INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

ORIGINAL ARTICLE

OPEN ACCESS

Received: 23/01/2025

Accepted: 11/03/2025

Published: 23/04/2025

Citation: Lavino PJ, Priya Dharsini AM (2025) Best Proximity Point Theorems on *b* − Multiplicative Metric Spaces. Indian Journal of Science and Technology 18(13): 1029-1037. https://doi.org/10.17485/IJST/v18i13.168

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Best Proximity Point Theorems on *b* − Multiplicative Metric Spaces

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Abstract

Objectives: To introduce b - multiplicative metric spaces and cyclic multiplicative rational contractions within the framework of best proximity point theorems. **Methods:** We define b - multiplicative metric spaces and prove best proximity point theorems for multiplicative proximal contractions, including the first and second kind, cyclic multiplicative rational contraction which extend banach's contraction principle to non-self mappings. **Findings:** In b - multiplicative metric spaces, the research proved the existence and uniqueness of the best proximity points for multiplicative proximal contractions of the first and second kind. We have also introduced cyclic multiplicative rational contractions. **Novelty:** The novelty of this work lies in introducing b-multiplicative metric spaces in the study of best proximity points and cyclic multiplicative rational contractions, thereby broadening the scope of proximity point theorems.

2020 Mathematics Subject Classification: 47H10, 54H25.

Keywords: b - Multiplicative metric spaces (b-MMS); Best proximity points (BPP); Cyclic multiplicative rational contraction; Multiplicative proximal contraction; Multiplicative proximal contraction of first and second kind

1 Introduction

Grossman and Katz initially developed the concept of multiplicative calculus. In 2008, Bashirov et al. introduced a new type of space called multiplicative metric spaces. The concept of a b - metric was first proposed by Bakhtin in 1989 and later formalized by Czerwik in 1993 as a generalization of metric spaces. Ali et al.[(1)] later introduced the innovative concept of b-multiplicative metric spaces.

In 2006, Eldred and Veeramani provided results regarding best proximity points for cyclic contraction mappings. In 2011, Sadiq Basha established best proximity point theorems for proximal contractions. In 2015, Mongkolkeha and Sintunavarat[(2)] introduced the concept of best proximity points for multiplicative proximal contractions in multiplicative metric spaces. That same year, Reny George et al.[(3)] proved the existence of best proximity points for cyclic contractions and generalized first- and second-kind proximal contractions in complete b-metric spaces. J. Jarvisvivin and A. Mary Priya Dharsini[(4)] recently studied the use of fixed-point theorems for differential equations in b -multiplicative metric spaces in 2024. Furthermore, a fixed-point theorem for non-self mappings of rational type in b -multiplicative metric spaces have been developed by Joselin et al.[5] in 2024. The analysis of various types of contractions for the existence of a best proximity point is presented in[6],[7],[8],[9],[10].

This paper introduces the concept of $b-$ multiplicative metric spaces in the context of best proximity point results. For multiplicative proximal mappings, including the first and second types, we prove the existence and uniqueness of best proximity points in the context of $b-$ multiplicative metric spaces. In addition, we introduce cyclic multiplicative rational mapping contractions.

2 Preliminaries

Definition 2.1:[1] A mapping $f: X\times X\rightarrow[1,\infty)$ is considered a $b-$ multiplicative metric on the non-void set X with the coefficient $s\geq 1$ if it fulfills any of the following conditions for $x,y,w\in X$,

(m1) $d(x,y)\geq 1$ if and only if $x=y$;

(m2) $d(x,y)=d(y,x)$;

(m3) $d(x,w)\leq[d(x,w)d(w,y)]^{s}$

The (X,d) is called a $b-MMS$.

Definition 2.2:[1] Let (X,d) be a $b-MMS$,$x\in X$ and $\epsilon>1$,

$B_{\epsilon}(x)=\{y\in X|d(x,y)<\epsilon\}$,

The multiplicative open ball with a radius of ϵ centered at x.

And multiplicative closed ball as

$\overline{B}_{\epsilon}(x)=\{y\in X|d(x,y)\leq\epsilon\}$

Definition 2.3:[1] Let (X,d) be a $b-MMS$,$\{x_n\}$ be a sequence in $X,x\in X$. If every multiplicative open ball $B_{\epsilon}(x)$, there exists a natural number N such that if $n\geq N\Rightarrow x_n\in B_{\epsilon}(x)$, then the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $\{x_n\}\rightarrow x(n\rightarrow\infty)$.

Definition 2.4:[1] In a $b-$ multiplicative metric spaces and a sequence $\{x_n\}$ is deemed a multiplicative Cauchy sequence if, for every $\epsilon>1$, there exists $N\in N$ such that $d(x_m,y_n)<\epsilon$ for $m,n\geq N$.

Definition 2.5:[1] We define a $b-MMS$ as complete when every multiplicative Cauchy sequence within it converges multiplicatively to a limit point $x\in X$.

Definition 2.6:[9] A subset K of a metric space X is boundedly compact if each bounded sequence in K has a subsequence converging to a point in K.

Definition 2.6:[12] Consider the multiplicative metric spaces. (X,d) and Let A,B be non-void subsets of X respectively. The mapping $f: A\rightarrow B$ is referred to as a multiplicative proximal contraction if there exists a real number $\lambda\in[0,1)$ such that

$d(x_1,fy_1)=d(A,B)$

$d(x_2,fy_2)=d(A,B)$

$for x_1,x_2,y_1,y_2\in A$

This implies $d(x_1,x_2)\leq d(y_1,y_1)^{\lambda}$

Definition 2.7:[9] Consider (X,d) be a $b-$ metric spaces with the coefficient $s\geq 1$ and A,B be non-void subsets of X respectively. The mapping $f: A\rightarrow B$ is termed as generalized proximal contraction of first kind if there exists a non-negative number a,b,c,e with $s(a+b)+s(s+1)c+e<1$ such that

$d(x_1,fy_1)=d(A,B)$

$d(x_2,fy_2)=d(A,B)$

$for x_1,x_2,y_1,y_2\in A$

Which implies $d(x_1,x_2)\leq ad(y_1,y_2)+bd(y_1,x_1)+cd(y_2,x_2)+e[d(y_1,x_2)d(y_2,x_1)]$.

Definition 2.8:[9] Let (X,d) be a $b-$ metric spaces with the coefficient $s\geq 1$ and A,B be

non-void subsets of X respectively. The mapping $f: A\rightarrow B$ is called generalized proximal contraction of second kind if there exists a non-negative number a,b,c,e with $s(a+b)+s(s+1)c+e<1$ such that

$d(x_1,fy_1)=d(A,B)$

$d(x_2,fy_2)=d(A,B)$

$for x_1,x_2,y_1,y_2\in A$

Which implies

$d(fx_1,fx_2)\leq ad(fy_1,fy_2)+bd(fy_1,fx_1)+cd(fy_2,fx_2)+e[d(fy_1,fx_2)d(fy_2,fx1)]$.

Definition 2.9:

Let A,B be nonempty subsets of a metric spaces (X,d), $f: A\cup B\rightarrow A\cup B$ is said to be cyclic contraction, if

(i) $f(A) \subseteq B$ and $f(B) \subseteq A$ and
(ii) $d(fx, fy) \leq kd(x,y) + (1-k)d(A,B)$

for some $k \in (0,1)$ and for all $x \in A, y \in B$.

3 Result and discussion

We use the notations. A_0, B_0 and $dist(A,B)$ for non-void subsets A and B within $b-$ multiplicative metric spaces (X,d).

Consider A and B as non-void subsets of a $b-$ multiplicative metric spaces (X,d), We will explore the subsequent concepts and notations that will be essential for our discussion.

$dist(A,B) := \inf\{d(x,y) : x \in A \wedge y \in B\}$,

$A_0 := \{x \in A : d(x,y) = d(A,B)$ for some $y \in B\}$,

$B_0 := \{y \in B : d(x,y) = d(A,B)$ for some $x \in A\}$.

Definition 3.1:

Let A be a non-void subset of a $b-$ multiplicative metric spaces (X,d) with the coefficient $s \geq 1$. A mapping $g : A \to A$ is stated to be an isometry if $d(gx, gy) = d(x,y)$ for all $x, y \in A$.

Definition 3.2:

Let A and B be non-void subsets of $b-$ multiplicative metric spaces (X,d). A point $x \in A$ is referred to as a best proximity point of a mapping $f : A \to B$ if it satisfies the condition that $d(x, fx) = d(A,B)$.

Lemma 3.1:

Let (X,d) be a $b-MMS$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ as $m, n \to \infty$.

Definition 3.3:

A subset A within a $b-$ multiplicative metric spaces (X,d) is considered approximately compact concerning B if for every sequence $\{x_n\}$ in A, the condition $d(y, x_n) \to d(y,A)$ as $n \to \infty$ holds for some $y \in B$, which ensures that there exists a convergent subsequence.

Definition 3.4:

Let (X,d) be a $b-MMS$ with the coefficient $s \geq 1$ and Let A, B be non-void subsets of X respectively. The mapping $f : A \to B$ is termed as multiplicative proximal contraction, if a real number exists $\lambda \in (0,1) \ni$

$$\begin{aligned} d(x_1, fy_1) &= d(A,B) \\ d(x_2, fy_2) &= d(A,B) \end{aligned} \tag{3.1}$$

$for x_1, x_2, y_1, y_2 \in A$

Under these conditions, the inequality

$d(x_1, x_2) \leq d(y_1, y_2)^{\lambda}$

holds.

Definition 3.5:

Let (X,d) be a $b-MMS$ with the coefficient $s \geq 1$ and Let A, B be non-void subsets of X respectively. The mapping $f : A \to B$ is referred to multiplicative proximal contraction of first kind if there exists a non-negative number a, b, c, e with $s(a+b) + s(s+1)c + e < 1 \ni$

$d(x_1, fy_1) = d(A,B)$

$$d(x_2, fy_2) = d(A,B) \tag{3.2}$$

$for x_1, x_2, y_1, y_2 \in A$

Which implies $d(x_1, x_2) \leq d(y_1, y_2)^a d(y_1, x_1)^b d(y_2, x_2)^c [d(y_1, x_2) d(y_2, x_1)]^e$.

Definition 3.6:

Let (X,d) be a $b-MMS$ with coefficient $s \geq 1$ and let $A \; and \; B$ be non-void subsets of X. A mapping $f : A \to B$ is termed a multiplicative proximal contraction of second kind on $b-MMS$ if there exists a non-negative constant a, b, c, e satisfying $s(a+b) + s(s+1)c + e < 1$ and the following conditions are met:

$$\begin{aligned} d(x_1, fy_1) &= d(A,B) \\ d(x_2, fy_2) &= d(A,B) \end{aligned} \tag{3.3}$$

$forall x_1, x_2, y_1, y_2 \in A$

Under these conditions, the inequality

$d\left(fx_1, fx_2\right) \le d\left(fy_1, fy_2\right)^a d\left(fy_1, fx_1\right)^b d\left(fy_2, fx_2\right)^c \left[d\left(fy_1, fx_2\right) d\left(fy_2, fx_1\right)\right]^e$

holds true.

Definition 3.7:

Let A, B be non-void closed subsets of a $b-$ multiplicative metric spaces X, A function $f : A \cup B \to A \cup B$ is defined as a cyclic multiplicative rational contraction, if it satisfies the conditions:

(i) f is cyclic;

(ii) there exists non-negative real numbers a, b, c, e in $(0, \frac{1}{s^3})$ such that

$d(fx, fy) \le d(x, y)^a \left[\frac{[1+d(x, fx)]d(y, fy)}{1+d(x, y)}\right]^b [d(x, fx)d(y, fy)]^c [d(x, fy)d(y, fx)]^e \; dist(A, B)^{1-(a+b+2c+2e)}$

For all $x, y \in A \cup B$.

Theorem 3.1:

Let (X, d) be a complete $b-MMS$ with $s \ge 1$, Let A and B be non-empty, closed subset of X respectively such that A is approximately compact with respect to B. Assume that A_0 and B_0 are non-void. Let $f : A \to B$ and $g : A \to A$ a map fulfilling the requirements listed below:

a) f is a multiplicative proximal contraction,
b) $f\left(A_0\right)$ is contained in B_0.
c) g is an isometry
d) $A_0 \subseteq g\left(A_0\right)$.

Then, there exists a unique element. x^* in A such that

$d\left(gx^*, fx^*\right) = d(A, B)$.

Additionally, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$d\left(gx_n, fx_{n-1}\right) = d(A, B)$

converges to the element x^*.

Proof:

Let x_0 be a fixed element in A_0. Consider $f\left(A_0\right) \subseteq B_0$ and $A_0 \subseteq g\left(A_0\right)$, there exists an element $x_1 \in A_0$ such that

$d\left(gx_1, fx_0\right) = dist(A, B)$

Since $f\left(A_0\right) \subseteq B_0$ and $A_0 \subseteq g\left(A_0\right)$, There is an element $x_2 \in A_0$ such that

$d\left(gx_2, fx_1\right) = dist(A, B)$

Since f is a multiplicative proximal contraction on $b-MMS$ and g is isometry,

$$\begin{aligned} d\left(x_2, x_1\right) &= d\left(gx_2, gx_1\right) \\ &\le d\left(x_1, x_0\right)^\lambda \end{aligned} \tag{3.4}$$

Once more, because $f\left(A_0\right) \subseteq B_0$ and $A_0 \subseteq g\left(A_0\right)$, An element exists $x_3 \in A_0 \ni$

$d\left(gx_3, fx_2\right) = dist(A, B)$

It is inferred from f is a multiplicative proximal contraction on $b-MMS$,g is isometry and (**3.4**) that

$d\left(x_3, x_2\right) = d\left(gx_3, gx_2\right)$

$\le d\left(x_2, x_1\right)^\lambda$

$\le d\left(x_1, x_0\right)^\lambda$

By employing the same method, for every n belonging to the natural numbers, we can identify. x_n, x_{n+1} that are elements of A_0 such that

$d\left(gx_n, fx_{n-1}\right) = dist(A, B)$

and

$$d\left(gx_{n+1}, fx_n\right) = dist(A, B) \tag{3.5}$$

$d\left(x_{n+1}, x_n\right) = d\left(gx_{n+1}, gx_n\right)$

$\le d\left(x_n, x_{n-1}\right)^\lambda$

$\le d\left(x_n, x_{n-1}\right)^{\lambda^2}$

....
$\leq d(x_n, x_{n-1})^{\lambda^n}$
For all $n \in N$. Afterward, we shall demonstrate that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in N$ with $m > n$, then
$d(x_m, x_n) \leq d(x_m, x_{m-1}) d(x_{m-1}, x_{m-2}) \ldots d(x_{n+1}, x_n)$
$= d(x_1, x_0)^{s\lambda^m \frac{1}{1-s\lambda}}$ where $s\lambda < 1$
As we let $m, n \to \infty$ in the preceding inequality, we find that $d(x_m, x_n) \to 1$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence. Since A is a closed subset of complete $b-MMS$, then the sequence $\{x_n\}$ will converge to some element $x \in A$. It is important to note that,
$d(gx, B) \leq d(gx, fx_n)$
$\leq [d(gx, gx_{n+1}) \bullet d(gx_{n+1}, fx_n)]^s$
$= [d(gx, gx_{n+1}) \bullet dist(A, B)]^s$
$\leq [d(gx, gx_{n+1}) \bullet d(gx, B)]^s$
For all $n \in N$. Given the continuity of g and the sequence $\{x_n\}$ converges to x, it follows that the sequence $\{gx_n\}$ also converges to gx, that is $d(gx, gx_n) \to 1$ as $n \to \infty$. Consequently $d(gx, fx_n) \to d(gx, B)$ as $n \to \infty$. Given that B is approximately compact with respect to A, there exists a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ that converges to some element $u \in B$. Furthermore, for each $k \in N$, we have

$$\begin{aligned} d(A,B) &\leq d(gx,u) \\ &\leq d\left(gx, gx_{n_k+1}\right).d\left(gx_{n_k+1}, fx_{n_k}\right).d\left(fx_{n_k}, u\right) \\ &\leq d\left(gx, gx_{n_k+1}\right).d(A,B).d\left(fx_{n_k}, u\right) \end{aligned} \quad (3.6)$$

Letting $k \to \infty$ in (**3.6**), we get $d(gx, u) = d(A, B)$ and hence $gx \in A_0$. From the fact that $A_0 \subseteq g(A_0)$, then $gx = gz$ for some $z \in A_0$. By the isometry of g, we get
$d(x, z) = d(gx, gz) = 1$
and thus $x = z$, that is, x is an element of A_0. Since, $f(A_0) \subseteq B_0$, there exists $x \in A$ such that

$$d(x^*, fx) = dist(A, B) \quad (3.7)$$

From (**3.5**), (**3.7**) and the multiplicative proximal contractive condtion of f, we get
$d(gx_{n+1}, x^*) \leq d(x_n, x)^{\lambda}$
For all $n \in N$. It results in that
$\lim_{n\to\infty} d(gx_{n+1}, x^*) \leq \lim_{n\to\infty} d(x_n, x)^{\lambda} = 1.$
This demonstrates that the sequence $\{gx_n\}$ converges to x^*. By lemma (3.1), we consider that. $gx = x^*$. Hence,
$d(gx, fx) = d(x^*, fx) = d(A, B)$.
Then, to demonstrate the uniqueness, suppose that there exist.$x_1 \in A$ with $x \neq x_1$ and
$d(gx_1, fx_1) = dist(A, B)$
Since g is an isometry and f is a multiplicative proximal contraction on $b-$ multiplicative metric spaces, Consequently,
$d(x, x_1) = d(fx, fx_1) \leq d(x, x_1)^{\lambda}$,
which contradicts itself. Therefore we get $x = x_1$. The proof is now complete.

Corollary 3.1:

Let (X, d) be a complete $b-MMS$ with $s \geq 1$ and Let A, B be non-void closed subsets of X such that A_0 and B_0 are non-void and B is approximately compact with respect to A. Consider a function $f : A \to B$ satisfies the following conditions:

a) f is a multiplicative proximal contraction;
b) $f(A_0) \subseteq B_0$.

Then there exists a unique point. $x^* \in A$ such that
$d(x^*, fx^*) = d(A, B)$.
Also, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by
$d(x_n, fx_{n-1}) = d(A, B)$
converges to x^*.

Theorem 3.2:

Consider (X, d) be a complete $b-MMS$ with $s \geq 1$, Let A and B be non-empty, closed subset of X respectively such that A is approximately compact with respect to B. Assume that A_0 and B_0 are non-void. Let A and B be a map fulfilling the following conditions:

a) f is a multiplicative proximal contraction of first kind,
b) $f(A_0)$ is contained in B_0.

Then there exists a unique element x exists in A such that
$d(x, fx) = d(A, B)$
and the sequence \{x_n\} converges to the best proximity point x, where x_0 is any fixed element in A_0 and $d(x_{n+1}, fx_n) = dist(A, B)$ for $n \geq 0$.

Proof:

Let $x_0 \in A_0$. As a result of $f(A_0)$ is contained in B_0, An element x_0 is assured to exist in A_0 that meets the condition
$d(x_1, fx_0) = d(A, B)$
Additionally, since fx_1 is a part of $f(A_0)$ which is contained in B_0, It can be inferred that there exists an element x_2 in A_0 such that
$d(x_2, fx_1) = d(A, B)$
This process can be extended indefinitely. Having chosen $\{x_n\}$ in A_0, there exits an element x_{n+1} in A_0 satisfying the condition that
$d(x_2, fx_1) = d(A, B)$
For every non-negative integer n. Given that f is a multiplicative proximal contraction of first kind,
$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^a d(x_{n-1}, x_n)^b d(x_n, x_{n+1})^c [d(x_{n-1}, x_{n+1}) d(x_n, x_n)]^e$
$\leq d(x_{n-1}, x_n)^a d(x_{n-1}, x_n)^b d(x_n, x_{n+1})^c [d(x_{n-1}, x_{n+1})]^e$
$\leq d(x_{n-1}, x_n)^a d(x_{n-1}, x_n)^b d(x_n, x_{n+1})^c [d(x_{n-1}, x_n) d(x_n, x_{n+1})^c]^e$
$d(x_n, x_{n+1})^{1-c-se} \leq d(x_{n-1}, x_n)^{a+b+se}$
$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^{\frac{a+b+se}{1-c-se}}$
$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^{\lambda}$
Where $\lambda = \frac{a+b+se}{1-c-se} < 1$.
$\therefore \{x_n\}$ is a Cauchy sequence. Because the spaces is complete, the sequence $\{x_n\}$ converges to x in A. Therefore,
$d(x, B) \leq d(x, fx_n) = \lim_{n\to\infty} d(x_{n+1}, fx_n) = d(A, B) \leq d(x, B)$.
Therefore $d(x, fx_n) \to d(x, B)$. Since B is approximately compact with respect to A, the sequence $\{fx_n\}$ has a subsequence $\{fx_{n_k}\}$ converging to some element $y \in B$. So, it follows that
$d(x, y) = \lim_{n\to\infty} d\left(x_{n_k+1}, fx_{n_k}\right) = d(A, B)$
Therefore, x must belong to A_0. Given that $f(A_0)$ is contained in B_0

$$d(u, fx) = dist(A, B) \tag{3.8}$$

for some element $u \in A$. Since f is multiplicative proximal contraction of first kind, and we know that $d(u, fx) = dist(A, B)$ and $d(x_{n+1}, fx_n) = d(A, B)$, it can be concluded that
$d(u, x_{n+1}) \leq d(x, x_n)^a d(u, x)^b d(x_n, x_{n+1})^c [d(x, x_{n+1}) d(x_n, u)]^e$
Letting $n \to \infty$,
$d(u, x) \leq d(u, x)^{b+e}$
This implies $u = x$.
Therefore, from (3.8) it can be that $d(x, fx) = d(A, B)$. Let us assume that there is another best proximity point x^* in A so that
$d(x^*, fx^*) = d(A, B)$.
Since f is multiplicative proximal contraction of the first kind, it follows that
$d(x, x^*) \leq d(x, x^*)^a d(x, x)^b d(x^*, x^*)^c [d(x, x^*) d(x, x^*)]^e d(x, x^*)$
$\leq d(x, x^*)^{a+2e}$
$(a + 2e) < 1$ imply $x = x^*$.

Corollary 3.2:

Let f be a self mapping on a complete $b-$ multiplicative metric spaces (X, d) with $s \geq 1$. Further, let us assume that there exist non-negative real numbers a, b, c, e with $s(a + b) + s(s + 1)c + e < 1$ and $d(x_1, x_2) \leq d(y_1, y_2)^a d(y_1, x_1)^b d(y_2, x_2)^c [d(y_1, x_2) d(y_2, x_2)]^e$ for all x_1, x_2 in the domain of the mapping f. Then the mapping f has a unique fixed point.

Theorem 3.3:

Let (X,d) be a complete $b-MMS$ with $s \geq 1$, Let A and B be non-empty, closed subset of X respectively such that A is approximately compact with respect to B. Assume that A_0 and B_0 are non-void. Let $f: A \rightarrow B$ be a map satisfying the following conditions:

a) f is a multiplicative proximal contraction of second kind,

b) $f(A_0)$ is contained in B_0.

Then, there exists a unique element x in A such that

$d(x,fx) = d(A,B)$

the sequence \{$x_n\}$ converges to the best proximity point x with x_0 being any chosen element from A_0 and $d(x_{n+1}, fx_n) = dist(A,B)$ for $n \geq 0$.

Further, if x^* is another best proximity point of f, then $fx = x^*$, hence f is a constant on the set of all best proximity points of f.

Proof:

Let $x_0 \in A_0$. Because $f(A_0)$ is included in B_0, there is guaranteed to be an element x_0 in A_0 that satisfies the condition

$d(x_1, fx_0) = d(A,B)$

Further, since fx_1 is a member of $f(A_0)$ which is contained in B_0, It can be deduced that there exists an element x_2 in A_0 such that

$d(x_2, fx_1) = d(A,B)$

This procedure can be extended indefinitely. Having selected $\{x_n\}$ in A_0, there exists an element x_{n+1} in A_0 satisfying the condition that

$d(x_{n+1}, fx_n) = d(A,B)$

For every non-negative integer n. Given that f is multiplicative proximal contraction of first kind, we have

$d\left(fx_n, fx_{n+1}\right) \leq d\left(fx_{n-1}, fx_n\right)^a d\left(fx_{n-1}, fx_n\right)^b d(fx_n, fx_{n+1})^c [d(fx_{n-1}, fx_{n+1}) d(fx_n, fx_n)]^e.$

$\leq d\left(fx_{n-1}, fx_n\right)^a d\left(fx_{n-1}, fx_n\right)^b d(fx_n, fx_{n+1})^c [d(fx_{n-1}, fx_{n+1})]^e$

$d\left(fx_n, fx_{n+1}\right) \leq d\left(fx_{n-1}, fx_n\right)^\lambda$

Where $\lambda = \frac{a+b+se}{1-c-se} < 1$.

Therefore, $\{fx_n\}$ is a Cauchy sequence. Because the spaces is complete, the sequence $\{fx_n\}$ converges to some element x in A. Therefore,

$d(y,A) \leq d(y, x_{n+1}) = \lim_{n\to\infty} d\left(x_{n+1}, fx_n\right) = d(A,B) \leq d(y,A)$.

Therefore $d(y, x_{n+1}) \rightarrow d(y,A)$. Since A is approximately compact with respect to B, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some element $y \in B$. So, it results that

$d(x,fx) = \lim_{n\to\infty} d\left(x_{n_k+1}, fx_{n_k}\right) = d(A,B)$

Let us consider another best proximity point x^* in A so that

$d(x^*, fx^*) = d(A,B)$

Because f is a multiplicative proximal contraction of the second kind, we have

$d(fx, fx^*) =\leq d(fx, fx^*)^{a+2e}$

$(a+2e) < 1$ imply $fx = fx^*$.

Theorem 3.4:

Let (X,d) be a complete $b-MMS$ with $s \geq 1$, Let A and B be non-empty, closed subset of X respectively such that A is approximately compact with respect to B. Assume that B_0 is non-void. Consider a function $f: A \rightarrow B$ that satisfies the following conditions:

a) f is both a multiplicative proximal contraction of the first kind and a proximal contraction of the second kind,

b) $f(A_0)$ is contained in B_0.

It follows that there exists a unique element x in A such that

$d(x,fx) = d(A,B)$

and the sequence \{$x_n\}$ converges to the best proximity point x, where x_0 is any fixed element in A_0 and $d(x_{n+1}, fx_n) = dist(A,B)$ for $n \geq 0$.

Proof:

Continuing with the approach in Theorem (3.2), it is possible to find a sequence $\{x_n\}$ in A_0 such that

$$d\left(x_{n+1}, fx_n\right) = d(A,B) \tag{3.9}$$

holds for every non-negative n. As indicated in Theorem (3.2), one can demonstrate that the sequence $\{x_n\}$ is a Cauchy sequence and thus converges to some element x in A. Additionally, as highlighted in Theorem (3.3), it can be asserted that the sequence $\{fx_n\}$ is a Cauchy sequence and hence converges to some element y in B. Therefore, it follows that

$$d(x,y) = \lim_{n\to\infty} d\left(x_{n+1}, fx_n\right) = dist(A,B) \tag{3.10}$$

Hence, x becomes an element of A_0. Since $f\left(A_0\right)$ is contained in B_0.

$$d(u,fx) = d(A,B) \tag{3.11}$$

for some element u in A. Since f is a multiplicative proximal contraction of the first kind, we have

$$d\left(u, x_{n+1}\right) = d\left(x, x_n\right)^a d(u,x)^b d\left(x_n, x_{n+1}\right)^c [d(x, x_{n+1}) d(x_n, u)]^e \tag{3.12}$$

Letting $n \to \infty$, $d(u,x) \leq d(u,x)^{(b+e)}$, which implies that x and u must be identical. Thus it follows that
$d(x,fx) = d(u,fx) = d(A,B)$.

Also, the uniqueness of the BPP of the mapping f follows as in Theorem (3.2). This completes the proof of the theorem.

Proposition 3.1:

Let A, B be non-void subsets of a $b-$ multiplicative metric spaces X. Suppose $f : A \cup B \to A \cup B$ is a cyclic multiplicative rational contraction. Then starting with any $x_0 \in A \cup B$, we have $d\left(x_n, x_{n+1}\right) \to d(A,B)$ where $x_{n+1} = fx_n$ for all $n \in N \cup \{0\}$.

Proof:

Let $x_0 \in A \cup B$. A sequence $\{x_n\}$ is defined by $x_{n+1} = fx_n$ for all $n \in N \cup \{0\}$. Then by Definition (3.1.1), we have

$d\left(x_n, x_{n+1}\right) = d\left(fx_{n-1}, fx_n\right)$

$\leq d\left(x_{n-1}, x_n\right)^a \left[\frac{[1+d(x_{n-1}, fx_{n-1})]d(x_n, fx_n)}{1+d(x_{n-1}, x_n)}\right]^b [d(x_{n-1}, fx_{n-1})d(x_n, fx_n)]^c [d(x_{n-1}, fx_n)d(x_n, fx_{n-1})]^e$
$dist(A,B)^{1-(a+b+2c+2e)}$

$\leq d\left(x_{n-1}, x_n\right)^a \left[\frac{[1+d(x_{n-1}, x_n)]d(x_n, x_{n+1})}{1+d(x_{n-1}, x_n)}\right]^b [d(x_{n-1}, x_n)d(x_n, x_{n+1})]^c [d(x_{n-1}, x_{n+1})d(x_n, x_n)]^e$
$dist(A,B)^{1-(a+b+2c+2e)}$

$\leq d\left(x_{n-1}, x_n\right)^a \left[d\left(x_n, x_{n+1}\right)\right]^b \left[d\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right]^c \left[d\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right]^{se} \ dist(A,B)^{1-(a+b+2c+2e)}$
$dist(A,B)^{1-(a+b+2c+2e)}$

$\leq d\left(x_{n-1}, x_n\right)^{a+c+se} \left[d\left(x_n, x_{n+1}\right)\right]^{b+c+se} dist(A,B)^{1-(a+b+2c+2e)}$

Which gives as

$d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right)^{\frac{a+c+se}{1-b-c-se}} dist(A,B)^{1-\frac{(a+b+2c+2e)}{1-b-c-se}}$

We note that $\frac{a+c+se}{1-b-c-se} < 1$. Then the above inequality becomes

$d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right)^{\frac{a+c+se}{1-b-c-se}} dist(A,B)^{\{1-\frac{(a+b+2c+2e)}{1-b-c-se}\}}$

Similarly, $d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-2}, x_{n-1}\right)^{\left(\frac{a+c+se}{1-b-c-se}\right)^2} dist(A,B)^{\left\{1-\left(\frac{(a+b+2c+2e)}{1-b-c-se}\right)^2\right\}}$

Continuing this process, we get

$d\left(x_n, x_{n+1}\right) \leq d\left(x_0, x_1\right)^{\left(\frac{a+c+se}{1-b-c-se}\right)^n} dist(A,B)^{\left\{1-\left(\frac{(a+b+2c+2e)}{1-b-c-se}\right)^n\right\}}$

Letting limit as $n \to \infty$, we have

$d\left(x_n, x_{n+1}\right) \to d(A,B)$.

Proposition 3.2:

Let A, B be non-void closed subsets of a complete $b-$ multipicative metric space X,
$f : A \cup B \to A \cup B$ be a cyclic multiplicative rational contraction map, let $x_0 \in A$ and define $x_{n+1} = fx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x,fx) = d(A,B)$.

Proof:

Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ and $\lim_{k\to\infty} x_{2n_k} = x$ for some $x \in A$. Now,

$d(A,B) \leq d\left(x, x_{2n_k-1}\right) \leq \left[d\left(x, x_{2n_k}\right) d\left(x_{2n_k}, x_{2n_k-1}\right)\right]^s$

Taking limit as $n \to \infty$ in the above inequality, we have
$d\left(x_{2n_k}, x_{2n_k-1}\right) \to dist(A, B)$
Since $d(A, B) \leq d\left(x_{2n_k}, fx\right) \leq d\left(x_{2n_k-1}, x\right)$
As $n \to \infty$, we have
$d(x, fx) = d(A, B)$.

Theorem 3.5:

Let A, B be non-void closed subsets of a $b-$ multiplicative metric spaces X and $f : A \cup B \to A \cup B$ is a cyclic multiplicative rational contraction. If either A or B is boundedly compact, then there exists $x \in A \cup B$ such that $d(x, fx) = d(A, B)$.

Proof. It follows directly from propositions (3.1), (3.2)

4 Conclusion

The importance of $b-$ multiplicative metric spaces in best proximity point theorems were studied in this work. We demonstrated the existence and uniqueness of best proximity points for multiplicative proximal contractions, including the first and second kind, within $b-$ multiplicative metric spaces by applying Banach's contraction principle to non-self mappings. We additionally introduced the idea of cyclic multiplicative rational contractions on the theorems of best proximity points.

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