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ORIGINAL RESEARCH

Generalization of IP Domination Number for Complement Graphs

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Abstract

Objectives: This research introduces Isolate Pendant domination parameter, a novel concept applied to complement graphs. **Methods:** This investigation explores Isolate Pendant domination in complement graphs, characterizing the minimum dominating set and satisfying isolate pendant domination conditions within induced graphs, by applying this parameter in $\bar{G}$ to yield insightful results. **Findings:** This study determines the minimum Isolate Pendant Dominating set (IPD-set) in complement graphs, characterizing exact values for complement graphs. Moreover, generalized results are derived for complement graphs. **Novelty:** This study pioneers new insights into the behavior of complement graphs, presenting exact values and results on the Isolate Pendant domination number, thereby making a profound impact on the advancement of graph theory.

Keywords: Isolate vertex; Pendant vertex; Complement graphs; IP Dominating set; IP domination number.

1 Introduction

In this study, restrict our graphs to be finite, simple, and undirected graphs denoted as G, which have no loops and no multiple edges. For general graph notations, follow[1,2]. In the graph $G = (V, E)$, the degree of a vertex v is defined to be the number of edges incident with v and is denoted by $deg(v)$. The minimum of $\{deg(v) \mid v \in V(G)\}$ is denoted by $\delta(G)$ and the maximum of $\{deg(v) \mid v \in V(G)\}$ is denoted by $\triangle(G)$. The graph G is said to be totally disconnected if for all $u, v \in V$ such that there is no edge $e = uv \in V$, that is $|e| = \emptyset$. If S is a subset of G, then the sub graph induced by S is the graph that has S as its set of vertices and contains all the edges of G that have both endpoints in S. It is denoted by $< S >$. A graph G is said to be regular if $\delta(G) = \triangle(G)$. A vertex v of a graph G is called a cut vertex if its removal increases the number of components. A bridge or cut edge of a graph is an edge whose removal increases the number of components. A vertex $v \in V(G)$ is called a pendant vertex if it has degree one. A vertex $v \in V(G)$ is called an isolated vertex if it has a degree zero. An edge incident to a pendant vertex is called a pendant edge. The connected acyclic graph is called a tree. The complement of a graph G is a graph $\bar{G}$ on the same vertices such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G. That is, the complement graph $\bar{G}$ is a graph with vertex set $V\,(G)$ and two vertices are adjacent in $\bar{G}$ if and only if $d_G(u,v)\ >\ 1$. From [3] the following definitions are observed, A set $D \subseteq V(G)$ is said to be a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of all dominating set is known as the domination number, it is denoted by $\gamma\,(G)$. Assigning colors to the vertices of a graph so that no two adjacent vertices share the same color. The chromatic number of a graph, denoted by $\chi(G)$, is the minimum number of colors required to properly color the graph.

The Isolate domination is an interesting concept introduced in [4]. A dominating set S of a graph G is said to be an isolate dominating set of G if $\langle S\rangle$ has at least one isolated vertex. An isolate dominating set S is said to be a minimal isolate dominating set if no proper subset of S is an isolate dominating set. The minimum cardinality of a minimal isolate dominating set of G is called the isolate domination number $\gamma_0(G)$. The concept of pendant domination was introduced in [5] and further studied in [6]. A dominating set S in G is called a pendant dominating set if $\langle S\rangle$ contains at least one pendant vertex. The least cardinality of a pendant dominating set is called the pendant domination number of G denoted by $\gamma_{pe}\,(G)$. Motivated by these concepts, the new domination parameter called Isolate Pendant Domination was introduced in [7]. The Isolate Pendant Dominating set (IPD-set) was determined and the Isolate Pendant Domination (IP Domination) number was found for some standard graphs and generalized graphs. Various domination parameters were discussed for complement graphs in [8,9]. Inspired by these ideas, this paper presents the exact values for the IP domination number for complement graph of some standard graphs and special graphs. Initial work has been done to establish the relation between $\gamma_{01}(G)$ & $\gamma_{01}(\overline{G})$. The IP domination number for the complement graph of G was observed in terms of chromatic number and minimum degree. Additionally, the IP domination number has been generalized for the complement graph of G.

2 Methodology

The following definition enabled us to discover significant findings.

Definition 2.1. IP Dominating set

A dominating set $I\ \subseteq\ V\,(G)$ is said to be the IPD-set if all the vertices in $\langle I\rangle$ is of degree either $0\ or\ 1$ such that there must exist two vertices $u,v\ \in\ \langle I\rangle$ where $deg\,(u) =\ 0\ \&\ deg(v)\ =\ 1$

Definition 2.2 . IP **Domination Number**

IP domination number of G, denoted by $\gamma_{01}(G)$ is the minimum cardinality of the IPD-set of G.

3 Isolate Pendant Domination Number for Complement of Standard Graphs

Theorem 3.1. Let P_n be a path with $n \geq 3$ then $\gamma_{01}(\overline{P}_n) = 3$.

Proof: Let $\overline{P}_n$ be the complement graph with the same number of vertices in P_n say $\{v_1, v_2, \ldots, v_n\}$ and the edges will be distinct from P_n so any two vertices are adjacent in $\overline{P}_n$ if they are not adjacent in P_n. This leads to a connected graph of $\overline{P}_n\,(n \geq 3)$. Here the subset $S = \{v_1, v_2, v_3\}\,.\subseteq V\,(P_n)V(\overline{P}_n)$. Now the dominating vertices will be the subset S, its enough to dominate all the vertices in $\overline{P}_n$. Thus, the *IP dominating set*$I = \{v_1, v_2, v_3\}$ and then the vertices of the graph induced by I are of $deg \leq 1$ and it has one isolated vertex and two pendant vertices. Therefore IP domination number $\gamma_{01}(\overline{P}_n) = 3$ for $n \geq 3$.

Theorem 3.2. For any cycle C_n with $n\ \geq 4$,$\gamma_{01}(\overline{C}_n) = 3$.

Proof: Let $\overline{C_n}$ be the complement graph with the same number of vertices in C_n say $\{u_1, u_2, \ldots, u_n\}$ and the edges distinct from C_n, if it has an edge then it is not adjacent in C_n.

The following cases are considered:

Case 1: For $n = 4$

The graph $\overline{C_4}$ is found to be disconnected, consisting of two components labeled X and Y.

Here $V\left(\overline{C}_4\right) = V(X) \cup V(Y), V(\overline{C}_4) = \{u_1, u_2, v_1, v_2\}$ and $E(\overline{C}_4) = \{u_1u_2, v_1v_2\}$. It is possible to choose any 3 vertices of $V(\overline{C}_4)$ which satisfies the IP domination condition. Thus $\gamma_{01}(\overline{C}_4) = 3$.

Case 2: For $n \geq 5$

The complement graph $\overline{C_n}$ is a connected graph with a vertex set $V(\overline{C}_n) = \{u_1, u_2, ..., u_n\}$. Here the subset $S = \{u_1, u_2, u_3\} \subseteq V(\overline{C}_n)$. Now the dominating vertices will be the subset S it is enough to dominate all the vertices in $\overline{C}_n$. Thus, the *IP dominating set* $I = \{u_1, u_2, u_3\}$ and then the vertices of the graph induced by I are of $deg \leq 1$ and it has one isolated vertex and two pendant vertices. Therefore IP domination number $\gamma_{01}(\overline{C}_n) = 3$ for $n \geq 5$.

Theorem 3.3. For any complete graph K_n with $n \geq 3$, $\gamma_{01}(\overline{K}_n)$ does not exist.

Proof: Let $\overline{K}_n$ be the complement graph of the complete graph K_n which has the same number of vertices as in K_n and $\overline{K}_n$ contains no edges. That is, $\overline{K}_n$ becomes totally disconnected graph. Then clearly it can be said that $< I >$ does not have the pendant vertices, so the IP domination number of $\overline{K}_n$ does not exist.

Theorem 3.4. For a wheel graph $(n \geq 6), \gamma_{01}\left(\bar{W}_n\right) = 3$.

Proof: The wheel graph has a universal vertex u connected to all the vertices of W_n. Let $\overline{W}_n$ be the complement graph with the same number of vertices in W_n say $\{u, v_1, ..., v_n\}$ and the edges distinct from W_n, any two vertices in $\overline{W}_n$ have adjacency if they are not adjacent in W_n. The resultant graph is a disconnected graph with two components which is an isolated vertex and a connected graph. Now the *IP* dominating vertices will be isolated vertex u in $\bar{W}_n$ and for the connected graph choose two adjacent vertices say v_1, v_4 in $\bar{W}_n$, which have distance three in W_n. Thus the *IP* dominating set $I = \{u, v_1, v_4\}$. Therefore, $\gamma_{01}\left(\bar{W}_n\right) = 3$.

4 Isolate Pendant Domination Number for Complement of Special Graphs

Theorem 4.1. For any ladder graph L_n of order $n \geq 3, \gamma_{01}(\overline{L}_n) = 3$.

Proof: Let $\overline{L}_n$ be the complement graph of the ladder graph and it is a connected graph with the same number of vertices as in L_n say $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and edges distinct from L_n any two vertices are adjacent in $\overline{L}_n$ if they are not adjacent in L_n. Now the dominating vertices will be any two adjacent vertices say u_1, v_2 in $\overline{L}_n$ which has distance two in L_n and choose one more vertex say v_1 which is not adjacent to the adjacent vertices u_1 and v_2. Thus $I = \{u_1, v_1, v_2\}$ is the minimum IPD-set. Therefore $\gamma_{01}(\overline{L}_n) = 3$.

Theorem 4.2. For any helm graph H_n of order $n \geq 3, \gamma_{01}(\overline{H}_n) = 3$.

Proof: Let $\overline{H}_n$ be the complement graph of the helm graph as a connected graph having the vertices of the cycle say $v_1, v_2, \ldots, v_n$, each node of the cycle connected to the vertex u and each node of the cycle adjoining a pendant edge with vertex set $\{u, v_1, v_2, ..., v_n, w_1, w_2, ..., w_n\}$. Let the minimum IPD-set be any two adjacent vertices say $v_i, v_j \in \{ v_1, v_2, .., v_n\}$ in H_n and one vertex say $w_k \in \{w_1, \ldots, w_n\}$ such that $k = i \; or \; j$ which satisfies the IP domination condition. Thus IPD-set $I = \{v_i, v_j, w_k\}$. Therefore $\gamma_{01}(\overline{H}_n) = 3$.

Observation 4.3.

1. For Complete Bipartite graph $K_{m,n} (m, n \geq 2), \gamma_{01}\left(\overline{K_{m,n}}\right) = 3$
2. For Flower graph FL_n of order $n \geq 3, \gamma_{01}\left(\overline{FL_n}\right) = 3$
3. For Banana tree graph $B(k, n) (k \geq 2, n \geq 3), \gamma_{01}(\overline{B(k, n)}) = 3$
4. For Triangular Ladder graph TL_n of order $n \geq 3, \gamma_{01}\left(\overline{TL_n}\right) = 3$.
5. For Coconut tree graph $C(m, n) (m, n \geq 2), \gamma_{01}(\overline{C(m, n)}) = 3$
6. For n- Barbell graph B_n of order $n \geq 3, \gamma_{01}\left(\overline{B_n}\right) = 3$.
7. For Gear graph G_n of order $n \geq 3, \gamma_{01}\left(\overline{G_n}\right) = 3$.

Observation 4.4. If $\overline{G}$ is a connected graph then $\gamma_{01}(G) \geq \gamma_{01}(\overline{G})$.

Observation 4.5. Let G be a connected graph. Then the following condition holds:

$$(i) \qquad \gamma_{01}\overline{(G)} \leq \delta(G) + 2$$

$$(ii) \qquad \gamma_{01}\overline{(G)} \leq \chi(G) + 1$$

Observation 4.6. If graph *G* has the IP domination number then $\overline{G}$ also has the IP domination number. Converse need not be true.

Example: A star graph $K_{1,n}$ is a complete bipartite graph with a single vertex belonging to one set and all the remaining vertices belonging to the other set. For $K_{1,n}$ IP domination number doesn't exist but the IP domination number exists for $\overline{K}_{1,n}$.

Analyzing the above theorems and observation, exact bounds for an arbitrary graph *G* have been found.

5 General Results of Isolate Pendant Domination Number for Complement Graphs

Theorem 5.1. If $\overline{G}$ is a connected graph then $\gamma_{01}\overline{(G)} = 3$
Proof: Let G be any graph $n \geq 5$. If $\overline{G}$ is a connected graph with the same number of vertices in G and the edge set will be two vertices are adjacent in $\overline{G}$ which are not adjacent in G. Then choose any two adjacent vertices say $u, v \in \overline{G}$ which has distance two in G and has no maximum degree in G and also choose one more vertex say w in $\overline{G}$ which is not adjacent to those adjacent vertices u & v. Thus the dominating set $I = \{w, u, v\}$ of $\overline{G}$ satisfy the IP domination condition is that the vertices of the graph induced by I is of $deg \leq 1$ and it has at least one isolated vertex and pendant vertex. Therefore $\gamma_{01}\overline{(G)} = 3$.
Corollary 5.1.1. If $\overline{G}$ is the complete graph K_n then IP domination does not exist.
Proof: Let $\overline{G}$ be the complete graph then the dominating set will be a singleton set. That is the set has one vertex which is enough to dominate all other vertices. According to the above theorem, vertex selection is not possible, and thus the IP domination condition is not satisfied. Therefore IP domination number does not exist.
Note 5.1.1.1. Every IP dominating set is a dominating set. Converse need not be true. corollary 5.1.1. is an example of this, it is a dominating set but not an IP dominating set.
Theorem 5.2. If $\overline{G}$ is a disconnected graph having j components of the complete graph then $\gamma_{01}\overline{(G)} = j+1$, where j is the number of components.
Proof: Let G be any graph. For this, if $\overline{G}$ is a disconnected graph that has j components of the complete graphs say $K_{n_1}, K_{n_2}, ..., K_{n_j}$. Choosing one vertex from each component yields a dominating set with j vertices it does not satisfy the IP domination condition, so choose one more vertex from any one of the component. Then the vertices of the graph induced by the set are of $deg \leq 1$ and it has at least one isolated vertex and pendant vertex. Hence the dominating set becomes the IP dominating set and $\gamma_{01}\overline{(G)} = j+1$.
Corollary 5.2.1. If G is a bipartite graph then $\overline{G}$ is a disconnected graph having exactly two components, thus $\gamma_{01}\overline{(G)} = 3$.
Proof: Let G be a bipartite graph, consisting of two distinct vertex sets. Then $\overline{G}$ is disconnected with exactly two components say K_{n_1} and K_{n_2}. Therefore, from the above theorem, it becomes, $\gamma_{01}\overline{(G)} = 3$.
Theorem 5.3. If $\overline{G}$ is disconnected graph is of the form $(mK_1 \cup H)$ where m is the number of universal vertex in G then

$$\gamma_{01}\left(\bar{G}\right) = \begin{cases} m+2, & if\ H\ is\ complete\ graph \\ m+3 & if\ H\ is\ any\ connected\ graph,\ H \cong \bar{C_5} \\ m+2 & otherwise \end{cases}$$

Proof: Let G be a graph, and the following cases arise.
Case 1: H is a complete graph.
If $\overline{G}$ is a disconnected graph of the form $(mK_1 \cup H)$, where m is a number of universal vertices in the graph G and H is a complete graph in $\overline{G}$. Consider the following subcases.
Subcase (i): Let m = 0
Then $\overline{G}$ becomes connected graph H. Since H is a complete graph, from corollary 5.1.1. IP domination does not exist.
Subcase (ii): Let $m \neq 0$.
Prove this subcase by mathematical induction.
If G has one universal vertex then $\overline{G}$ becomes a disconnected graph of the form $K_1 \cup H$. The complement graph $\overline{G}$ contains an isolated vertex say w, which is the required selection and in H choose any two vertices say u & v which satisfying the IP domination condition is that the vertices of the graph induced by I is of $deg \leq 1$ and it must have at least one isolated vertex and pendant vertex. Thus the IP dominating set $I = \{w, u, v\}$. Therefore $\gamma_{01}\overline{(G)} = 1+2 = 3$. If G has two universal vertices then $\overline{G}$ becomes a disconnected graph of the form $(2K_1 \cup H)$. Since $\overline{G}$ contains two isolated vertices, which are required to be chosen and in H choose any two vertices that satisfy the condition of IPD-set. Thus $\gamma_{01}\overline{(G)} = 2+2 = 4$.
Let us assume for $(m-1)$ universal vertex in G which have $\gamma_{01}\overline{(G)} = (m-1)+2$.
Now prove for m universal vertex in G.
If G has m universal vertex then $\overline{G}$ is a disconnected graph of the form $(mK_1 \cup H)$. Since $\overline{G}$ contains m isolated vertices, which are required to be chosen that m vertices and in H it is enough to choose two vertices to satisfy the condition of IPD-set. Therefore, $|I| = m$ (isolated vertices) $+$ any two vertices in $H = m+2$.
Case 2: H is any connected graph $H \cong \overline{C_5}$.
If $\overline{G}$ is a disconnected graph of the form $(mK_1 \cup H)$ where m is a number of universal vertices in G and H is any connected graph that is isomorphic to $\overline{C_5}$. The following subcases arise.
Subcase (i): Let $m = 0$
Then $\overline{G}$ becomes connected graph H. By using theorem 5.1., the IP domination number is 3. ie).$\gamma_{01}\overline{(G)} = 3$.

Subcase (ii): For $m \neq 0$

In this subcase, $\overline{G}$ is a disconnected graph. Prove this subcase by mathematical induction. If G has one universal vertex then $\overline{G}$ becomes a disconnected graph of the form $K_1 \cup H$. Since $\overline{G}$ has one isolated vertex, it must choose that vertex and by using theorem 5.1., choose the remaining vertices to satisfy the IPD-set condition. From theorem 5.1., the IP domination number for H is 3 and have one isolated vertex. Thus $\gamma_{01}\overline{(G)} = 3+1 = 4$. If G has two universal vertices then $\overline{G}$ becomes a disconnected graph of the form $(2K_1 \cup H)$. Since $\overline{G}$ contains two isolated vertices, which are required to be chosen and again by theorem 5.1., the IP domination number for H is 3. Thus $\gamma_{01}\overline{(G)} = 2+3 = 5$.

Let us assume for $(m-1)$ universal vertex in $\overline{G}$, then $\gamma_{01}\overline{(G)} = (m-1)+3$.

Prove for m universal vertex in G.

If G has m universal vertex then $\overline{G}$ is a disconnected graph of the form $(mK_1 \cup H)$. Since $\overline{G}$ contains m isolated vertices, which are required to be chosen that m vertices and by theorem 5.1., the IP domination number for H is 3. Therefore $\gamma_{01}\overline{(G)} = m+3$.

Case 3: H is any connected graph

If $\overline{G}$ is a disconnected graph of the form $(mK_1 \cup H)$ where m is a number of universal vertices in G and H is any connected graph that is not isomorphic to $\overline{C}_5$. Suppose G has $m \geq 1$ universal vertices. Since $\overline{G}$ is a disconnected graph that has j components, in these $(j-1)$ components are isolated and one component is the connected graph. Since G has m Universal vertices, that $'j-1'$ components have m isolated vertices and one connected component say H in $\overline{G}$. For the IP dominating set, one must choose that m isolated vertices in $\overline{G}$ and also choose any two adjacent vertices, say $u, v \in \overline{G}$ which have a distance of at least three in all possible ways of path in G except universal vertex. Thus, these vertices dominate all the vertices in $\overline{G}$ and also satisfy our IP domination condition. $|I| = m(\text{isolated vertices}) + \{u, v\} = m+2$.

6 Conclusion

This study focuses on determining the IPD-set and the IP domination number for the complement graphs of various standard and special graphs. Furthermore, the results obtained have led to generalized findings for IP domination number of complement graphs. Also we conclude that if $\bar{G}$ is a connected graph, then $\gamma_{01}(\bar{G}) = 3$ Future research will expand on this concept by investigating the characterization of the complement of IP domination with other parameters.

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