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ORIGINAL ARTICLE

A Meaningful Construction of New Circular Distribution for Applications in Geomorphology

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Abstract

Objectives: This work introduces a novel circular probability distribution-the Double Truncated Wrapped Exponential (DTWE) distribution highlighting the importance of circular statistics with cyclical characteristics contrary to usual linear data. **Methods**: The DTWE distribution is developed using the principle of truncation on the wrapped exponential distribution, which satisfies the principles of circularity. The properties of the distribution, such as the trigonometric mean, skewness, and kurtosis, are derived to enhance interpretability. Parameter estimation is carried out using Maximum Likelihood Estimation, Least Squares, and Weighted Least Squares methods. The goodness-of-fit is carried out, which makes DTWE distribution comparable to other well-known circular probability models. **Findings**: The numerical results of the simulation study across sample sizes $(n = 30, 50, 100, 1000)$ and parameter values $(\theta = 0.5, 1, 2)$ demonstrate that the DTWE distribution achieves accurate and consistent parameter estimation. For $\theta = 0.5$ and $n = 30$, the key performance metrics, such as the bias, Mean Square Error (MSE), and standard deviation (SD) for MLE outperform the LS and WLS methods by approximately 20%. Similarly, for $\theta = 2$ and $n = 1000$, the MLE achieves greater consistency reducing the bias, MSE, and SD by more than 30%. Real world data analysis shows that the DTWE distribution captures the cyclical patterns in ecological and geological data perfectly and gives meaningful insights into directional behaviours. **Novelty**: This study introduces a novel truncation-based framework for constructing circular probability distributions. The new distribution provides a distinctive approach for evaluating the circular data in ecological and geological datasets.

Keywords: Directional Statistics; Truncation; Exponential Distribution; Circular Distribution; MLE

1 Introduction

Unlike linear data, circular data is measured on a circular scale, where the start and end points of the scale are the same. Examples of circular data are angles (from 0° to 360°), times of day (from 0 to 24 hours), and directions (e.g., wind directions). Further, it has a natural beginning and end so that the extreme values are next to each other (e.g., 0° is the same as 360°). Because of this wrapping nature, special statistical methods are required for analysis, since traditional linear statistics are not capable of handling the characteristics of circular data. There are many fields of study such as Ecology, Meteorology, and Oceanography generating the data often involves directions (e.g., wind directions, animal movement directions), Circular distributions are necessary for evaluating periodic data like time-of-day or seasonal data, where the beginning and end points are the same. In Epidemiology, the seasonal nature of disease outbreaks can be described by circular statistics used in this study thereby enabling identification and analysis to make informed decisions. In the field of Geology, circular distributions are used to analyze features like strike directions of faults, while in oceanography, they are used to study ocean current directions. These fields rely on circular statistics to model the phenomena accurately[(1)]. Urban geomorphological analyses, as demonstrated in recent studies[(2)] exemplifies the recent and critical use of geomorphological analyses. The broader relevance of circular studies is shown by the growth of circular economy as a prominent study field[(3)].

Truncated distributions are highly efficient for data analysis[(4)]. The truncated distribution is a probability distribution that is obtained by removing some portion from the sample space and the truncation can be done either right (below some point) or left (above some point), or both[(5)]. Truncation is a statistical technique where a probability distribution is constrained within a certain range or interval which means that data beyond this range, is eliminated. Thus, this strategy serves form several aims. It is often used to describe the limitations represented by the practicalities of the real world, where sometimes individual values are out of reach or simply do not apply, steering the research in the right direction. For example, in reliability studies the distribution of time to failure may only be calculated where time to failure is between two times, that is, between time t and time $t + \Delta t$. Some authors have noted that truncation can enhance the levels of efficiency in the estimation of parameters by diminishing the effect of such large values that have no effects on statistical operations. Furthermore, it highlights a desirable ability of modelling, namely, the capability to show realistic constraints such as the impossibility of non-positive values of the physical measurement. For this reason, a distribution may be censored at zero to provide a better characterization. Truncation is widely used in such areas as Survival analysis or Econometrics to deal with censored observations whose mean is within a certain range are observable[(6)]. Moreover, truncation improves the rate of computation by reducing the amount of data to be analysed, and makes the statistical models simpler, hence increasing the accuracy and meaning of computations. It may also have to happen to ensure that distributional assumptions are met especially when a distribution should not be beyond a certain level.

A distribution with pdf $g(x;\Omega)$ with its corresponding cdf $G(x;\Omega)$ where Ω is a parameter vector of the random variable X. Given that the value of the variable X is within the interval $[a,b]$ where $-\infty < a \leq x \leq b < \infty$, then the conditional distribution on $a \leq x \leq b$ is the double truncated distribution. Then the probability density function (pdf) and cumulative density function (cdf) of the random variable X are respectively,

$$f(x;\Omega) = g\left(\frac{x}{a \leq x \leq b;\Omega}\right) = \frac{g(x;\Omega)}{G(b;\Omega) - G(a;\Omega)} \tag{1}$$

and

$$F(x;\Omega) = \int_a^x f(x;\Omega)dx = \frac{G(x;\Omega) - G(a;\Omega)}{G(b;\Omega) - G(a;\Omega)} \tag{2}$$

where $a \leq x \leq b, -\infty < a < b < \infty$.

The primary objective of this work is to create and enhance the flexibility of the baseline distribution called double truncated wrapped exponential (DTWE) distribution for modelling circular data using the truncation technique. In circular distributions, this is particularly helpful when addressing real-world data that demonstrates boundaries or constraints. Truncation aids in the removal of outliers or extraneous ranges inside the circular domain, hence enhancing model accuracy. The removal of outliers or outlying ranges in the circular domain enhances the accuracy and reliability of model truncation so that it better reflects the data patterns. Such constraints are absent in existing circular probability distributions, which limits their application on naturally bounded datasets. The present study introduces the technique of truncation as a novel approach to wrapped distributions; thus, being the first study of its kind. The proposed method is shown to be effective in overcoming the limitations of existing models by improving the goodness of fit, accuracy, and interpretability for real-world circular data. In the field of circular statistics, the DTWE distribution introduces truncation as a methodological addition to the circular statistical toolbox. It bridges the gap between circular statistical treatments and empirical data because of its flexible and unbounded nature.

In this study we are examining the utilization of circular probability distributions to analyze the directional data of 100 outwash stones from an outwash terrace along the Fox River, near Cary, Illinois, adjacent to Lake Wisconsin. The pebbles, deposited by glacial meltwater, possess directional features that can be examined by circular statistics to enhance comprehension of the geomorphological processes that formed the region. One more dataset having the movement patterns of turtles is analyzed here. Their directional pattern significantly influences sediment transport and landscape development, as their interactions with soil and base facilitate sediment redistribution. Directional choices may indicate adaptations to differing energy gradients in aquatic or terrestrial environments, offering insights into historical terrain evolution and habitat stability. Utilizing circular probability distributions enables deeper ecological interpretations and reveals patterns in natural systems that may otherwise be neglected.

2 Methodology

2.1 The truncated version of the wrapped distribution

The wrapping method is an acknowledged technique for deriving a circular distribution from a distribution family[7]. Therefore, the wrapped distributions are quite significant when modelling circular data. The pdf and cdf of the wrapped exponential distribution which was introduced by Jammalamadaka and Kozubowski are given by

$$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{3}$$

and

$$F(\theta) = \frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{4}$$

where $\lambda > 0$ and $\theta \in$.

The key goal of this study is to achieve a more adaptable distribution than the wrapped exponential which can be utilized for the best modelling of circular data. As a result, by employing **equations (3)** and **(4)** in the truncation method, we get the pdf and cdf of a double truncated wrapped exponential random variable Θ as follows:

$$f_{\Theta}(x;\theta) = \frac{\theta e^{-x\theta}}{e^{-a\theta}-e^{-b\theta}} \tag{5}$$

and

$$F_{\Theta}(x;\theta) = \frac{e^{-a\theta}-e^{-x\theta}}{e^{-a\theta}-e^{-b\theta}} \tag{6}$$

respectively, where $x \in [a,b]; a,b \in [0,2\pi)$ and $\theta > 0$.

The DTWE distribution with parameters θ, a and b shall henceforth be denoted as $\Theta\ DTWE(\theta,a,b)$ which is distributed by a random variable Θ. Figure 1 and Figure 2 depicts several feasible forms of the pdf and their corresponding cdf of a DTWE distribution for various parameter values of θ, a and b.

3 Results and Discussion

3.1 Characteristic function and Trigonometric moments

Circular distributions can be described using the characteristic function (cf) in an approach that is very similar to that of distributions over the real line. The characteristic function of $DTWE(\theta,a,b)$ distribution is

$$\varphi_{\theta}(p) = \varphi_p = E\left(e^{ip\Theta}\right)$$

$$E\left(e^{ip(\theta+2\pi)}\right) = e^{2ip\pi}E\left(e^{ip\theta}\right)$$

where $i=\sqrt{-1}$, $p=\pm1,\pm2,\ldots$ i.e. p can have only integer values.

An alternative method of expressing the double truncated wrapped exponential distribution's p^{th} trigonometric moment is

$$\varphi_p = E\left(e^{ip\Theta}\right) = E$$
$$\varphi_p = E\left(e^{ip\Theta}\right)$$
$$¿E¿$$

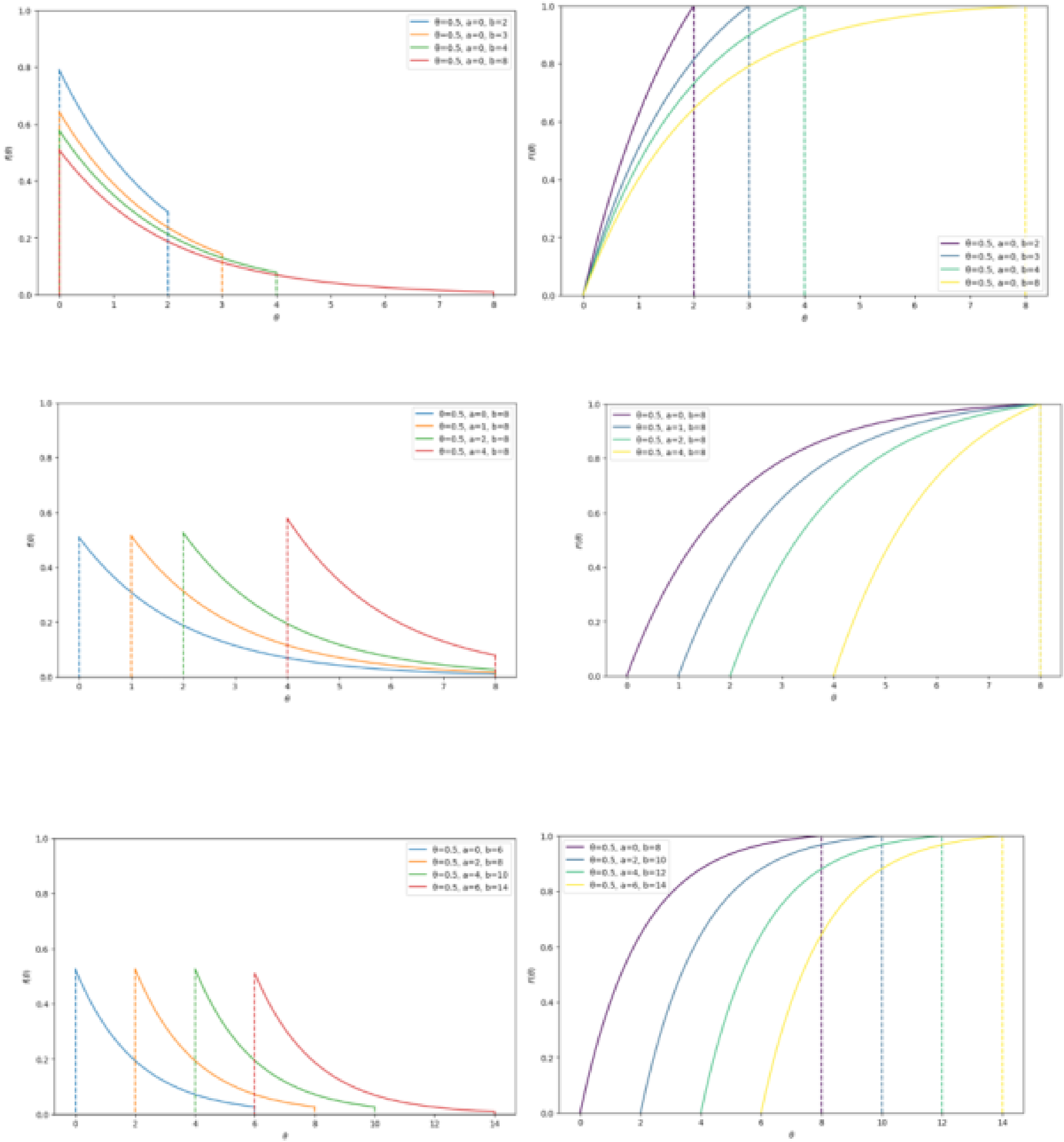

Fig 1. Linear Plots of the pdf and cdf of DTWE distribution for different values of θ, a and b

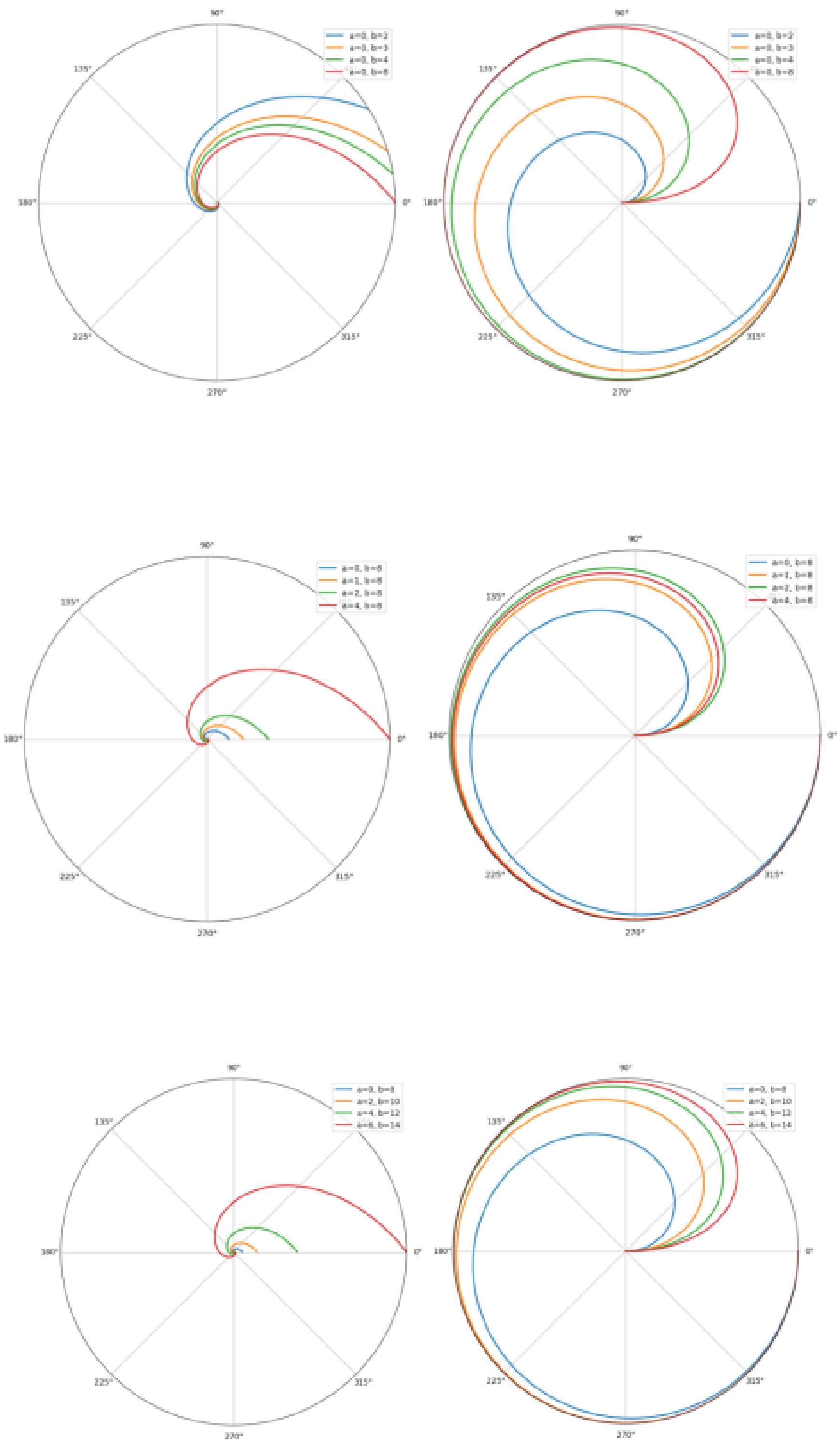

Fig 2. Polar plots of the pdf and cdf of DTWE distribution for different values of θ, a and b.

where $\alpha_p = E(\cos(p\theta))$ and $\beta_p = E(\sin(p\theta))$.

While considering the complex plane, φ_p is the mean resultant vector with length ρ_p and direction μ_p which can be represented as below. Here *atan(.)* is the quadrant inverse tangent function

$$\mu_p = a\tan\left(\alpha_p {\beta_p}^{-1}\right) = \begin{cases} \tan^{-1}\left(\frac{\beta_p}{\alpha_p}\right), \alpha_p > 0, \beta_p \geq 0 \\ \frac{\pi}{2, \alpha_p = 0, \beta_p > 0} \\ \tan^{-1}\left(\frac{\beta_p}{\alpha_p}\right) + \pi, \alpha_p < 0 \\ \tan^{-1}\left(\frac{\beta_p}{\alpha_p}\right) + 2\pi, \alpha_p \geq 0, \beta_p < 0 \\ undefined, \alpha_p = 0, \beta_p = 0 \end{cases}$$

The p^{th} trigonometric moment for the double truncated wrapped exponential distribution can be expressed as follows:

$$\begin{aligned} \varphi_p &= \frac{\theta^2}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + p^2\right)} \\ &\quad + i\frac{p\theta}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + p^2\right)} \\ \varphi_p &= \frac{\theta}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + p^2\right)}[\theta + ip] \end{aligned} \tag{7}$$

where $i = \sqrt{-1}, p = \pm 1, \pm 2, \ldots$

Here the non-central trigonometric moments which are the p^{th} cosine and sine moments are provided by

$$\alpha_p = \frac{\theta^2}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + p^2\right)} \tag{8}$$

and

$$\beta_p = \frac{\theta p}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + p^2\right)} \tag{9}$$

3.2 Location and Dispersion

The mean direction of the distribution is given by μ_p at $p = 1$. The mean resultant length is used to quantify the circular data's spread around the mean and is given by ρ_p at $p = 1$. More data are concentrated towards the mean, the closer the mean resultant is to 1. To obtain the parameters μ_p and ρ_p, we can use the representation

$$\mu_p = \text{atan}\left(\alpha_p {\beta_p}^{-1}\right) \text{ and } \rho_p = \sqrt{{\alpha_p}^2 + {\beta_p}^2}$$

Using this we get

$$\mu_p = atan\left(\frac{\theta}{p}\right)$$

$$\rho_p = \frac{\theta\left(\theta^2 + p^2\right)^{\frac{1}{2}}}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + p^2\right)}$$

The mean direction and the resulting length of $DTWE(\theta, a, b)$ distribution is provided, respectively by

$$\mu = \mu_1 = atan(\theta) \tag{10}$$

and

$$\rho = \rho_1 = \frac{\theta\left(\theta^2 + 1\right)^{\frac{1}{2}}}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2 + 1\right)} \tag{11}$$

Similar to the mean in linear models, the mean direction vector offers insight on the distribution's mean. This vector's length represents the typical standard deviation or variance in linear models, and it serves as a measure of dispersion around the mean. The DTWE distribution's circular variance and standard deviation are provided, respectively, by

$$V = 1 - \rho$$

$$V = 1 - \frac{\theta\left(\theta^2+1\right)^{\frac{1}{2}}}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2+1\right)} \tag{12}$$

$$\sigma = \sqrt{-2\ln\rho}$$

$$\sigma = \sqrt{-2\ln\left(\frac{\theta\left(\theta^2+1\right)^{\frac{1}{2}}}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2+1\right)}\right)} \tag{13}$$

Then the central trigonometric moments are
$\overline{\alpha_p} = \rho_p \cos\left(\mu_p - p\mu_1\right)$ and $\overline{\beta_p} = \rho_p \sin\left(\mu_p - p\mu_1\right)$

$$\overline{\alpha_p} = \frac{\theta\left(\theta^2+p^2\right)^{\frac{1}{2}}}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2+p^2\right)} \cos\left(atan\left(\frac{\theta}{p}\right) - patan\theta\right) \tag{14}$$

$$\overline{\beta_p} = \frac{\theta\left(\theta^2+p^2\right)^{\frac{1}{2}}}{\left(e^{-a\theta} - e^{-b\theta}\right)\left(\theta^2+p^2\right)} \sin\left(atan\left(\frac{\theta}{p}\right) - patan\theta\right) \tag{15}$$

The quantile function of DTWE can be easily obtained from the solution of equation $F(\theta) - u = 0$ with respect to θ as

$$Q(u) = \frac{-1}{\theta}\ln\left(e^{-a\theta}(1-u) + ue^{-b\theta}\right) \tag{16}$$

where $u \in (0,1)$.

The median direction of a circular distribution is a value M that satisfies

$$\int_0^M f(\theta)d\theta = \int_M^{2\pi} f(\theta)d\theta = 0.5$$

The median of DTWE can be effectively obtained by the equation $Q(0.5) = M$ and is given as

$$M = \frac{\ln 2}{\theta} - \frac{1}{\theta}\ln\left(e^{-a\theta} + e^{-b\theta}\right) \tag{17}$$

3.3 Skewness and Kurtosis

One way to determine the asymmetry is by calculating the skewness coefficient.

The skewness coefficient of the DTWE distribution is provided by

$$\eta_1 = \frac{\overline{\beta}_2}{V^{\frac{3}{2}}}$$

$$\eta_1 = \frac{\frac{\theta\left(\theta^2+4\right)^{\frac{1}{2}}}{\left(e^{-a\theta}-e^{-b\theta}\right)\left(\theta^2+4\right)}\sin\left(\mu_2-2\mu_1\right)}{\left[1-\frac{\theta\left(\theta^2+1\right)^{\frac{1}{2}}}{\left(e^{-a\theta}-e^{-b\theta}\right)\left(\theta^2+1\right)}\right]^{\frac{3}{2}}} \tag{18}$$

The kurtosis coefficient of the DTWE distribution is provided by

$$\eta_2 = \frac{\overline{\alpha}_2-\rho^4}{V^2}$$

$$\eta_2 = \frac{\overline{\alpha}_2-\rho^4}{\left[1-\frac{\theta\left(\theta^2+1\right)^{\frac{1}{2}}}{\left(e^{-a\theta}-e^{-b\theta}\right)\left(\theta^2+1\right)}\right]^{2}} \tag{19}$$

For different values of θ, Table 1 shows some estimated values for various characteristic features of the DTWE distribution. The data distribution becomes more symmetric and concentrated in the mean direction as θ increases.

Table 1. Computed values of different characteristics of the DTWE distribution across varying θ

		Parameter θ			
		0.1	**0.5**	**1**	**2**
Characteristic Trigonometric moments	α_1	0.05462	0.3163953	0.5782588	0.8149259
	α_2	0.01375725	0.09305745	0.2313035	0.5093287
	β_1	0.5462035	0.6327907	0.5782588	0.4074629
	β_2	0.2751449	0.3722298	0.4626071	0.5093287
Central Trigonometric moments	$\overline{\alpha_1}$	0.5489277	0.7074815	0.8177815	0.9111148
	$\overline{\alpha_2}$	0.2724207	0.2977839	0.2313035	0.1018657
	$\overline{\beta_1}$	0.00000	0.00000	0.00000	0.00000
	$\overline{\beta_2}$	-0.04099932	-0.2419494	-0.4626071	-0.7130602
Mean direction	μ	0.09966865	0.4636476	0.7853982	1.107149
Resultant length	ρ	0.5489277	0.7074815	0.8177815	0.9111148
Circular Variance	**V**	0.4510723	0.2925185	0.1822185	0.08888516
Circular Standard deviation	σ	1.095252	0.831918	0.6342872	0.4314773
Coefficient of Skewness	η_1	-0.1353342	-1.529306	-5.947349	-26.90805
Coefficient of Kurtosis	η_2	0.8926592	0.5522394	-6.503668	-74.33013

3.4 Invariance Properties

Circular data probability distributions often conform to a universal framework, defined by a closed-form density function and the unit circle as the domain. Nonetheless, circular data possess unique characteristics that must be acknowledged in any research. Circular data is defined by the lack of a fixed zero or starting point, with its inherent direction assigned arbitrarily. While established circular distributions may be effective, neglecting the considerations of initial direction and orientation can lead to erroneous findings. For instance, most studies define the zero point as the eastward direction and quantify rotation positively, whilst others identify north as the zero direction and measure rotation clockwise. Circular distributions must exhibit two characteristics: invariance under alterations of initial direction (ICID) and invariance under modifications of system orientation (ICO). These qualities are crucial for inference that is independent of the reference frame[8].

A circular distribution characterized by the probability density function $g(\theta)$ is ICID and ICO if and only if, for $\delta = \{-1, 1\}$, $\xi \epsilon 2\pi j_{j=0}^{l-1}$, and $\theta^* = \delta(\theta + \xi)$, the probability density functions $g(\theta)$ and $g(\theta^*)$ belong to the same parametric family, meaning their characteristic functions must be identical in form. The characteristic function value that generates the wrapped distribution corresponds to the p^{th} trigonometric moment of the wrapped distribution. Therefore, an alternative representation of the p^{th} trigonometric moment for the DTWE distribution is as follows:

$$\varphi_{\theta^{i}}(p) = e^{ip\delta\xi}\varphi_{\theta}(p\delta)$$

$$= \frac{\theta}{(e^{-a\theta} - e^{-b\theta})(\theta^2 + p^2)} [\cos(p\delta\xi) - i\sin(p\delta\xi)][\theta + ip] \tag{20}$$

Since only when $\delta = 1$ and $\xi = 0$ do the real and imaginary components of equations (3) and (15) equalize, the DTWE distribution is not both ICID and ICO. The distribution's invariant form is provided by

$$g(x^*, \theta, \delta, \xi) = \frac{\theta e^{-\theta(\delta x^* - \xi)}}{(e^{-a\theta} - e^{-b\theta})} \tag{21}$$

3.5 Estimation

This section will address the estimation of the unknown parameter using three distinct methods such as Maximum Likelihood, Least Squares, and Weighted Least Squares.

3.5.1. *Maximum likelihood estimation*

Let $x_1, x_2, \ldots x_n$ be a random sample of size *n* drawn from the DTWE distribution with pdf $DTWE(\theta, a, b)$, then the log likelihood function denoted by "L" is given by

$$L\left(\frac{\theta}{x_1, x_2, \ldots x_n}\right) = \prod_{i=1}^{n} f(x, \theta)$$

$$= n\log\theta - \theta\sum_{i=1}^{n} x_i - n\log\left(e^{-a\theta} - e^{-b\theta}\right) \tag{22}$$

The log-likelihood function's derivative about the parameter θ is

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum x_i - n\frac{ae^{-a\theta} - be^{-b\theta}}{e^{-a\theta} - e^{-b\theta}} \tag{23}$$

Taking the partial derivative of the above with respect to θ, then equating it to zero

$$\frac{\partial \log L}{\partial \theta = 0}$$

Equating **equation (23)** to zero and solving with respect to θ gives the MLEs of θ. There is no explicit solution so we can use numerical methods to solve them.

3.5.2. *Least squares estimation*

To derive the least squares estimates of the DTWE (θ, a, b) distribution, we will look at the ordered random sample from this distribution. The least squares estimate of the unknown parameter of the DTWE (θ, a, b) distribution, denoted as $\hat{\theta}_{LS}$ is derived by minimizing

$$\sum_{j=1}^{n}\left(\frac{e^{-a\theta} - e^{-x_{(i)}\theta}}{e^{-a\theta} - e^{-b\theta}} - \frac{j}{n+1}\right)^2$$

with respect to θ where $\frac{j}{n+1}$ is the expectation of the empirical distribution function of the ordered data.

A prominent version of the least squares method is the weighted least squares, which exhibits reduced bias compared to regular least squares. The WLS estimates of the parameter of the DTWE (θ, a, b) distribution is derived by reducing

$$\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left(\frac{e^{-a\theta} - e^{-x_{(i)}\theta}}{e^{-a\theta} - e^{-b\theta}} - \frac{j}{n+1} \right)^2$$

with respect to θ.

3.6 Simulation study

This section presents Monte Carlo simulation studies to demonstrate and compare the estimated performances of the MLE, LS, and WLS estimators developed in the previous part of this paper. Here, we derive 10,000 DTWE samples to examine their performance and the behaviour of the maximum likelihood estimators along with least square estimates and weighted least square estimates. With $n = 30, 50, 100, 1000$ as the sizes of the samples, the parameter θ taken as 0.5, 1 and 2, we perform the simulation study to get the mean value, Bias, mean squared error (MSE) and the standard deviation (SD).

As demonstrated in Table 2, bias, the SD and MSE values decline as the sample size increases for all values of the parameter. These results support the validity and precision of the estimations, as well as their consistency. As sample size increases and reaches 1000, the average estimate value is much closer to the true value thereby showing the consistency and unbiasedness of the MLE. Further, the findings of the simulation indicate that both LS and WLS estimators possess these attributes as well. But, as seen in the simulated results of this study, the ML estimators are more precise than both LS and WLS estimators because the ML estimators have lower MSE values than the LS and WLS estimators.

3.7 Applications

This section analyzes two circular data sets using the inferential approaches outlined in the previous sections**.**

3.7.1. Orientation of pebbles dataset

Pebble orientation in river systems is influenced by shape, size, flow velocity, and riverbed geomorphology. Larger, elongated pebbles tend to align with orientation to minimize resistance, while smaller or flat pebbles in low-energy conditions settle randomly. High-energy flows and steep slopes will orient pebbles downstream; turbulence or interactions with sediment may disrupt this orientation and lead to different orientations. Rolling, sliding, and imbrication are some of the transport modes that affect the orientation of pebbles. Zero-degree orientations from "Horizontal axes of 100 outwash pebbles" indicate alignment with gentle flows, suggesting deposition in low-energy environments. The removal of this layer illuminates the inclined pebbles left from turbulent conditions, providing perspectives on stronger hydrodynamic forces and past environmental dynamics.

Excluding the zero-degree pebbles from the dataset allows for a more nuanced analysis of the remaining data, emphasizing the distribution of inclined pebbles and their correlation with more dynamic depositional processes. This exclusion alters the interpretation of stream velocity and energy levels by removing pebbles that represent smoother, low-energy conditions and focusing on those that reflect a more turbulent environment. Such an approach can help reconstruct past environmental conditions, as pebbles with more angular or inclined orientations are often associated with faster water flow and more rapid sediment deposition. Using models like the exponential distribution can further enhance this analysis by estimating the distance or time until pebbles reach a specific orientation, thereby providing a statistical framework to interpret the hydrodynamic forces at play during sediment deposition.

To demonstrate the applicability of our proposed model in this study, we will consider one dataset from Krumbein in[9]; which contains direction of samples of 100 washout pebbles from an outwash terrace along the Fox River near Cary, Illinois, near Lake Wisconsin. To assess the comparative adequacy of our proposed model against established models in the literature, we utilize wrapped Akash (WA)[10], wrapped Shanker (WS)[11], wrapped Lindley (WL)[12] and wrapped exponential (WE)[13] distributions. In order to make a comparison, Akaike information criterion (AIC), Bayesian information criterion (BIC), Akaike information criterion corrected (AICC), consistent AIC (CAIC), Hannon and Quinn's information criterion (HQIC) and Kolmogorov-Smirnov (KS) values for the DTWE, WA, WS, WL and WE distributions are given in Table 3. According to **Table 3**, the DTWE distribution has the smallest AIC, BIC, AICC, CAIC, HQIC and KS than the others. Thus, we can clearly say that the DTWE distribution gives better fit than the wrapped Akash (WA), wrapped Shanker (WS), wrapped Lindley (WL) and wrapped exponential (WE) distributions. Figure 3 illustrate the circular plot of the data.

Table 2. Simulated results for the DTWE distribution at varying θ values

	Method	n	$\hat{\theta}$	Bias	MSE	SD
$\theta = 0.5$	MLE	30	0.467638	0.032362	0.016456	0.12413
		50	0.457419	0.042581	0.010374	0.092526
		100	0.453188	0.046812	0.006591	0.066334
		1000	0.449107	0.050893	0.003001	0.020263
	LS	30	0.652715	1.152715	4.734101	1.845359
		50	0.459649	0.959649	1.838769	0.958041
		100	0.365473	0.875473	1.193729	0.666847
		1000	0.376481	0.866481	0.860721	0.304143
	WLS	30	0.135685	0.635685	0.4043	0.104147
		50	0.119926	0.619926	0.387147	0.082432
		100	0.107726	0.607726	0.376126	0.053276
		1000	0.090082	0.590082	0.359044	0.014284
$\theta = 1$	MLE	30	1.030946	0.030946	0.041427	0.201171
		50	1.020811	0.020811	0.024075	0.153758
		100	1.005816	0.005816	0.011088	0.105141
		1000	1.000017	0.000017	0.000994	0.031531
	LS	30	0.397918	0.602082	0.37106	0.092503
		50	0.400326	0.599674	0.363394	0.061525
		100	0.399903	0.600097	0.361453	0.03655
		1000	0.400913	0.599087	0.359024	0.010906
	WLS	30	0.230274	0.877412	0.617297	0.157541
		50	0.195612	0.835711	0.665505	0.135884
		100	0.164289	0.804388	0.708698	0.101412
		1000	0.122588	0.769726	0.771298	0.038031
$\theta = 2$	MLE	30	2.072132	0.072132	0.15419	0.385989
		50	2.053967	0.053967	0.088556	0.292649
		100	2.014791	0.014791	0.039566	0.198361
		1000	2.001512	0.001512	0.003598	0.059961
	LS	30	0.348627	1.651373	2.730285	0.057041
		50	0.345793	1.654207	2.738208	0.042515
		100	0.346689	1.653311	2.734356	0.030321
		1000	0.347382	1.652618	2.731239	0.009662
	WLS	30	0.50126	1.49874	2.495674	0.108375
		50	0.485765	1.578939	2.371676	0.09618
		100	0.462626	1.537374	2.302159	0.090325
		1000	0.421061	1.514235	2.257967	0.051247

Table 3. An overview of the orientation patterns observed in the pebbles dataset

Model	Estimates	AIC	BIC	AICC	HQIC	CAIC	K-S
DTWE	$\hat{\theta} = 1.5538$ $\hat{a} = 0.3490$ $\hat{b} = 2.7925$	217.039	224.855	217.28950	201.876	194.224	0.6100
WA	$\hat{\theta} = 1.2570$	229.229	231.834	229.270	224.174	221.624	0.9024
WS	$\hat{\theta} = 1.1499$	236.566	239.171	236.606	231.511	228.960	0.9091
WL	$\hat{\theta} = 1.1811$	236.638	239.243	236.679	231.583	229.030	0.9102
WE	$\hat{\theta} = 0.7925$	240.223	242.828	240.264	235.168	232.618	0.9127

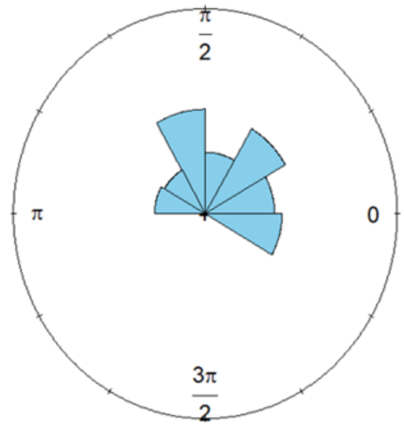

Fig 3. Circular Rose Diagram of the data

3.7.2. Turtle dataset

The dataset contributed by Dr. E. Gould of the Johns Hopkins School of Hygiene[14] comprises directional measurements indicating the trajectories of turtle's post-treatment intervention. This study seeks to examine the possible presence of a favoured direction in the movement patterns of turtles. It is essential to acknowledge that certain individuals may demonstrate misunderstanding between forward and backward motions, which could complicate the assessment of their directional preferences. Analyzing these directional data provides insights into the turtles' behavioural responses to treatment and their navigational preferences, enhancing our understanding of the ecological and biological elements that influence their movement dynamics. Through the examination of the directional patterns displayed by turtles, researchers can deduce the impact of the surrounding landscape, encompassing elements such as terrain, vegetation, and hydrology, on their movements. Turtles, while traversing their habitat, may facilitate sediment redistribution, hence affecting local geomorphology through their interactions with soil and substrate.

The removal of orientations near zero in the turtle movement dataset enhances the analysis by excluding low-energy or neutral movements. These directions probably indicate instances where turtles show negligible directional bias or indecision and hence do not substantially enhance the comprehension of the proposed preferable direction. The omission of such data enhances the emphasis on turtles exhibiting clearer directional preferences, minimizing the likelihood of confusion between forward and backward movements. This truncation for a more precise examination of whether the turtles exhibit a genuine preferential orientation post-treatment. The study demonstrates more pronounced trends in the data by reducing the impact of ambiguous or neutral movements. To facilitate comparison, the AIC, BIC, Criterion AICC, CAIC, HQIC, and KS values for the DTWE, WA, WS, WL, and WE distributions are presented in Table 4. Table 4 indicates that the DTWE distribution exhibits the lowest values for AIC, BIC, AICC, CAIC, HQIC, and KS compared to the other distributions. Consequently, it is evident that the DTWE distribution outperforms the WA, WS, WL, and WE distributions in terms of fit. The Figure 4 displays the circular plot of the data.

Table 4. An overview of the direction patterns observed in the turtles dataset

Model	Estimates	AIC	BIC	AICC	HQIC	CAIC	K-S
DTWE	$\hat{\theta} = 0.5903$ $\hat{a} = 0.2268$ $\hat{b} = 6.1086$	214.556	221.548	214.889	199.762	192.564	0.6175
WA	$\hat{\theta} = 1.1185$	247.155	249.486	247.209	242.223	239.824	0.9822
WS	$\hat{\theta} = 0.7601$	239.664	241.995	239.718	234.733	232.334	0.9747
WL	$\hat{\theta} = 0.7482$	241.417	243.748	241.471	236.486	234.087	0.9760
WE	$\hat{\theta} = 0.4228$	243.294	245.625	243.348	238.363	235.964	0.9656

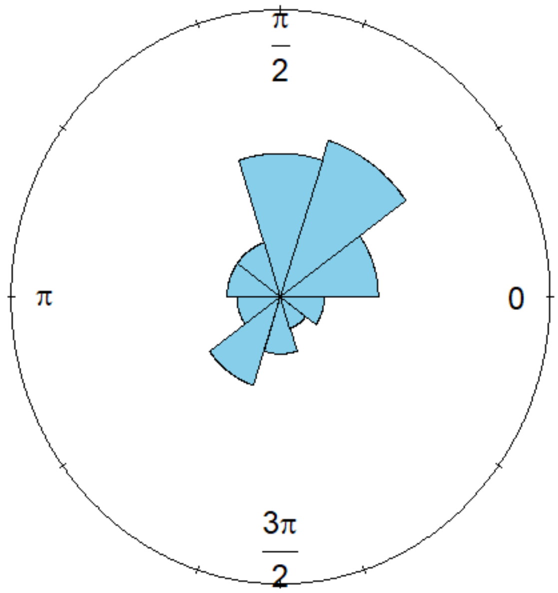

Fig 4. Circular Rose Diagram of the data

4 Conclusion

This study highlights the pivotal role of circular statistics in analyzing directional data, demonstrating its distinct advantages over conventional linear methods. The study improves the accuracy of ecological and geological analyses by creating a new circular probability distribution that is tailored to the properties of angular data. This is especially useful for figuring out the direction of outwash pebbles from the Fox River and the preferred direction of turtle's post-treatment. This research demonstrates that the newly developed circular probability distribution offers a better fit for the outwash pebble data and the turtle dataset than previous distributions documented in the literature. The comparison metrics, such as AIC, BIC, AICC, CAIC, HQIC and KS values, consistently indicate that the new distribution surpasses alternatives in accuracy and model fit. We have applied the statistical techniques, including maximum likelihood estimation, Least Squares, and Weighted Least Squares, to estimate the unknown parameters. This validates the efficacy of the innovative method, especially the incorporation of truncation in circular distributions, as a significant advancement in the domain of circular statistics. Our research strength is the development of a strong model for improving parameter estimation accuracy, especially in applications using circular data. However, one downside of the method is that it is computationally intensive, which may prove daunting for larger datasets. Hence, future work could be geared towards optimizing the algorithm so that it becomes computationally less heavy but retains model accuracy. This study illustrates the effectiveness of the proposed distribution in encapsulating the directional characteristics of ecological and geological data, presenting possible applications in several scientific fields.

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References

1) Jammalamadaka SR, Sengupta A. Topics in circular statistics. World Scientific Publishing Co. Pte. Ltd. 2001.
2) Dodge Y. The Oxford Dictionary of Statistical Terms. Oxford University Press. 2003.
3) Pica A, Lämmle L, Burnelli M, Monte MD, Donadio C, Faccini F, et al. Urban Geomorphology Methods and Applications as a Guideline for Understanding the City Environment. *Land*. 2024;13(7):1–27. doi:https://doi.org/10.3390/land13070907.
4) Çimen Ö. Development of a Circular Building Lifecycle Framework: Inception to Circulation. *Results in Engineering*. 2023;17:1–20. doi:https://doi.org/10.1016/j.rineng.2022.100861.

5) Singh SK, Singh U, Sharma VK. The Truncated Lindley Distribution: Inference and Application. *Journal of Statistics Applications & Probability*. 2014;3(2):219–228. doi:https://doi.org/10.12785/jsap/030212.
6) Thiamsorn I, Aryuyuen S. The Truncated Ishita Distribution: Properties and Application. *International Journal of Applied Mathematics*. 2020;33(1):99–108. doi:https://doi.org/10.12732/ijam.v33i1.8.
7) Zinhom E, Nassar MM, Radwan SS, Elmasry A. The wrapped XLindley distribution. *Environmental and Ecological Statistics*. 2023;30(4):669–686. doi:https://doi.org/10.1007/s10651-023-00579-2.
8) Mastrantonio G, Lasinio GJ, Maruotti A, Calise G. Invariance properties and statistical inference for circular data. *Statistica Sinica*. 2019;29(1):67–68. doi:https://doi.org/10.5705/ss.202016.0067.
9) Krumbein WC. Preferred Orientation of Pebbles in Sedimentary Deposits. *J Geol*. 1939;47:673–706. Available from: https://www.journals.uchicago.edu/doi/abs/10.1086/624827.
10) Al-Khazaleh AM. Wrapped Akash Distribution. *Electronic Journal of Applied Statistical Analysis*. 2021;p. 305–317. doi:http://dx.doi.org/10.1285/i20705948v14n2p305.
11) Al-Meanazel ATR, Al-Khazaleh AMH. Wrapped shanker distribution. *Italian Journal of Pure And Applied Mathematics*. 2021;46:184–194. Available from: https://ijpam.uniud.it/online_issue/202146/17%20Meanazel-Khazaleh.pdf.
12) Joshi S, Jose KK. Wrapped Lindley distribution. *Communication in Statistics- Theory and Methods*. 2017;47(5):1013–1021. doi:https://doi.org/10.1080/03610926.2017.1280168.
13) Jammalamadaka SR, Kozubowski TJ. New families of wrapped distributions for modeling skew circular data. *Commun Stat Theory Methods*. 2004;33(9):2059–2074. Available from: https://www.tandfonline.com/doi/abs/10.1081/STA-200026570.
14) Stephens MA. Techniques for directional data. Dept. of Statist., Stanford University. 1969.