but not a metric space."

Example 2.4.

"Let (X, d) be a metric space and define d^*(x, y) = $(d\,(x,\ y))^p$ where p $>$ 1 is a real number. Then, (X, d^*, s) is a b-metric space with s = 2^{p-1} but not a metric space."

Example 2.5.

"Let X = l_p(R), 0$<$ p $<$ 1where l_p(R) = {$\{x_n\} \subset$ R, $\sum_{n=1}^{\infty}|x_n|^p < \infty$} and define d(x, y) = $(\sum_{n=1}^{\infty}|x_n - y_n|^p)^{\frac{1}{p}}$, for x = $\{x_n\}$, y = $\{y_n\}$ $\in l_p$(R). Then, (X, d, s) is a b-metric space with s = $2^{\frac{1}{p}}$ but not a metric space."

Definition 2.6.

"Let (X, d, s) be a b-metric space.

(a) A sequence $\{x_n\}$ in X is called **convergent** if and only if there exists x $\in$ X such that for each $\epsilon > 0$ there exists m $\in N$ such that d(x_n, x) $<\epsilon$ for all n $\geq$ m.

(b) The sequence $\{x_n\}$ in X is said to be **Cauchy** if and only if for each $\epsilon > 0$ there exists $n_0 \in N$ such that d(x_n, x_m) $<\epsilon$ for all n, m $\geq n_0$."

Definition 2.7.

"The b-metric space (X, d, s) is called **complete** if every Cauchy sequence in X is convergent in X."

Definition 2.8.

"Let (X, d, s) be a b-metric space and S, T: X $\rightarrow$ X be two mappings. The mapping S and T are said to be **weakly compatible** if they commute at their coincident points, i.e., TSx = STx whenever Sx = Tx."

3 Main Results

Theorem 3.1.

Let (X, d, s) be a complete b-metric space and T be a self-map of X satisfying

$$[1 + pd(x, y)]d^2(Tx, Ty) \leq \frac{p}{s} max\,\{\tfrac{1}{2}[d^2(x, Tx)d(y, Ty) + d(x, Tx)d^2(y, Ty)], d(x, Tx)d(x, Ty)d(y, Tx), d(x, Ty)d(y, Tx)d(y, Ty)\} + m(x, y) - \varphi(m(x, y)), \quad (1)$$

where

m(x, y) = max{d^2(x, y), d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx),

p $\geq$ 0 is a real number and φ: [0,∞) $\rightarrow$ [0,∞) is a continuous function with φ(t) = 0 if and only if t = 0 and φ(t) > 0 for each t > 0. Then, T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X. We define a sequence $\{x_n\}$ in X by

$x_{n+1} = Tx_n,\ n \geq 0.$ (2)

If $x_n = x_{n+1}$, for some n, then, T has trivially a fixed point. Therefore, we assume that $x_n \neq x_{n+1}$ for all n $\in N$. For convenience, we write

$d(x_n, x_{n+1}) = d_n$ (3)

First, we show that $\{d_n\}$ is a monotonically decreasing sequence, i.e., $d_{n+1} \leq d_n$.

Case 1: If n is odd. Take n = 2m-1. We prove that $d_{2m} \leq d_{2m-1}$.

Now, taking x = x_{2m-1} , y = x_{2m} in Inequality (1), we have

[1 + pd(x_{2m-1}, x_{2m})]d^2(Tx_{2m-1}, Tx_{2m})

$\leq \frac{p}{s}$max {$\frac{1}{2}$[d^2(x_{2m-1}, Tx_{2m-1})d(x_{2m}, Tx_{2m}) + d(x_{2m-1}, Tx_{2m-1}) d^2(x_{2m}, Tx_{2m})],

d(x_{2m-1}, Tx_{2m-1})d(x_{2m-1}, Tx_{2m})d(x_{2m}, Tx_{2m-1}), d(i, Ti)d(i, Ti)d(i,Ti)} + m(x_{2m-1}, x_{2m}) – φ(m(x_{2m-1}, x_{2m})),

where

m(x_{2m-1}, x_{2m}) = max{d^2(x_{2m-1}, x_{2m}), d(x_{2m-1}, Tx_{2m-1})d(x_{2m}, Tx_{2m}),

$\frac{1}{2s}$[d(x_{2m-1}, Tx_{2m-1})d(x_{2m-1}, Tx_{2m})+d(x_{2m}, Tx_{2m-1})d(x_{2m}, Tx_{2m})]}

Now using Inequality (2), we get

[1 + pd(x_{2m-1}, x_{2m})]d^2(x_{2m}, x_{2m+1})

$\leq \frac{p}{s}$max {$\frac{1}{2}$[d^2(x_{2m-1}, x_{2m})d(x_{2m}, x_{2m+1}) + d(x_{2m-1}, x_{2m}) d^2(x_{2m}, x_{2m+1})],

d(i, i)d(i, i)d(i,i)} + m(x_{2m-1}, x_{2m}) – φ(m(x_{2m-1}, x_{2m})),
where
m(x_{2m-1}, x_{2m}) = max{$d^2(x_{2m-1}, x_{2m})$, d(x_{2m-1}, x_{2m})d(x_{2m}, x_{2m+1}), d(x_{2m-1},x_{2m+1})d(x_{2m}, x_{2m}),
$\frac{1}{2s}$[d(x_{2m-1}, x_{2m})d(x_{2m-1}, x_{2m+1})+d(x_{2m}, x_{2m})d(x_{2m}, x_{2m+1})]}.
Then,

$$[1 + pd_{2m-1}]d^2{}_{2m} \le \frac{p}{s} max\left\{\frac{1}{2}[d^2{}_{2m-1}\, d_{2m} + d_{2m-1}\, d^2{}_{2m}],\ 0, 0\right\} + m(x_{2m-1}, x_{2m}) - \phi(m(x_{2m-1}, x_{2m}) \quad (4)$$

where,
m(x_{2m-1}, x_{2m}) = max{d^2_{2m-1} , $d_{2m-1}d_{2m}$, 0, $\frac{1}{2s}$[d_{2m-1}d(x_{2m-1}, x_{2m+1}) + 0]}.
$\le$ max{d^2_{2m-1} , $d_{2m-1}d_{2m}$, 0, $\frac{1}{2s}$[d_{2m-1} s(d_{2m-1}+d_{2m})]}.
[By triangle inequality of b-metric space]
If $d_{2m-1} < d_{2m}$, then
m(x_{2m-1}, x_{2m}) $<$ max{d^2_{2m}, d^2_{2m}, 0, d^2_{2m}} = d^2_{2m}
So Inequality (4) becomes
d^2_{2m} + p$d_{2m-1}d^2_{2m}$$<$$\frac{p}{s}$max {$\frac{1}{2}$[$d_{2m-1}d^2_{2m}$ +$d_{2m-1}d^2_{2m}$], 0, 0} + d^2_{2m} - $\varphi(d^2_{2m})$.
$\le$ p $d_{2m-1}d^2_{2m}$+ d^2_{2m} - (d^2_{2m}). [gives -φ() -φ()]
which gives $\varphi(d^2_{2m}) < 0$, a contradiction.
Thus, $d_{2m} \le d_{2m-1}$.

Case 2: If n is even, taking n = 2m and putting x = x_{2m} , y = x_{2m+1} in Inequality (1) and proceeding as above we can easily obtain that $d_{2m+1} \le d_{2m}$.

Therefore, {d_n} is a monotonically decreasing sequence of non-negative real numbers and hence convergent.

$Let \lim_{n\to\infty} d_n = k\, for\, some\, k \ge 0$ (5)

We assert that k = 0. Using Inequality (1), we have
[1 + pd(x_n, x_{n+1})]d^2(Tx_n, Tx_{n+1})
$\le$$\frac{p}{s}$max {$\frac{1}{2}$[$d^2$($x_n$, T$x_n$)d($x_{n+1}, Tx_{n+1}$) + d($x_n$, T$x_n$) d^2(x_{n+1}, Tx_{n+1})], d(x_n, Tx_n)d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n),
d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)d(x_{n+1},Tx_{n+1})}
+ m(x_n, x_{n+1}) – φ(m(x_n, x_{n+1})) = $\frac{p}{s}$max {$\frac{1}{2}$[d^2(x_n, x_{n+1})d(x_{n+1}, x_{n+2})+ d(x_n, x_{n+1}) d^2(x_{n+1}, x_{n+2})], d(x_n, x_{n+1})d(x_n, x_{n+2})d(x_{n+1}, x_{n+1}),
This implies that

$$(1 + pd_n)\, d^2_{n+1} \le \frac{p}{s} max\left\{\frac{1}{2}[d^2_n d_{n+1} + d_n d^2_{n+1}], 0, 0\right\} + m(x_n, x_{n+1}) - \varphi(m(x_n, x_{n+1})).\ (6)$$

where
m(x_n, x_{n+1}) = max{$d^2(x_n, x_{n+1})$, d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1}),d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n),
$\frac{1}{2s}$[d(x_n, Tx_n)d(x_n, Tx_{n+1})+d(x_{n+1}, Tx_n) d(x_{n+1}, Tx_{n+1})]}
= max{$d^2(x_n, x_{n+1})$, d(x_n, x_{n+1})d(x_{n+1}, x_{n+2}), d(x_n,x_{n+2})d(x_{n+1},x_{n+1}),
$\frac{1}{2s}$[d(x_n, x_{n+1})d(x_n, x_{n+2})+d(x_{n+1}, x_{n+1})d(x_{n+1}, x_{n+2})]}
= max{d^2_n , $d_n d_{n+1}$, 0, $\frac{1}{2s}$[d_nd(x_n, x_{n+2}) + 0]}.
Now, using triangle inequality of b-metric space, taking limit as $n \to \infty$ and using Inequality (5) we have,

$$\lim_{n\to\infty} m(x_n, x_{n+1}) \le max\{k^2, k^2, 0, \tfrac{1}{2s} k\, s[k + k]\} = k^2\ (7)$$

Now taking limit as $n \to \infty$ in Inequality (6) and using Inequality (7), we have
(1+pk) $k^2 \le \frac{p}{s}k^3$ + k^2- $\varphi(k^2)$$\le$ pk^3 + $k^2\varphi$ (k^2)
which gives $\varphi(k^2) \le$ 0, which implies that k = 0.
Therefore,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0\ (8)$$

Now, we show that {x_n} is a cauchy sequence in X. Let m, n $\in$ N, m$>$ n, then
d(x_n, x_m) $\le$s {d(x_n, x_{n+1}) + d(x_{n+1}, x_m)}
$\le$s d(x_n, x_{n+1}) + s^2{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)}
$\le$s d(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) +……………..
= sd_n + $s^2 d_{n+1}$ + $s^3 d_{n+2}$ +…………………………………………
Taking limit as $n \to \infty$ and using Inequality (8), we get $\lim_{n\to\infty}$ d(x_n, x_m) = 0
Hence, {x_n} is a cauchy sequence in X. But X is a complete b- metric space and therefore there exists some x $\in$ X such that

$$\lim_{n\to\infty} x_n = x\ (9)$$

Now, we prove that x is a fixed point of T.

Replacing y by x_n in inequality (1), we obtain

$[1 + pd(x, x_n)]d^2(Tx, Tx_n) \leq \frac{p}{s}\max\{\frac{1}{2}[d^2(x, Tx)\, d(x_n, Tx_n) + d(x, Tx)\, d^2(x_n, Tx_n)],$

$d(x, Tx)d(x, Tx_n)d(x_n, Tx), d(x, Tx_n)d(x_n, Tx)d(x_n, Tx_n)\}$

$+ m(x, x_n) - \varphi(m(x, x_n)),$

$$[1 + pd(x, x_n)]d^2(Tx, x_{n+1}) \leq \frac{p}{s}max\{\frac{1}{2}[d^2(x, Tx)\, d(x_n, x_{n+1}) + d(x, Tx)d^2(x_n, x_{n+1})], d(x, Tx)d(x, x_{n+1})d(x_n, Tx), d(x, x_{n+1})d(x_n, Tx)d(x_n, x_{n+1})\} + m(x, x_n) - \varphi(m(x, x_n)), \quad (10)$$

where

$m(x, x_n) = \max\{d^2(x, x_n), d(x, Tx)d(x_n, Tx_n), d(x, Tx_n)d(x_n, Tx),$

$\frac{1}{2s}[d(x, Tx)d(x, Tx_n) + d(x_n, Tx)d(x_n, Tx_n)]\},$

$= \max\{d^2(x, x_n), d(x, Tx)d(x_n, x_{n+1}), d(x, x_{n+1})d(x_n, Tx), \frac{1}{2s}[d(x, Tx)d(x, x_{n+1}) + d(x_n, Tx)d(x_n, x_{n+1})]\},$

Now taking limit as $n \to \infty$ and using Inequality (8) and Inequality (9) we get

$\lim\limits_{n\to\infty} m(x, x_n) = max\{0, 0, 0, \frac{1}{2s}[0 + 0]\} = 0$ (11)

Now taking limit as $n \to \infty$ in Inequality (10) and using Inequality (11), we have

$(1+ 0)\, d^2(Tx, x) \leq \frac{p}{s}\max\{\frac{1}{2}[0+ 0], 0, 0\} + 0 - \varphi(0) = 0$

This implies that $d^2(Tx, x) \leq 0$. Thus, $d^2(Tx, x) = 0$ and consequently, $d(Tx, x) = 0$.

Thus, Tx = x.

Uniqueness : Let x and y be two fixed points of T. Then by using Tx = x and Ty = y, inequality (1) reduces to

$$[1 + pd(x, y)]d^2(x, y) \leq \frac{p}{s}max\{\frac{1}{2}[d^2(x, x)d(y, y) + d(x, x)d^2(y,y)], d(x, x)d(x, y)d(y, x), d(x, y)d(y, x)d(y, y)\} + m(x, y) - \varphi(m(x, y)) \quad (12)$$

where

$m(x, y) = \max\{d^2(x, y), d(x, x)d(y, y), d(x, y)d(y, x), \frac{1}{2s}[d(x, x)d(x, y) + d(y, x)d(y, y)]\}$

$= \max\{d^2(x, y), 0, d^2(x, y), 0\}$

$= d^2(x, y)$

So, Inequality (12) becomes

$[1 + pd(x, y)]d^2(x, y) \leq \frac{p}{s}\max\{0, 0, 0\} + d^2(x, y) - \varphi(d^2(x, y)),$

$pd^3(x, y) \leq -\varphi(d^2(x, y)),$

$pd^3(x, y) + \varphi(d^2(x, y)) \leq 0$, , which is only possible when $d^2(x, y) = 0$.

This implies that x = y. Hence, T has a unique fixed point.

Now, we furnish an example that shows the existence of a unique fixed point for a self map in complete b- metric space X(X is not a metric space as shown in example 2.3). The example establishes the generalization of our result over the result of Murthy and Prasad[12].

Example 3.2

Let X = {0, 1, 3}, and d: X×X $\to [0, \infty)$ be defined by d(x, y) = $(x - y)^2$. Then, (X, d) is complete b-metric space with s = 2. Consider the mapping T: X $\to$ X defined by T(0) = 1, T(1) = 1, T(3) = 0 and $\varphi(t) = \frac{t}{3}$. Now, we check the inequality (1) for different points of X.

Case 1: If x = 0 and y = 1, then (1) gives after simplification, $0 \leq \frac{2}{3}$ which is true.

Case 2: If x = 1 and y = 3, then we have $1 + 4p \leq 18p + 16 - \varphi(16) = 18p + \frac{32}{3}$, i.e., $14p + \frac{29}{3} \geq 0$ which holds for $p \geq 0$.

Case 3: If x = 0 and y = 3, then after simplifying (1), we get $1 + 9p \leq \frac{45}{2}p + 54$, i.e., $27p + 106 \geq 0$, which is true since $p \geq 0$.

Thus, all the conditions of Theorem 3.1 are satisfied. Clearly, 1 is a unique fixed point of T.

Corollary 3.3.

Taking s = 1 in Theorem 3.1, we get Theorem 1.3 and further taking p = 0 we get Corollary 4 of Murthy and Prasad[12].

Corollary 3. 4.

Let (X, d, s) be a complete b-metric space and T be a self-map of X satisfying

$d^2(Tx, Ty) \leq m(x, y) - \varphi(m(x, y)),$

where

$m(x, y) = \max\{d^2(x, y), d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx), \frac{1}{2s}[d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)],$

$p \geq 0$ is a real number and $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function with $\varphi(t) = 0$ if and only if t = 0 and $\varphi(t) > 0$ for each t > 0. Then, T has a unique fixed points in X.

Now, we extend theorem 3.1 by finding a common fixed points result for two self maps using generalized φ-weak contractive condition with cubic terms involving p in the setting of b-metric space.

Theorem 3.5.

Let (X, d, s) be a complete b-metric space and S, T be self-maps of X such that T(X) $\subset$ S(X) and the pair {S, T} is weakly compatible. Also,

$$[1 + pd(Sx, Sy)]d^2(Tx, Ty) \leq \frac{p}{s}max\,\{\tfrac{1}{2}[d^2(Sx, Tx)d(Sy, Ty) + d(Sx, Tx)d^2(Sy, Ty)], \\ d(Sx, Tx)d(Sx, Ty)d(Sy, Tx),\ d(Sx, Ty)d(Sy, Tx)d(Sy, Ty)\}\ + m(Sx, Sy) - \varphi(m(Sx, Sy)), \quad (13)$$

where

m(Sx, Sy) = max{d^2(Sx, Sy), d(Sx, Tx)d(Sy, Ty), d(Sx, Ty)d(Sy, Tx), $\frac{1}{2s}$[d(Sx, Tx)d(Sx, Ty)+ d(Sy, Tx)d(Sy, Ty)]},

p $\geq$ 0 is a real number and φ: [0,∞) $\to$ [0,∞) is a continuous function with φ(t) = 0 if and only if t = 0 and φ(t) > 0 for each t > 0. Then, S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since T(X) $\subset$ S(X) Therefore, we can find some x_1 such that Tx_0 = Sx_1 = y_1. We define a sequence $\{y_n\}$ in X by

$y_n = Tx_{n-1} = Sx_n$, $n \geq 0$. (14)

For convenience, we write

$d(y_n, y_{n+1}) = d_n$ (15)

First, we show that $\{d_n\}$ is a monotonically decreasing sequence, i.e., $d_{n+1} \leq d_n$.

Case 1: If n is odd. Take n = 2m-1. We prove that $d_{2m} \leq d_{2m-1}$.

Now, taking x = x_{2m-1} , y = x_{2m} in Inequality (13), we have

[1 + pd(Sx_{2m-1}, Sx_{2m})]d^2(Tx_{2m-1}, Tx_{2m})

$\leq \frac{p}{s}$max {$\frac{1}{2}$[d^2(Sx_{2m-1}, Tx_{2m-1})d(Sx_{2m}, Tx_{2m}) + d(Sx_{2m-1}, Tx_{2m-1}) d^2(Sx_{2m}, Tx_{2m})],

d(Sx_{2m-1}, Tx_{2m-1})d(Sx_{2m-1}, Tx_{2m})d(Sx_{2m}, Tx_{2m-1}), d(Sx_{2m-1}, Tx_{2m})d(Sx_{2m}, Tx_{2m-1})d(Sx_{2m},Tx_{2m})}

+ m(Sx_{2m-1}, Sx_{2m}) – φ(m(Sx_{2m-1}, Sx_{2m})),

Where

m(Sx_{2m-1}, Sx_{2m}) = max{d^2(Sx_{2m-1}, Sx_{2m}), d(Sx_{2m-1}, Tx_{2m-1})d(Sx_{2m}, Tx_{2m}),

d(Sx_{2m-1},Tx_{2m})d(Sx_{2m},Tx_{2m-1}),

$\frac{1}{2s}$[d(Sx_{2m-1}, Tx_{2m-1})d(Sx_{2m-1}, Tx_{2m})+d(Sx_{2m}, Tx_{2m-1})d(Sx_{2m}, Tx_{2m})]}

Now using Inequality (14) , we get

[1 + pd(y_{2m-1}, y_{2m})]d^2(y_{2m}, y_{2m+1})

$\leq \frac{p}{s}$max {$\frac{1}{2}$[d^2(y_{2m-1}, y_{2m})d(y_{2m}, y_{2m+1}) + d(y_{2m-1}, y_{2m}) d^2(y_{2m}, y_{2m+1})],

d(y_{2m-1}, y_{2m})d(y_{2m-1}, y_{2m+1})d(y_{2m}, y_{2m}), d(y_{2m-1}, y_{2m+1})d(y_{2m}, y_{2m})d(y_{2m},y_{2m+1})}

+ m(y_{2m-1}, y_{2m}) – φ(m(y_{2m-1}, y_{2m})),

where

m(y_{2m-1}, y_{2m}) = max{d^2(y_{2m-1}, y_{2m}), d(y_{2m-1}, y_{2m})d(y_{2m}, y_{2m+1}), d(y_{2m-1},y_{2m+1})d(y_{2m}, y_{2m}),

$\frac{1}{2s}$[d(y_{2m-1}, y_{2m})d(y_{2m-1}, y_{2m+1})+d(y_{2m}, y_{2m})d(y_{2m}, y_{2m+1})]}.

Then,

$$[1 + pd_{2m-1}]d_{2m}^2 \leq \frac{p}{s}max\,\{\tfrac{1}{2}[d_{2m-1}^2\,d_{2m} + d_{2m-1}\,d_{2m}^2],\ 0, 0\} \\ + m(y_{2m-1}, y_{2m}) - \varphi(m(y_{2m-1}, y_{2m})), \quad (16)$$

where,

m(y_{2m-1}, y_{2m}) = max{d_{2m-1}^2 , $d_{2m-1}d_{2m}$, 0, $\frac{1}{2s}$[d_{2m-1}d(x_{2m-1}, x_{2m+1}) + 0]}.

$\leq$ max{d_{2m-1}^2 , $d_{2m-1}d_{2m}$, 0, $\frac{1}{2s}$[d_{2m-1} s(d_{2m-1}+d_{2m})]}.

[By triangle inequality of b-metric space]

If $d_{2m-1} < d_{2m}$, then

m(x_{2m-1}, x_{2m}) < max{d_{2m}^2, d_{2m}^2, 0, d_{2m}^2} = d_{2m}^2

So Inequality (15) becomes

d_{2m}^2 + p$d_{2m-1}d_{2m}^2$ < $\frac{p}{s}$max {$\frac{1}{2}$[$d_{2m-1}d_{2m}^2$ +$d_{2m-1}d_{2m}^2$], 0, 0} + d_{2m}^2 - φ(d_{2m}^2).

$\leq$ p $d_{2m-1}d_{2m}^2$+ d_{2m}^2 - φ(d_{2m}^2). [$d_{2m-1}^2 < d_{2m}^2\ gives\ -\varphi(d_{2m}^2) < -\varphi(d_{2m-1}^2)$]

which gives $\varphi(d_{2m}^2) < 0$, a contradiction.

Thus, $d_{2m} \leq d_{2m-1}$

Case 2: If n is even, taking n = 2m and putting x = x_{2m} , y = x_{2m+1} in Inequality (1) and proceeding as above we can easily obtain that $d_{2m+1} \leq d_{2m}$.

Therefore, $\{d_n\}$ is a monotonically decreasing sequence of nonnegative real numbers. Now, proceeding as in theorem 3.1, we can easily prove that

$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ and $\{y_n\}$ is a cauchy sequence in X.

But X is a complete b-metric space and therefore there exists some y ∈ X such that

$\lim_{n\to\infty} y_n = y$ (17)

Since, y ∈ T(X) and T(X) ⊂ S(X), therefore, there exist a point u ∈ X such that y = Su. Now, we prove that Tu = y. For this replacing x by u and y by x_n in inequality (13), we obtain

$[1 + pd(Su, Sx_n)]d^2(Tu, Tx_n)$

$\leq \frac{p}{s}\max\{\frac{1}{2}[d^2(Su, Tu)d(Sx_n, Tx_n) + d(Su, Tu)\, d^2(Sx_n, Tx_n)],$

$d(Su, Tu)d(Su, Tx_n)d(Sx_n, Tu),\ d(Su, Tx_n)d(Sx_n, Tu)d(Sx_n, Tx_n)\} + m(Su, Sx_n) - \varphi(m(Su, Sx_n)),$

$m(Su, Sx_n) = \max\{d^2(Su, Sx_n), d(Su\, Tu)d(Sx_n, Tx_n), d(Su, Tx_n)d(Sx_n, Tu),$

$\frac{1}{2s}[d(Su, Tu)d(Su, Tx_n)+d(Sx_n, Tu)d(Sx_n, Tx_n)]\}.$

Thus

$[1 + pd(y, y_n)]d^2(Tu, y_{n+1})$

$\leq \frac{p}{s}\max\{\frac{1}{2}[d^2(y, Tu)d(y_n, y_{n+1}) + d(y, Tu)\, d^2(y_n, y_{n+1})],$

$d(y, Tu)d(y, y_{n+1})d(y_n, Tu),\ d(y, y_{n+1})d(y_n, Tu)d(y_n, y_{n+1})\} + m(y, y_n) - \varphi(m(y, y_n)),$

$m(y, y_n) = \max\{d^2(y, y_n), d(y, Tu)d(y_n, y_{n+1}), d(y, y_{n+1})d(y_n,Tu),$

$\frac{1}{2s}[d(y, Tu)d(y, y_{n+1})+d(y_n, Tu)d(y_n, y_{n+1})]\}$

Now taking limit as $n \to \infty$ and using Inequality (17) , we get

$d^2(Tu, y) \leq \frac{p}{s}\max\{0, 0, 0\} + m(y, y_n) - \varphi(m(y, y_n))$, where

$m(y, y_n) = \max\{0, 0,0, 0\} = 0$

Therefore, $d^2(Tu, y) \leq 0$, which gives y = Tu.

Hence, Su = y = Tu. Thus, u is coincidence point of S and T. Now, since {S, T} is weakly compatible, therefore, STu = TSu which gives Sy = Ty.

Now, we claim that y is common fixed point of S and T. For this, we replace x by u in Inequality (13)

$[1 + pd(Su, Sy)]d^2(Tu, Ty) \leq \frac{p}{s}\max\{\frac{1}{2}[d^2(Su, Tu)d(Sy, Ty) + d(Su, Tu)\, d^2(Sy, Ty)],$

$d(Su, Tu)d(Su, Ty)d(Sy, Tu), d(Su, Ty)d(Sy, Tu)d(Sy, Ty)\} + m(Su, Sy) - \varphi(m(Su, Sy)),$

where

$m(Su, Sy) = \max\{d^2(Su, Sy), d(Su, Tu)d(Sy, Ty), d(Su, Ty)d(Sy, Tu),$

$\frac{1}{2s}[d(Su, Tu)d(Su, Ty)+ d(Sy, Tu)d(Sy, Ty)]\},$

Since Su = y = Tu and Ty = Sy, therefore,

$[1 + pd(y, Sy)]d^2(y, Sy) \leq \frac{p}{s}\max\{\frac{1}{2}[d^2(y, y)d(Sy, Sy)+ d(y, y)\, d^2(Sy, Sy)],$

$d(y, y)d(y, Sy)d(Sy, y), d(y, Sy)d(Sy, y)d(Sy, Sy)\} + m(y, Sy) - \varphi(m(y, Sy)),$

where

$m(y, Sy) = \max\{d^2(y, Sy), d(y, y)d(Sy, Sy), d(y, Sy)d(Sy, y),$

$\frac{1}{2s}[d(y, y)d(y, Sy)+ d(Sy, y)d(Sy, Sy)]\} = d^2(y, Sy).$

Therefore,

$d^2(y, Sy) + pd^3(y, Sy) \leq \frac{p}{s} . 0 + d^2(y, Sy) - \varphi(d^2(y, Sy))$. This implies that

$pd^3(y, Sy) + \varphi(d^2(y, Sy)) \leq 0$, which gives d (y, Sy) = 0 and hence y = Sy = Ty.

Thus, y is common fixed point of S and T

Uniqueness : Let z be another common fixed point of S and T. Then using Inequality (13) , we have

$[1 + pd(Sz, Sy)]d^2(Tz, Ty) \leq \frac{p}{s}\max\{\frac{1}{2}[d^2(Sz, Tz)d(Sy, Ty)+ d(Sz, Tz)\, d^2(Sy, Ty)],$

$d(Sz, Tz)d(Sz, Ty)d(Sy, Tz), d(Sz, Ty)d(Sy, Tz)d(Sy, Ty)\}$

$+ m(Sz, Sy) - \varphi(m(Sz, Sy)),$

where

$m(Sz, Sy) = \max\{d^2(Sz, Sy), d(Sz, Tz)d(Sy, Ty), d(Sz, Ty)d(Sy, Tz),$

$\frac{1}{2s}[d(Sz, Tz)d(Sz, Ty)+ d(Sy, Tz)d(Sy, Ty)]\},$

$[1 + pd(z, y)]d^2(z, y) \leq \frac{p}{s}\max\{\frac{1}{2}[d^2(z, z)d(y, y)+ d(z, z)\, d^2(y, y)],$

$d(z, z)d(z, y)d(y, z), d(z, y)d(y, z)d(y, y)\} + m(z, y) - \varphi(m(z, y))$, (18)

where

$m(z, y) = \max\{d^2(z, y), d(z, z)d(y, y), d(z, y)d(y, z),$

$\frac{1}{2s}[d(z, z)d(z, y)+ d(y, z)d(y, y)]$

$= d^2(z, y).$

Solving inequality (18), we have z = y.

Consequently, y is a unique common fixed point of S and T.

Now, we give an example in support of our theorem:

Example 3.6.

Let X = {0, 1, 3, 4}, and d: X×X $\rightarrow$ [0, ∞) be defined by d(x, y) = $(x-y)^2$. Then, (X, d) is complete b-metric space with s = 2. Let φ(t) = $\frac{t}{3}$ and consider the mappings S, T: X $\rightarrow$ X defined by

T(0) = 1, T(1) = 1, T(3) = 0, T(4) = 1

S(0) = 3, S(1) = 1,S(3) = 3, S(4) = 0.

Then, T(X) $\subset$ S(X) and {S, T} are weakly compatible as 1 is the only coincident point of S and T and ST(1) = TS(1). Now, we check the inequality (13) for different points of X.

Case 1: If x = 0 and y = 1, then (13) gives after simplification, 0 $\leq \frac{32}{3}$ which is true.

Case 2 : If x = 0 and y = 3, then after simplifying (13), we get 1 $\leq$ 162p + 36 - φ(36) , i.e., 162p + 23$\geq$ 0, which is true since p $\geq$ 0.

Case 3 : If x = 0 and y = 4, then we have 0 $\leq$ 8p + 54 which holds for p $\geq$ 0.

Case 4 : If x = 1 and y = 3, then we have 1 + 4p $\leq$ 18p + 16- φ(16) = 18p + $\frac{32}{3}$, i.e., 14p + $\frac{29}{3}$ $\geq$ 0 which holds for p $\geq$ 0.

Case 5 : If x = 1 and y = 4, then after simplifying (13), we get 0 $\leq \frac{p}{2}$. 0 +1, which is true.

Case 6 : If x = 3 and y = 4, then we have 1 + 9p $\leq \frac{45}{2}$p + 81- φ(81) = $\frac{45}{2}$p + 54, i.e., $\frac{27}{2}$p + 53$\geq$ 0 which holds for p $\geq$ 0.

Thus, all the conditions of Theorem 3.5 are satisfied. Clearly, 1 is a unique common fixed point of S and T.

Corollary 3.7.

If we take S = I(Identity Map) in theorem 3.5, we obtain Theorem 3.1 and in addition if we take s = 1, we get Theorem 1.3 of Murthy and Prasad[12].

Remark 3.8.

Theorem 3.1 and Theorem 3.5 extend and generalize the corresponding results of Agrawal et. al.[6], Hao and Guan[7], Jain and Kaur[8], Petrusel et. al.[10] and Guan H, Li J[9] etc.

4 Conclusion

In the present paper, firstly, we have established a fixed point theorem using a generalized φ-weak contractive condition involving cubic terms of d(x, y) in the setting of b-metric space and then we have generalized the results by finding a unique common fixed point of two self mappings. The results proved in this paper generalize and extend various known existing results. Further, some suitable examples are also given which show the generality of our proven results over the existing ones.

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