INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Raising Series to A Power

S Sriram[1]*, A David Christopher[1]

1 PG & Research Department of Mathematics, National College, affiliated to Bharathidasan University, Trichy, Tamil Nadu, India

Check for updates

OPEN ACCESS

Received: 29-08-2023

Accepted: 05-03-2024

Published: 31-05-2024

Editor: ICDMMMDE-2023 Special Issue Editors: Dr. G. Mahadevan & Prof. Dr. P. Balasubramaniam

Citation: Sriram S, Christopher AD (2024) Raising Series to A Power. Indian Journal of Science and Technology 17(SP1): 124-135. https://doi.org/10.17485/IJST/v17sp1.249

* **Corresponding author**.

sriram.priya02@yahoo.com

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objective. Let $F(q) = \sum_{n=0}^{\infty} a_n q^n$ be a formal power series and let $\alpha \in C$. In this note, we consider the function $F(q)^\alpha$. We find that if $F(q)^\alpha$ has a series expansion at $q = 0$, then its coefficients are polynomials in α. The coefficients of these polynomials were found to be a weighted composition sum. **Methods.** The method to arrive at this representation involves logarithmic derivative and exponential representation. **Findings.** As a consequence of this, new identities involving partition functions and binomial coefficients were obtained. Further, a particular class of Dirichlet series is found to have the form of an exponential function. Consequently, identities involving Riemann zeta function values were obtained. **Novelty.** The present work generalizes a class of functions considered by D'Arcais. Divisor-sum identities involving partition functions and exponential representation of Dirichlet series of this article were new to the literature.

Keywords: Polynomials; Partitions; Dirichlet Series; DivisorSum; Power Series

1 Introduction

The motivation of this study stems from the following product-to-sum representation:

$$q\prod_{m=1}^{\infty}(1-q^m)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

where an infinite product, namely, $\prod_{m=1}^{\infty}(1-q^m)$ is raised to the power 24. The function $\tau(n)$ was defined by Ramanujan and is known as Ramanujan's tau function [1]. The product $\prod_{m=1}^{\infty}(1-q^m)$ is known as Euler's product and has an interesting series expansion which was observed by Euler. This paper is all about a generalization of this definition. In the first part of this paper, we consider the class of functions having formal power series expansion:

$$F(q) = a(0) + a(1)q + \cdots.$$

Raising $F(q)$ to a complex power α and expressing $F(q)^\alpha$ as a formal power series in q we have

$$F(q)^\alpha = 1 + g_\alpha(1)q + g_\alpha(2)q^2 + \cdots.$$

The coefficients $g_{\alpha}(n)$ are found to be polynomials in α of degree n. The coefficients of these polynomials are found to be composition-sums.

Further, we substitute the term $\sum_{n=1}^{\infty}(1-q^n)^{-1}$ in place of $F(q)$ and obtain interesting identities. Recently, many papers mainly by Bernard Heim et al. (see [2], [3]) have been appeared related to this class of functions given below:

$$\prod_{n=1}^{\infty}(1-q^n)^{-\alpha}=\sum_{m=0}^{\infty}\tau_{\alpha}(m)q^m.$$

This kind of study forms the core part of section 2 and 3.

The study of Dirichlet series of arithmetic functions is an interesting branch of number theory. In section 4, we consider the following definition.

Definition 1. Let $F(q)^{\alpha}=1+g_{\alpha}(1)q+g_{\alpha}(2)q^2+\cdots$. Let $n=p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}\geq 2$ be the prime factorisation of n. We define

$$h_{g_{\alpha}}(n)=g_{\alpha}(\beta_1+1)\cdots g_{\alpha}(\beta_t+1)$$

and $h_{g_{\alpha}}(1)=1$.

As the main findings of section 4, we found that the Dirichlet series of $h_{g_{\alpha}}(n)$ given by

$$D_{h_{g_{\alpha}}}(\gamma)=\sum_{n=1}^{\infty}\frac{h_{g_{\alpha}}(n)}{n^{\gamma}}$$

is an exponential function in α. Here too plugging some well-known functions gives new infinite product identities especially involving Riemann zeta function values.

2 Methodology

Polynomial Representation

Representation of the coefficients of $F(q)^{\alpha}$ as the polynomials in l (whose coefficients are composition-sums) is essential to obtain the main result of this article.

Theorem 2. Let $F(q)=a(0)+a(1)q+\cdots$ with $a(0)=1$. Denote

$$logF(q)=b(1)q+b(2)q^2+\cdots.$$

Define, for every $\alpha\in C$, $G_{\alpha}(q)=(F(q))^{\alpha}$. If $G_{\alpha}(q)$ can be represented as a series like

$$G_{\alpha}(q)=1+g_{\alpha}(1)q+g_{\alpha}(2)q^2+\cdots,$$

then for $n\geq 1$, we have

$$g_{\alpha}(n)=\sum_{s=1}^{n}\frac{\alpha^s}{s!}\sum_{\substack{a_1+\cdots+a_s=n\\ a_i\in N}}b(a_1)\cdots b(a_s).---(1)$$

Proof . We have

$$\begin{aligned}1+g_{\alpha}(1)q+g_{\alpha}(2)q^2+\cdots&=G_{\alpha}(q)\\&=\left(e^{\log F(q)}\right)^{\alpha}\\&=e^{\alpha\log F(q)}\\&=1+\frac{\alpha}{1!}\log F(q)+\frac{\alpha^2}{2!}\log F(q)^2+\cdots\\&=1+\sum_{n=1}^{\infty}\left(\sum_{s=1}^{n}\frac{\alpha^s}{s!}\sum_{\substack{a_1+\cdots+a_s=n\\ a_i\in\mathrm{N}}}b(a_1)\cdots b(a_s)\right)q^n.\end{aligned}$$

Now equating the coefficient of the like powers of q finishes the proof.

Theorem 3. The coefficients $a(n)$ and $b(n)$ of Theorem 2 can be related as follows:

$$(n+1)\,a(n+1) = \sum_{k=0}^{n}(k+1)b(k+1)\,a(n-k)\,. - - - (2)$$

Proof. We have

$$logF(q) = b(1)\,q + b(2)\,q^2 + \cdots .$$

Differentiating the above, we have

$$\frac{F'(q)}{F(q)} = b(1) + 2b(2)\,q + \cdots .$$

This gives

$$a(1) + 2a(2)\,q + \cdots = (a(0) + a(1)\,q + \cdots)\,(b(1) + 2b(2)\,q + \cdots)\,.$$

Now equating the coefficients of the like powers of q gives the expected relation.

Remark 4 . From Theorem 3 one can see that if $b(n)$ is an integer sequence, then $a(n)$ will be an integer sequence.

3 Results and Discussion

Partition Identities

We recall few basics of partition theory. Let n be a positive integer. By a partition of n we mean a non-increasing sequence of positive integers whose sum equals n. Each element of this sequence is called a part. If each distinct part, say a_i, appears f_i times in a partition of n then we denote that partition by $n = a_1^{f_1} \cdots a_r^{f_r}$. If $f_1 = f_2 = \cdots = f_r = 1$ then that partition of n is said to be distinct.

Definition 5. Let n be a positive integer. The number of partitions of n is denoted by $p(n)$, and the number of distinct partitions of n is denoted by $D(n)$.

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)\,q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)}$$

with the convention that $p(0) = 1$.

The generating function for $D(n)$ is given by

$$\sum_{n=0}^{\infty} D(n)\,q^n = \prod_{m=1}^{\infty} (1+q^m)$$

with the convention that $D(0) = 1$.

Now we illustrate on plugging special values in Theorem 2 and Theorem 3.

Define $F(q) = p(0) + p(1)\,q + p(2)\,q^2 + \cdots$. Then we have

$$logF(q) = log\prod_{n=1}^{\infty}(1-q^n)^{-1}$$

$$= \sum_{n=1}^{\infty} log\frac{1}{1-q^n}$$

$$= -\sum_{n=1}^{\infty} log(1-q^n)$$

$$= -\sum_{n=1}^{\infty} -\left(q^n + \frac{q^{2n}}{2} + \frac{q^{3n}}{3} + \cdots\right)$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^{mn}}{m}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{d|k}\frac{1}{d}\right)q^k$$

$$= \sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k,$$

where $\sigma(n) = \sum_{d|n} d$. Thus, we have $b(n) = \frac{\sigma(n)}{n}$ while plugging $a(n) = p(n)$ in Theorem 2. Now Theorem 3 implies that

$$(n+1)\,p(n+1) = \sum_{k=0}^{n}\sigma(k+1)\,p(n-k)\,. - - - (3)$$

If one defines

$$\prod_{n=1}^{\infty}(1-q^n)^{-\alpha} = 1 + \tau_\alpha(1)\,q + \tau_\alpha(2)\,q^2 + \cdots,$$

then Theorem 2 implies that

$$\tau_\alpha(n) = \sum_{s=1}^{n}\frac{\alpha^s}{s!}\sum_{\substack{a_1+\cdots+a_s=n\\ a_i\in N}}\frac{\sigma(a_1)}{a_1}\cdots\frac{\sigma(a_s)}{a_s}. - - - (4)$$

The class of polynomials $\tau_\alpha(n)$ are called D'Arcais polynomials (refer[4]). The search for reducibility criterions of these polynomials over the ring of integers is of special interest and one can see a lot of papers appearing in this direction (refer[2,5–7]), the main reason behind this search is that the non-vanishing of the polynomials at $\alpha = -24$ for each n is equivalent to Lehmer's conjecture on Ramanujan's tau function which is still open. As the relations (3) and (4) are well-known we take the above derivation as an illustration, and proceed in similar fashion taking into account the other partition-generating functions.

Define $F(q) = D(0) + D(1)\,q + D(2)\,q^2 + \cdots$. Then we have

$$\begin{aligned} logF(q) &= log\prod_{n=1}^{\infty}(1+q^n) \\ &= \sum_{n=1}^{\infty} log(1+q^n) \\ &= \sum_{n=1}^{\infty}\left(q^n - \frac{q^{2n}}{2} + \frac{q^{3n}}{3} + \cdots\right) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{q^{mn}}{m} \end{aligned}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{d|k}(-1)^{d-1}\frac{1}{d}\right)q^k$$

$$= \sum_{k=1}^{\infty}\sigma_s(k)\,q^k,$$

where $\sigma_s(n)=\sum_{d|n}(-1)^{d-1}\frac{1}{d}$. Now from Theorem 3 we have the following recurrence identity for $D(n)$.

Theorem 6. Let n be a positive integer. We have

$$(n+1)D(n+1)=\sum_{k=0}^{n}(k+1)\sigma_s(k+1)D(n-k).$$

An application of Theorem 2 gives the following result.

Theorem 7. Let n be a positive integer. We have

$$D(n)=\sum_{t=1}^{n}\frac{1}{t!}\sum_{\substack{a_1+\cdots+a_t=n\\ a_i\in N}}\sigma_s(a_1)\cdots\sigma_s(a_t),$$

where $\sigma_s(n)=\sum_{d|n}(-1)^{d-1}\frac{1}{d}$.

Euler's partition theorem states that the number of distinct partitions of n is equal to the number of partitions of n with odd parts. This statement can be expressed in terms of generating functions as follows:

$$\prod_{n=1}^{\infty}(1+q^m)=\prod_{m=1}^{\infty}\left(1-q^{2m-1}\right)^{-1}.$$

Now consider the following equalities:

$$\begin{aligned} log\prod_{m=1}^{\infty}\left(1-q^{2m-1}\right)^{-1} &= -\sum_{m=1}^{\infty}log\left(1-q^{2m-1}\right)\\ &= -\sum_{m=1}^{\infty}-\left(q^{2m-1}+\frac{q^{2(2m-1)}}{2}+\frac{q^{3(2m-1)}}{3}+\cdots\right)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q^{n(2m-1)}}{n} \end{aligned}$$

$$=\sum_{k=1}^{\infty}\sigma_o(k)q^k,$$

where

$$\sigma_o(k)=\sum_{\substack{d\mid k\\ d\equiv 1\ (mod\ 2)}}\frac{d}{k}.$$

Then in view of the Theorem 2 we have another recurrence identity for $D(n)$.

Theorem 8. Let n be a positive integer. We have

$$(n+1)D(n+1)=\sum_{k=0}^{n}(k+1)\sigma_o(k+1)D(n-k).$$

Now comparing Theorem 6 and Theorem 8 we have the following theorem.

Theorem 9. Let n be a positive integer. We have

$$\sum_{d|n}(-1)^{d-1}\frac{1}{d}=\sum_{\substack{d\mid n\\ d\equiv 1\ (mod\ 2)}}\frac{d}{n}.$$

Consider the infinite products:

$$T_q=\prod_{n=1}^{\infty}(1+q^n)=e^{log\prod_{n=1}^{\infty}(1+q^n)}=e^{A_q}$$

and

$$T_q^* = \prod_{n=1}^{\infty}(1-q^n) = e^{log\prod_{n=1}^{\infty}(1-q^n)} = e^{A_q^*}.$$

From the previous observations, one can have

$$A_q = log\prod_{n=1}^{\infty}(1+q^n) = \sum_{k=1}^{\infty}\sigma_s(k)\,q^k$$

and

$$A_q^* = log\prod_{n=1}^{\infty}(1-q^n) = -\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k.$$

Now we have

$$\begin{aligned} T_qT_q^* &= e^{A_q}e^{A_{q^*}} \\ &= e^{A_q+A_{q^*}}. \end{aligned}$$

On the other hand, we have

$$T_qT_q^* = e^{A^*_{q^2}}.$$

Now equating the coefficients of like power of q gives the following result.

Theorem 10. Let n be a positive integer. Then we have

$$\sigma_s(n) - \frac{\sigma(n)}{n} = \begin{cases} -\frac{\sigma\left(\frac{n}{2}\right)}{\frac{n}{2}} & \text{if } n \equiv 0 \ (mod\ 2); \\ 0 & \text{otherwise.} \end{cases}$$

Our next concern is over a particular class of partitions, namely, regular partitions which is defined as follows.

Definition 11. Let n be a positive integer and let $l \geq 2$ be a positive integer. Then the l-regular partition of n is defined to be the partition of n, none of its part is divisible by l. The number of l-regular partitions of n is denoted by $p_l(n)$.

The study of regular partitions is predominant in additive number theory. In recent development, many arithmetic properties of l-regular partition functions were obtained. Abinash[8] studied the 3-divisibility of 3 and 9 regular partition functions. Cherubini et al.[9] studied the parity of 8 regular partition function.

The generating function for l-regular partitions of n is given by

$$\sum_{n=0}^{\infty}p_l(n)\,q^n = \prod_{m=1}^{\infty}\frac{1-q^{lm}}{1-q^m}$$

with the convention that $p_l(0) = 1$.

Consider the following equalities:

$$\begin{aligned} \log\prod_{m=1}^{\infty}\frac{1-q^{lm}}{1-q^m} &= \sum_{m=1}^{\infty}\left(\log(1-q^{lm}) - \log(1-q^m)\right) \\ &= \sum_{m=1}^{\infty} -\left(q^{lm} + \frac{q^{2lm}}{2} + \frac{q^{3lm}}{3} + \cdots\right) + \left(q^m + \frac{q^{2m}}{2} + \frac{q^{3m}}{3} + \cdots\right) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{q^{mn}}{n} - \frac{q^{lmn}}{n}\right) \\ &= \sum_{k=1}^{\infty}\left(\sigma^{-1}(k) - \sigma_l(k)\right)q^k, \end{aligned}$$

where

$$\sigma^{-1}(n) = \sum_{d|n} \frac{1}{d}$$

and

$$\sigma_l^{-1}(n) = \sum_{ld|n} \frac{1}{d}.$$

Based on these observations, theorem 3 gives the following recurrence identity for $p_l(n)$.

Theorem 12. Let n be a positive integer and let $l \geq 2$ be a positive integer. We have

$$(n+1)\,p_l(n+1) = \sum_{k=0}^{n} (k+1)\left(\sigma^{-1}(k+1) - \sigma_l^{-1}(k+1)\right) p_l(n-k).$$

The infinite product form of the generating function of certain partition functions allows us to employ Theorem 3 to obtain partition function identities. This kind of derivation may be extended to many other partition functions.

Binomial Identities

Define $F(q) = 1+q$. In accordance with the notations of Theorem 2 we can write

$$a(n) = \begin{cases} 1 & \text{if } n = 0, 1; \\ 0 & \text{otherwise,} \end{cases}$$

and considering the following expansion:

$$log(1+q) = q - \frac{q^2}{2} + \frac{q^3}{3} - \cdots,$$

we can write

$$b(n) = \frac{(-1)^{n-1}}{n}.$$

Now for a positive integer k, we have

$$(1+q)^k = \binom{k}{0} + \binom{k}{1} q + \cdots + \binom{k}{k} q^k.$$

When the above substitutions were made in Theorem 2 we have the following result.

Theorem 13. Let n and k be two positive integers. We have

$$\sum_{s=1}^{n} \frac{k^s}{s!} \sum_{a_1+\cdots+a_s=n} \frac{(-1)^{a_1+\cdots+a_s-s}}{a_1\cdots a_s} = \begin{cases} \binom{k}{n} & \text{if } n \leq k; \\ 0 & \text{otherwise.} \end{cases}$$

In order to arrive at another identity, we define $F(q) = 1+q+q^2+\cdots = \frac{1}{1-q}$. In line with Theorem 2 we can write $a(n) = 1$ for every non-negative integer n. For a positive integer k, we have

$$F(q)^k = (1-q)^{-k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} q^n$$

We observe that

$$logF(q) = -log(1-q) = q + \frac{q^2}{2} + \frac{q^3}{3} + \cdots.$$

Now again in line with Theorem 2 we can write $b(n) = \frac{1}{n}$. Now an appeal to Theorem2 gives the following result.

Theorem 14. Let n and k be two positive integers. We have

$$\binom{n+k-1}{k-1} = \sum_{s=1}^{n} \frac{k^s}{s!} \sum_{a_1+\cdots+a_s=n} \frac{1}{a_1\cdots a_s}.$$

Class of Dirichlet series which behaves like an exponential function

One can see from the literature that Dirichlet series, power series and Riemann zeta function have harmonious interplay. B. Q. Li and J. Steuding[10] obtained asymptotic formula for the counting function for fixed points of Dirichlet series and Riemann zeta function. L. M. Navas et al.[11] obtained an analytic continuation of Dirichlet series. Parth Chavan et al. proposed a general formula linearizing the convolution of Dirichlet series with modified weights[12]. In this section, we consider a class of Dirichlet series mentioned in the Definition 1, namely, $D_{h_{g_\alpha}}(\gamma)$. For a given $\gamma \in C$, we will show that $D_{h_{g_\alpha}}(\gamma)$ is an exponential function of α. To acheive that end, we employ Euler's product formula.

Lemma 15. Let f be a multiplicative function. If its Dirichlet series exist for $\gamma \in C$, then we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^{\gamma}} = \prod_{p-prime} \left(1 + \frac{f(p)}{p^{\gamma}} + \frac{f(p^2)}{p^{2\gamma}} + \cdots\right).$$

To present the main results of this section, we need the following definition.

Definition 16. Let $\gamma \in C$. We define

$$S(\gamma) = \sum\nolimits_{p-prime} \frac{1}{p^{\gamma}}.$$

Theorem 17. Let F and g_α be as in the Theorem 2. We have

$$D_{h_{g_\alpha}}(\gamma) = \eta^{\alpha},$$

where

$$\eta = \prod_{p-prime} F\left(p^{-\gamma}\right).$$

Proof. Let n and m be two relatively prime positive integers greater than 1. Let $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}$ and $m = q_1^{\delta_1} q_2^{\delta_2} \cdots q_s^{\delta_s}$ be the prime factorisations of n and m respectively. Then from the definition of h_{g_α} we have

$$\begin{aligned} h_{g_\alpha}(nm) &= g_\alpha(\beta_1+1)\cdots g_\alpha(\beta_t+1)\, g_\alpha(\delta_1+1)\cdots g_\alpha(\delta_s+1) \\ &= h_{g_\alpha}(n)\, h_{g_\alpha}(m). \end{aligned}$$

Thus h_{g_α} is multiplicative. Because of this multiplicative nature of h_{g_α}, we can involve h_{g_α} in Euler's product formula for Dirichlet series as follows:

$$\begin{aligned} D_{h_{g_{\alpha+\beta}}}(\gamma) &= \sum_{n=1}^{\infty} \frac{h_{g_{\alpha+\beta}}(n)}{n^{\gamma}} \\ &= \prod_{p-prime} \left(1 + \frac{h_{g_{\alpha+\beta}}(p)}{p^{\gamma}} + \frac{h_{g_{\alpha+\beta}}(p^2)}{p^{2\gamma}} + \cdots\right) \\ &= \prod_{p-prime} \left(1 + \frac{g_{\alpha+\beta}(1)}{p^{\gamma}} + \frac{g_{\alpha+\beta}(2)}{p^{2\gamma}} + \cdots\right) \\ &= \prod_{p-prime} F\left(p^{-\gamma}\right)^{\alpha+\beta} \\ &= \left(\prod_{p-prime} F\left(p^{-\gamma}\right)^{\alpha}\right)\left(\prod_{p-prime} F\left(p^{-\gamma}\right)^{\beta}\right) \\ &= \prod_{p-prime} \left(1 + \frac{g_{\alpha}(1)}{p^{\gamma}} + \frac{g_{\alpha}(2)}{p^{2\gamma}} + \cdots\right) \prod_{p-prime} \left(1 + \frac{g_{\beta}(1)}{p^{\gamma}} + \frac{g_{\beta}(2)}{p^{2\gamma}} + \cdots\right) \\ &= \prod_{p-prime} \left(1 + \frac{h_{g_\alpha}(p)}{p^{\gamma}} + \frac{h_{g_\alpha}(p^2)}{p^{2\gamma}} + \cdots\right) \prod_{p-prime} \left(1 + \frac{h_{g_\beta}(p)}{p^{\gamma}} + \frac{h_{g_\beta}(p^2)}{p^{2\gamma}} + \cdots\right) \\ &= \left(\sum_{n=1}^{\infty} \frac{h_{g_\alpha}(n)}{n^{\gamma}}\right)\left(\sum_{n=1}^{\infty} \frac{h_{g_\beta}(n)}{n^{\gamma}}\right) \\ &= D_{h_{g_\alpha}}(\gamma) D_{h_{g_\beta}}(\gamma). \end{aligned}$$

Since $D_{h_{g_0}} = 1$, the above relation implies that the Dirichlet series $D_{h_{g_\alpha}}(\gamma)$ is an exponential function. As one can see from the above equalities that the base of this exponential function is $\prod_{p-prime} F\left(p^{-\gamma}\right)$. Now the proof is completed. ∎

In what follows we consider the following definitions of functions.

Definition 18. Let $\alpha \in C$. A class of functions, denoted $\tau_\alpha(n)$, is defined as follows:

$$q\prod_{m=1}^{\infty}(1-q^m)^{-\alpha} = \sum_{n=1}^{\infty}\tau_\alpha(n)\,q^n.$$

Definition 19. Let $\alpha \in C$. A class of functions, denoted $\tau_\alpha^*(n)$, is defined as follows:

$$q\prod_{m=1}^{\infty}(1+q^m)^{\alpha} = \sum_{n=1}^{\infty}\tau_\alpha^*(n)\,q^n.$$

Now in accordance with Theorem 17 we can write

$$\begin{aligned} D_{h_{\tau_\alpha}}(\gamma) &= \textstyle\prod_{p-prime}\left(1-p^{-\gamma}\right)^{-\alpha}\left(1-p^{-2\gamma}\right)^{-\alpha}\cdots \\ &= \left(\tfrac{1}{\zeta(\gamma)\zeta(2\gamma)\cdots}\right)^{-\alpha}. \end{aligned}$$

Denote

$$\delta = \frac{1}{\zeta(\gamma)\,\zeta(2\gamma)\cdots}.$$

Recall that

$$S(\gamma) = \sum_{p-prime} p^{-\gamma}.$$

Now we have

$$\begin{aligned} \log\delta &= -(\log\zeta(\gamma)+\log\zeta(2\gamma)+\cdots) \\ &= -\left(\log\prod_{p-prime}\frac{1}{1-p^{-\gamma}}+\log\prod_{p-prime}\frac{1}{1-p^{-\gamma}}+\cdots\right) \\ &= -\left(\sum_{p-prime}\log\frac{1}{1-p^{-\gamma}}+\sum_{p-prime}\log\frac{1}{1-p^{-2\gamma}}+\cdots\right) \\ &= -\left[\left(\sum_{p-prime}\frac{1}{p^\gamma}+\frac{1}{2}\frac{1}{p^{2\gamma}}+\cdots\right)+\frac{1}{2}\left(\sum_{p-prime}\frac{1}{p^{2\gamma}}+\frac{1}{2}\frac{1}{p^{4\gamma}}+\cdots\right)+\cdots\right] \\ &= -\left[\sum_{p-prime}\left(\frac{1}{p^\gamma}+\frac{1}{p^{2\gamma}}+\cdots\right)+\frac{1}{2}\sum_{p-prime}\left(\frac{1}{p^{2\gamma}}+\frac{1}{p^{4\gamma}}+\cdots\right)+\cdots\right] \\ &= -\left[\sum_{p-prime}\left(\frac{1}{p^\gamma}\frac{1}{1-\frac{1}{p^\gamma}}+\frac{1}{2}\frac{1}{p^{2\gamma}}\frac{1}{1-\frac{1}{p^{2\gamma}}}+\cdots\right)\right] \\ &= -\sum_{p-prime}\left[\frac{\sigma(1)}{1}p^{-\gamma}+\frac{\sigma(2)}{2}p^{-2\gamma}+\cdots\right] \\ &= -\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]. \end{aligned}$$

$$= -\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]$$

This gives

$$\delta = e^{-\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]}.$$

Consequently, $D_{h_{\tau_\alpha}}(\gamma)$ becomes

$$D_{h_{\tau_\alpha}}(\gamma) = e^{-\alpha\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]}.$$

This is recorded in the following theorem.

Theorem 20. We have

$$D_{h_{\tau_\alpha}}(\gamma) = e^{-\alpha\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]},$$

In the course of proof of the theorem above we have arrived at the following identity.

Corollary 21. We have

$$\frac{1}{\zeta(\gamma)\zeta(2\gamma)\cdots} = e^{-\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]}.$$

Our next concern is over representing $D_{h_{\tau_\alpha^*}}(\gamma)$ as an exponential function.

Consider

$$\begin{aligned} D_{h_{\tau_\alpha^*}}(\gamma) &= \prod_{p-prime}(1+p^{-\gamma})^{\alpha}\left(1+p^{-2\gamma}\right)^{\alpha}\cdots \\ &= \prod_{p-prime}\frac{1}{(1-p^{-\gamma})^{\alpha}\left(1-p^{-3\gamma}\right)^{\alpha}\cdots} \\ &= (\zeta(\gamma)\zeta(3\gamma)\cdots)^{\alpha}. \end{aligned}$$

Define

$$\lambda = \zeta(\gamma)\zeta(3\gamma)\cdots.$$

Consider

$$log\lambda = log\zeta(\gamma)+log\zeta(3\gamma)+\cdots$$

$$= log\prod_{p-prime}\frac{1}{1-p^{-\gamma}}+log\prod_{p-prime}\frac{1}{1-p^{-3\gamma}}+\cdots$$

$$\begin{aligned} &= \sum_{p-prime}\left(log\frac{1}{1-p^{-\gamma}}+log\frac{1}{1-p^{-3\gamma}}+\cdots\right) \\ &= \sum_{p-prime}\left[\left(p^{-\gamma}+\tfrac{1}{2}p^{-3\gamma}+\cdots\right)+\left(p^{-2\gamma}+\tfrac{1}{2}p^{-6\gamma}+\cdots\right)+\cdots\right] \\ &= \sum_{p-prime}\left[\frac{p^{-\gamma}}{1-p^{-2\gamma}}+\frac{1}{2}\frac{p^{-2\gamma}}{1-p^{-4\gamma}}+\cdots\right] \\ &= \sum_{p-prime}\frac{1}{2}\left[\frac{p^{-\gamma}}{1-p^{-\gamma}}+\frac{1}{2}\frac{p^{-2\gamma}}{1-p^{-2\gamma}}+\cdots\right]+\sum_{p-prime}\frac{1}{2}\left[\frac{p^{-\gamma}}{1+p^{-\gamma}}+\frac{1}{2}\frac{p^{-2\gamma}}{1+p^{-2\gamma}}+\cdots\right] \\ &= \sum_{p-prime}\frac{1}{2}\left[\frac{\sigma(1)}{1}p^{-\gamma}+\frac{\sigma(2)}{2}p^{-2\gamma}+\cdots\right]+\sum_{p-prime}\frac{1}{2}\left[\sigma_s(1)p^{-\gamma}+\sigma_s(2)p^{-2\gamma}+\cdots\right] \\ &= \frac{1}{2}\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]+\frac{1}{2}\left[\sigma_s(1)S(\gamma)+\sigma_s(2)S(2\gamma)+\cdots\right] \end{aligned}$$

This gives

$$\lambda = e^{\frac{1}{2}\left(\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]+\frac{1}{2}(\sigma_s(1)S(\gamma)+\sigma_s(2)S(2\gamma)+\cdots]},$$

and

$$D_{h_{\tau_\alpha^*}}(\gamma) = e^{\frac{\alpha}{2}\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]+\frac{\alpha}{2}\left[\sigma_s(1)S(\gamma)+\sigma_s(2)S(2\gamma)+\cdots\right]}.$$

This is recorded in the following theorem.

Theorem 22. We have

$$D_{h_{\tau_\alpha^*}}(\gamma) = e^{\frac{\alpha}{2}\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]+\frac{\alpha}{2}\left[\sigma_s(1)S(\gamma)+\sigma_s(2)S(2\gamma)+\cdots\right]}.$$

In the proof of the theorem above we have the following identities.

Corollary 23. We have

$$\zeta(\gamma)\zeta(3\gamma)\cdots = e^{\frac{1}{2}\left[\frac{\sigma(1)}{1}S(\gamma)+\frac{\sigma(2)}{2}S(2\gamma)+\cdots\right]+\frac{1}{2}\left[\sigma_s(1)S(\gamma)+\sigma_s(2)S(2\gamma)+\cdots\right]}.$$

Corollary 24. We have

$$\left[\left(\frac{1}{2}\frac{\sigma(1)}{1}+\frac{1}{2}\sigma_s(1)-\sigma_{\frac{1}{2}}(1)\right)S(\gamma)+\left(\frac{1}{2}\frac{\sigma(2)}{2}+\frac{1}{2}\sigma_s(2)-\sigma_{\frac{1}{2}}(2)\right)S(2\gamma)+\cdots\right]=0,$$

where

$$\sigma_{\frac{1}{2}}(n) = \begin{cases} \frac{\sigma\left(\frac{n}{2}\right)}{\frac{n}{2}} & \text{if } n \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

4 Conclusion

This study has raised a formal power series to a complex power. Polynomial representation of its coefficients was observed. Subsequently, new number theoretic identities were obtained when limiting formal power series to some well-known generating functions. This kind of study may be extended to existing vast collection of generating functions to derive identities or to study the properties of numerous arithmetic functions.

Declaration

Presented in 9th INTERNATIONAL CONFERENCE ON DISCRETE MATHEMATICS AND MATHEMATICAL MODELLING IN DIGITAL ERA (ICDMMMDE-2023) during March 23-25, 2023, Organized by the Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram - 624302, Dindigul, Tamil Nadu, India. ICDMMMDE-23 was supported by : GRI-DTBU, CSIR.

References

1) Mordell LJ. Note on Certain Modular Relations Considered by Messrs. Ramanujan, Darling, and Rogers. *Proceedings of the London Mathematical Society*. 1922;s2-20(1):408–416. Available from: https://dx.doi.org/10.1112/plms/s2-20.1.408.
2) Heim B, Neuhauser M, Weisse A. Records on the vanishing of Fourier coefficients of powers of the Dedekind eta function. *Research in Number Theory*. 2018;4(3). Available from: https://dx.doi.org/10.1007/s40993-018-0125-y.
3) Zouareg Y, Benoumhani M. On a conjecture of Heim and Neuhauser on some polynomials arising from modular forms and related to Fibonacci polynomials. *The Ramanujan Journal*. 2022;58(1):183–201. Available from: https://dx.doi.org/10.1007/s11139-021-00515-7.
4) Heim B, Neuhauser M. The Dedekind eta function and D'Arcais-type polynomials. *Research in the Mathematical Sciences*. 2020;7(1):3–3. Available from: https://dx.doi.org/10.1007/s40687-019-0201-5.
5) Heim B, Neuhauser M. Estimate for the largest zeros of the D'Arcais polynomials. *Research in the Mathematical Sciences*. 2024;11(1):1–1. Available from: https://dx.doi.org/10.1007/s40687-023-00412-z.
6) Heim B, Neuhauser M, Tröger R. Zeros of recursively defined polynomials. *Journal of Difference Equations and Applications*. 2020;26(4):510–531. Available from: https://dx.doi.org/10.1080/10236198.2020.1748022. doi:10.1080/10236198.2020.1748022.
7) Heim B, Neuhauser M, Tröger R. Zeros transfer for recursively defined polynomials. *Research in Number Theory*. 2023;9(4). Available from: https://dx.doi.org/10.1007/s40993-023-00480-8.
8) Abinash S. On 3-divisibility of 9- and 27-regular partitions. *The Ramanujan Journal*. 2022;57(3):1193–1207. Available from: https://dx.doi.org/10.1007/s11139-021-00463-2.
9) Cherubini G, Mercuri P. Parity of the 8-regular partition function. *The Ramanujan Journal*. 2024;63(3):715–722. Available from: https://dx.doi.org/10.1007/s11139-023-00784-4.

10) Li BQ, Steuding J. Fixed points of the riemann zeta function and dirichlet series. *Monatshefte für Mathematik*. 2022;198(3):581–589. Available from: https://dx.doi.org/10.1007/s00605-022-01709-x.
11) Navas LM, Ruiz FJ, Varona JL. A connection between power series and Dirichlet series. *Journal of Mathematical Analysis and Applications*. 2021;493(2):124541–124541. Available from: https://dx.doi.org/10.1016/j.jmaa.2020.124541.
12) Chavan P, Chavan S, Vignat C, Wakhare T. Dirichlet series under standard convolutions: variations on Ramanujan's identity for odd zeta values. *The Ramanujan Journal*. 2022;59(4):1245–1285. Available from: https://dx.doi.org/10.1007/s11139-022-00624-x.