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ORIGINAL RESEARCH

Solving Two-Dimensional Helmholtz and Poisson Equations Using Double Laplace Transform Method

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Abstract

Objectives: To explore the efficacy of the double Laplace transform technique in solving 2D Helmholtz and Poisson equations. **Methods:** The double Laplace transform clearly converts the 2D Helmholtz and Poisson equations into an algebraic calculation in the Laplace domain that can be solved easily. **Findings:** The double Laplace transform method offers exact solutions to the Helmholtz and Poisson equations by resolving a series of specific and understandable examples. **Novelty:** This research highlights the potential of double Laplace transform methods as a competent and flexible instrument for resolving 2D Helmholtz and Poisson equations.

Mathematics Subject Classifications: 35R09, 35R11, 44A10, 44A30.

Keywords: 2D Helmholtz Equation; Poisson Equation; Double Laplace Transform; Inverse Double Laplace Transform; Exact Solution

1 Introduction

The Helmholtz equation and Poisson's equation are two significant partial differential equations whose solutions have a varied range of applications in various fields. Many problems in Physics and Engineering lead to one of these equations. The Helmholtz equation was studied by a German Physicist and Physician named Hermann von Helmholtz in 1860. He was the first to derive the solution of boundary value problems related to this equation. Helmholtz equation is a versatile equation that is used in the study of optics, seismology, tsunamis, volcanic eruptions, medical imaging, electromagnetism, thermodynamics, acoustics, and industrial applications. This equation provides a mathematical framework for understanding and studying wave-like phenomena. Many complex situations can be analyzed by using the Helmholtz equation along with suitable boundary conditions.

The Poisson's equation was studied by a French Mathematician and Physicist Simeon Denis Poisson. He was the first to publish his solutions on Poisson's equation in the year 1813. Poisson equation is a generality of Laplace's equation. This equation has many applications in Physics related to electrostatics, theoretical physics, astronomy, modeling and simulation, heat flow, and fluid dynamics.

Many researchers have studied these two partial differential equations associated with different areas and obtained their solutions and applicability in numerous fields.

Despite significant advancement in solving these equations through analytical and numerical methods, a number of difficulties still exist. A promising substitute to fill these gaps is the double Laplace transform method, which reduces two-dimensional Helmholtz and Poisson equations to a simplified algebraic system.

Finite difference method and finite elements method[1] are implemented to find an approximate solution for the 2D Helmholtz equation. Localized singular boundary method[2] is employed for answering 2D Laplace and Helmholtz equations in intricate areas. The double Laplace transform method[3] is employed to obtain the resolution of Poisson partial differential equations. The authors in[4] have considered discrete sine transforms solver based on a sixth-order Compact finite difference methods for answering Poisson equations. 3D reconstruction process[5] for multi-visual animated images constructed on Poisson's equation theory.

A meshless numerical system[6], which is the mixture of the localized process of fundamental answers, the recursive composite multiple reciprocity technique, and the unphysical nodes, is offered to agreement with 2D and 3D boundary value problems, leading through Helmholtz and Poisson Equations. Authors in[7] have studied the numerical solution of a 2D inhomogeneous Helmholtz equation using the meshless radial basis function method.

Sixth-order compact approximations[8–10] are discussed for the resolution of Poisson equations and Helmholtz equation. The authors in[11] analyze both 2D and 3D cases of the Helmholtz equation and present several tactics for solving the Helmholtz equation. Recently the author in[12] utilized single Laplace transform combined with domain-boundary element method to compute numerical solutions for initial boundary value problems governed by the variable coefficient modified Helmholtz-type equation. The authors in[13] proposed an augmented matched interface and boundary method with the fast Fourier transform acceleration for three-dimensional Helmholtz interface problems

Consider the inhomogeneous two-dimensional Helmholtz's equation is of the form:

$$\frac{\partial^2 v(x,y)}{\partial x^2}+\frac{\partial^2 v(x,y)}{\partial y^2}+kv(x,y)=h(x,y),\ \ (x,y)\in\Omega, \tag{1}$$

where $\Omega \subseteq R^2$ and $\partial\Omega$ is its boundary.

If $k=\lambda^2>0$, the mathematical problem (1) is referred to as an oscillatory Helmholtz formula and λ stands the wave number and if $k<0$, the mathematical problem (1) is referred to as a monotone Helmholtz formula.

The unidentified amount $v(x,y)$ usually denotes a pressure field in the frequency domain and $h(x,y)$ is a source function. Suppose that $v(x,y)$ and $h(x,y)$ are suitably smooth functions.

Associated with (1), we consider the conditions

$$v(x,0)=f_1(x),\ \ v_y(x,0)=f_2(x),\ \ v(0,y)=g_1(y),\ \ v_x(0,y)=g_2(y). \tag{2}$$

For $k=0$ in (1), we have the two-dimensional Poisson equation

$$\frac{\partial^2 v(x,y)}{\partial x^2}+\frac{\partial^2 v(x,y)}{\partial y^2}=h(x,y),\ \ (x,y)\in\Omega\subseteq R^2, \tag{3}$$

In this scientific article, we use double Laplace transform technique to solve inhomogeneous two-dimensional Helmholtz's equation of the type (1) and two-dimensional Poisson equation of the type (2) related to the conditions (3). The following is how the remaining paper is organized: Section 2 describes the double Laplace transform method and shows how to deduce the general analytical solution for (1) as well as (2) step by step. A few numerical examples are also provided in Section 3, which shows how double Laplace transform is applied to a two-dimensional Helmholtz's equation and Poisson equation. Finally, some conclusions are drawn in Section 4.

2 Methodology

2.1 Basic Definitions and Assumptions of double Laplace transform:

The double Laplace transform of the function $f(x,y)$ is defined by

$$L_xL_y(f(x,y)\}=\bar{f}(p,s)=\int_0^\infty e^{-px}\int_0^\infty e^{-sy}f(x,y)\,dydx, \tag{4}$$

whenever that integral exists.

Assumptions of double Laplace transform:

1) The double Laplace transform is directly applicable if the given partial differential equation must be linear.

2) If a function $f(x,y)$ is a continuous function in every finite interval $(0,\ X)$ and $(0,\ Y)$and of exponential order $exp(ax+by)$, then the double Laplace transform of $f(x,y)$ exists for all p and s provided $Re\,p > a\ and\ Re\ s\ > b$.

3) Initial and boundary conditions should be specified and well behaved to ensure the transformed equation is solvable.

2.2 Description of the proposed DLT method

The double Laplace transform approach for obtaining the general solution $v(x,y)$ of inhomogeneous two-dimensional Helmholtz's equation (1) as well as Poisson equation (3) are developed in this segment.

Transforming (1) by double Laplace transform, we obtain

$$p^2\bar{v}(p,s) - p\bar{v}(0,s) - \bar{v}_x(0,s) + s^2\bar{v}(p,s) - s\bar{v}(p,0) - \bar{v}_y(p,0) + k\bar{v}(p,s) = \bar{h}(p,s), \tag{5}$$

where

$$\bar{h}(p,s) = L_x L_y(h(x,y)] = \int_0^\infty e^{-px}\int_0^\infty e^{-sy}h(x,y)\,dydx, \tag{6}$$

Further, transforming (2) by single Laplace transform, we get

$$\bar{v}(p,0) = \bar{f}_1(p),\ \ \bar{v}_y(p,0) = \bar{f}_2(p),\ \ \bar{v}(0,s) = \bar{g}_1(s),\ \ \bar{v}_x(0,s) = \bar{g}_2(s). \tag{7}$$

By putting (7) in (5) and we obtain by simplifying

$$\bar{v}(p,s) = \frac{1}{(p^2+s^2+k]}\left(\bar{h}(p,s) + p\bar{g}_1(s) + \bar{g}_2(s) + s\bar{f}_1(p) + \bar{f}_2(p)\right]. \tag{8}$$

We obtain the solution of inhomogeneous two-dimensional Helmholtz's equation (1) by using the inverse DLT to (8).

$$v(x,y) = L_x{}^{-1}L_y{}^{-1}\left(\frac{1}{(p^2+s^2+k]}\left(\bar{h}(p,s) + p\bar{g}_1(s) + \bar{g}_2(s) + s\bar{f}_1(p) + \bar{f}_2(p)\right]\right]. \tag{9}$$

By putting $k=0$ in (9), we attain the solution of inhomogeneous two-dimensional Poisson equation (3):

$$v(x,y) = L_x{}^{-1}L_y{}^{-1}\left(\frac{1}{(p^2+s^2]}\left(\bar{h}(p,s) + p\bar{g}_1(s) + \bar{g}_2(s) + s\bar{f}_1(p) + \bar{f}_2(p)\right]\right]. \tag{10}$$

In this case, we assume that the inverse DLT of (9) and (10) exists.

3 Results and Discussion

In this part, we provide examples to exhibit the applicability of the previous technique.

Example 3.1: By changing $k=-\alpha$ and $h(x,y) = 4-\alpha(x-y)^2$ in (1), we have a 2D inhomogeneous Helmholtz equation:

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} - \alpha v(x,y) = 4-\alpha(x-y)^2, \tag{11}$$

based on the conditions

$$v(x,0) = f_1(x) = x^2,\ v_y(x,0) = f_2(x) = -2x,\ \ v(0,y) = g_1(y) = y^2,\ \ v_x(0,y) = g_2(y) = -2y.\quad (11) \tag{12}$$

Transforming (12) by single Laplace transform, we get

$$\bar{f}_1(p) = \frac{2}{p^3},\ \ \bar{f}_2(p) = \frac{-2}{p^2},\ \ \bar{g}_1(s) = \frac{2}{s^3},\ \ \bar{g}_2(s) = \frac{-2}{s^2}. \tag{13}$$

Transforming $h(x,y)$ by DLT, we get

$$\bar{h}(p,s) = \frac{4}{ps} - \alpha\left(\frac{2}{p^3 s} - \frac{2}{p^2 s^2} + \frac{2}{ps^3}\right]. \tag{14}$$

Replacing the above in (9), we get solution of (11):

$$v(x,y) = L_x^{-1} L_y^{-1} \left(\frac{1}{(p^2+s^2-\alpha]} \left(\frac{4}{ps} - \alpha \left(\frac{2}{p^3 s} - \frac{2}{p^2 s^2} + \frac{2}{ps^3} \right] + p\frac{2}{s^3} - \frac{2}{s^2} + s\frac{2}{p^3} - \frac{2}{p^2} \right] \right]. \tag{15}$$

Computing, we get desired analytical solution:

$$v(x,y) = L_x^{-1} L_y^{-1} \left(\frac{2}{p^3 s} - \frac{2}{p^2 s^2} + \frac{2}{ps^3} \right] = (x-y)^2. \tag{16}$$

Fig. 1 shows the precise solution $v(x,y) = (x-y)^2$ utilizing a variety of values of $0 \le x \le 2$ and $0 \le y \le 2$.

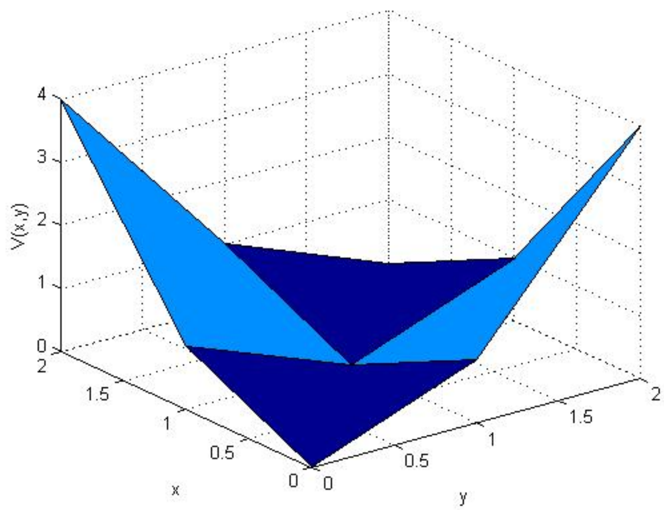

Fig 1. Precise solution $v(x,y) = (x-y)^2$

Example 3.2: By changing $k = 1$ and $h(x,y) = 0$ in (1), we have 2D homogeneous Helmholtz equation:

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} + v(x,y) = 0, \tag{17}$$

based on the conditions

$$v(x,0) = f_1(x) = cos(0.5)x,\ v_y(x,0) = f_2(x) = -\sqrt{0.75}sin(0.5\,x),\ \ v(0,y) = g_1(y) = cos(\sqrt{0.75})\,y,\ \ v_x(0,y) = g_2(y) = -0.5sin(\sqrt{0.75}\,y). \tag{18}$$

Transforming (18) by single Laplace transform, we get

$$\bar{f}_1(p) = \frac{p}{p^2+0.25},\ \ \bar{f}_2(p) = \left(-\sqrt{0.75}\right)\frac{(0.5)}{(p^2+0.25)},\ \ \bar{g}_1(s) = \frac{s}{s^2+0.75},\ \ \bar{g}_2(s) = (-0.5)\left(\frac{\sqrt{0.75}}{s^2+0.75}\right). \tag{19}$$

Transforming $h(x,y)$ by DLT, we get

$$\bar{h}(p,s) = 0. \tag{20}$$

Replacing above in (9), we get solution of (17):

$$v(x,y) = L_x^{-1} L_y^{-1} \left(\frac{1}{(p^2+s^2+1]} \left(p\left(\frac{s}{s^2+0.75}\right) + (-0.5)\left(\frac{\sqrt{0.75}}{s^2+0.75}\right) + s\left(\frac{p}{p^2+0.25}\right) + (-0.5) \left(\frac{\sqrt{0.75}}{s^2+0.75}\right). \right.\right. \tag{21}$$

Computing, we get desired analytical solution:

$$v(x,y) = L_x^{-1} L_y^{-1}\left(\frac{ps - (0.5)\left(\sqrt{0.75}\right)}{(p^2+0.25)(s^2+0.75)}\right] = cos\left(0.5x + \sqrt{0.75}\,y\right). \tag{22}$$

Fig. 2 shows the precise solution $v(x,y) = cos\left(0.5x + \sqrt{0.75}\,y\right)$ utilizing a variety of values of $0 \le x \le 2$ and $0 \le y \le 3$.

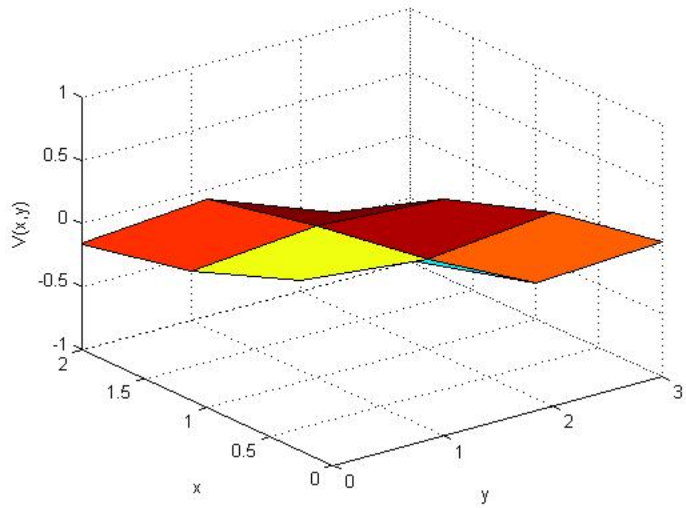

Fig 2. Precise solution $v(x,y) = cos\left(0.5x + \sqrt{0.75}\,y\right)$

Example 3.3: By changing $k = 0$ and $h(x,y) = xe^y$ in (1), we have non-homogeneous Poisson equation:

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} = xe^y, \tag{23}$$

subject to the conditions

$$v(x,0) = f_1(x) = x + cosx,\ v_y(x,0) = f_2(x) = x + cosx,\ v(0,y) = g_1(y) = e^y,\ v_x(0,y) = g_2(y) = e^y. \tag{24}$$

Transforming (24) by single Laplace transform, we get

$$\bar{f}_1(p) = \frac{1}{p^2} + \frac{p}{p^2+1} = \bar{f}_2(p),\ \ \bar{g}_1(s) = \bar{g}_2(s) = \frac{1}{s-1}. \tag{25}$$

Transforming $h(x,y)$ by DLT, we get

$$\bar{h}(p,s) = \frac{1}{p^2(s-1)}. \tag{26}$$

Replacing above in (10), we get solution of (23):

$$v(x,y) = L_x^{-1} L_y^{-1}\left(\frac{1}{(p^2+s^2]}\left(\frac{1}{p^2(s-1)} + p\left(\frac{1}{s-1}\right) + \frac{1}{s-1} + s\left(\frac{1}{p^2} + \frac{p}{p^2+1}\right) + \left(\frac{1}{p^2} + \frac{p}{p^2+1}\right)\right]\right]. \tag{27}$$

Computing, we get desired analytical solution:

$$v(x,y) = L_x^{-1} L_y^{-1}\left(\frac{1}{p^2(s-1)} + \frac{p}{(s-1)(p^2+1)}\right] = xe^y + e^y cosx. \tag{28}$$

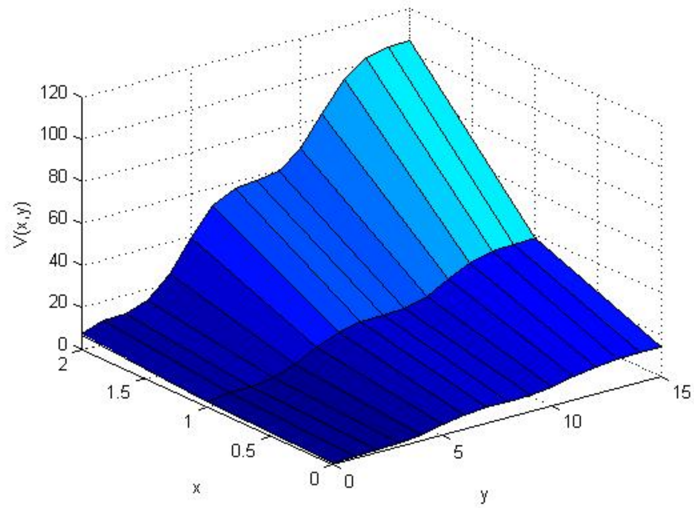

Fig 3. Precise solution $v(x,y) = xe^y + e^y cosx$

Fig. 3 shows the precise solution $v(x,y) = xe^y + e^y cosx$ utilizing a variety of values of $0 \le x \le 2$ and $0 \le y \le 15$.

Example 3.4: By changing $k = 0$ and $h(x,y) = sin(\pi x)sin(\pi y)$ in (1), we have 2D Poisson equation:

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} = sin(\pi x)sin(\pi y), \tag{29}$$

based on the conditions

$$v(x,0) = f_1(x) = 0,\ v_y(x,0) = f_2(x) = \frac{-1}{2\pi}sin(\pi x),\ \ v(0,y) = g_1(y) = 0,\ \ v_x(0,y) = g_2(y) = \frac{-1}{2\pi}sin(\pi y). \tag{30}$$

Transforming (30) by single Laplace transform, we get

$$\bar{f}_1(p) = 0,\ \ \bar{f}_2(p) = \frac{-1}{2(p^2+1)},\ \ \bar{g}_1(s) = 0,\ \ \bar{g}_2(s) = \frac{-1}{2(s^2+1)}. \tag{31}$$

Transforming $h(x,y)$ by DLT, we get

$$\bar{h}(p,s) = \left(\frac{\pi}{p^2+\pi^2}\right)\left(\frac{\pi}{s^2+\pi^2}\right). \tag{32}$$

Replacing above in (10), we get solution of (29):

$$v(x,y) = L_x{}^{-1}L_y{}^{-1}\left(\frac{1}{(p^2+s^2]}\left(\left(\frac{\pi}{p^2+\pi^2}\right)\left(\frac{\pi}{s^2+\pi^2}\right) - \frac{1}{2(s^2+1)} - \frac{1}{2(p^2+1)}\right]\right]. \tag{33}$$

Computing, we get desired analytical solution:

$$v(x,y) = L_x{}^{-1}L_y{}^{-1}\left(\frac{-1}{2}\left(\frac{1}{p^2+\pi^2}\right)\left(\frac{1}{s^2+\pi^2}\right)\right] = \frac{-1}{2\pi^2}sin(\pi x)sin(\pi y). \tag{34}$$

Fig. 4 shows the precise solution $v(x,y) = \frac{-1}{2\pi^2}sin(\pi x)sin(\pi y)$ utilizing a variety of values of $0 \le x < 10$ and $0 \le y < 10$.

4 Conclusion

The double Laplace transform technique proves to be a powerful and efficient approach for solving 2D Helmholtz and Poisson equations. By transforming original two-dimensional Helmholtz and Poisson equations into algebraic equations decreases

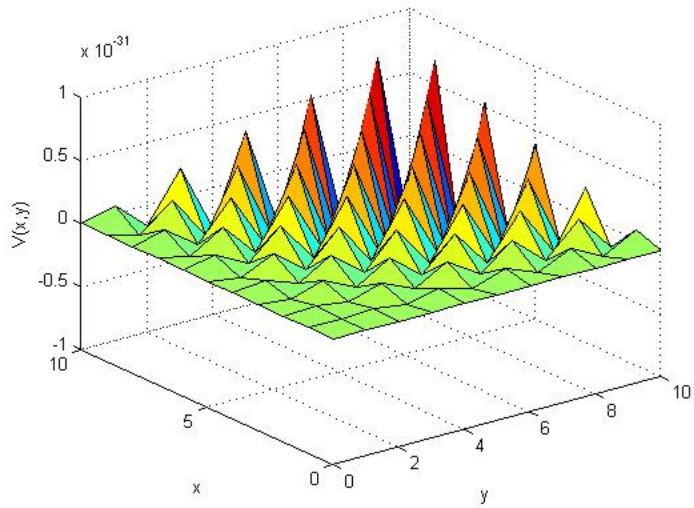

Fig 4. Precise solution $v(x,y) = \frac{-1}{2\pi^2} sin(\pi x) sin(\pi y)$

computational complexity and increases solution efficiency. It can be inferred that the Double Laplace transform approach yields the precise solution in each of the examples examined in Section 4 following a few computational steps. Some cutting-edge problems in signal-system, fluid dynamics and elasticity dealing with integral and partial differential equations will be discussed in a subsequent paper.

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