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RESEARCH ARTICLE

Discrete Labeling of Graphs

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Abstract

Objectives: To introduce a new type of labeling called Discrete Labeling. This labeling aims at providing the maximum output whenever the inputs are distinct with rational distribution of neighbours thereby exhibiting the stronger form of Cordial labeling. This work serves as an aid in decision-making. **Methods:** The discrete labels 0 and 1 are cordially assigned to the vertices such that the edges receive labels depending on the incident vertex labels using EX-OR operation with the condition that for every vertex the cardinality of neighbours labeled 0 and 1 differs by at most 1. **Findings:** This study proposes the discrete labeling of some standard and special graphs. Also it provides the cases for which certain path, star and cycle related graphs admit discrete labeling. **Novelty:** This labeling uses EX-OR operation which reduces the complexity of having two swords in one sheath. Apart from keeping the distinct vertex labels and edge labels difference minimal, the cardinality of the neighbouring labels of every vertex is also taken into account.

Keywords: Graph Parameters; Discrete; Trees; Cycle; Complete Graphs

1 Introduction

Labeling of graphs is a function that maps the vertex set (edge set) to the set of labels. Here, the domain and codomain are the set of vertices and {0, 1} respectively. Motivated by the cordial labeling[(1)] in this paper, we have defined a new type of labeling called "Discrete labeling" which assigns labels using EX-OR operations taking the neighbouring labels into consideration. This newly proposed labeling helps in characterization of neighbours which will further be helpful in determining highly effective, non-repetitive groups. The graph (V,E) discussed here are simple, connected and undirected. The terminologies and symbols used in this paper are in accordance with[(2)].

Note: In this paper, neighbours of v denote the adjacent vertices of the vertex $v$and

$\bar{0}= 1$, $\bar{1}= 0$

2 Methodology

Definition 2.1

Let $G(V,E)$be a simple, connected, undirected graph.G is said to have a discrete labeling if there exist functions $d : V \to \{0,1\}$ and $e : E \to (0,1\}$ defined by

$$e(uv) = \begin{cases} 0; d(u) = d(v) \\ 1; d(u) \neq d(v) \end{cases} \quad u,v \in V \text{ for which}$$

$$|n_d(0) - n_d(1)| \leq 1 \tag{i}$$

$$|n_e(0) - n_e(1)| \leq 1 \tag{ii}$$

$$\left|n_{N(v)}(0) - n_{N(v)}(1)\right| \leq 1 \forall v \in V \tag{iii}$$

where $n_d(x)$ *and* $n_e(x)$ denote the number of vertices and edges with $d(u) = x$ and $e(uv) = x$; $x \in (0,1\}$ respectively and $n_{N(v)}(0)$ *and* $n_{N(v)}(1)$ denote the number of neighbours of the vertex v labeled 0 and 1 respectively. A Graph G is discrete if it admits discrete labeling[3]

Example:

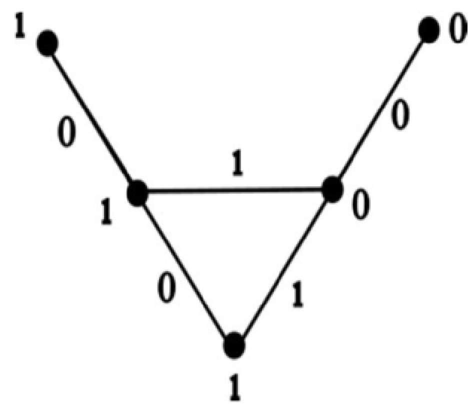

Fig 1. Bull graph

In Figure 1 , $|n_d(0) - n_d(1)| = |2-3| = 1$ and $|n_e(0) - n_e(1)| = |3-2|$. Also, for every vertex the cardinality of the neighbouring vertices labeled 0 and 1 differs by at most 1. Hence Bull graph admits discrete labeling.

3 Results and Discussion

Discrete Labeling of Some Standard Graphs

Theorem 3.1.1

Path P_n admits discrete labeling

Proof: Let the vertices of P_n be $u_1, u_2, \ldots, u_n$[4]. Define a function $f : V \to \{0,1\}$ by

$$f(u_1) = 0$$
$$f(u_{2i}) = f(u_{2i+1}) = \begin{cases} 1 & \text{when } i \text{ is odd} \\ 0 & \text{when } i \text{ is even} \end{cases}$$

Label the edges 1 if the vertex labels are distinct and 0 otherwise.

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed in Table 1 :

Table 1. Edge Conditions of P_n

Case	$\|V\|$	$\|E\|$	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$
$n = 0 \ mod\ 4$	$4t$	$4t-1$	$2t$	$2t$	$(2t-1)$	$2t$
$n = 1 \ mod\ 4$	$4t+1$	$4t$	$2t+1$	$2t$	$2t$	$2t$
$n = 2 \ mod\ 4$	$4t+2$	$4t+1$	$2t+1$	$2t+1$	$2t$	$2t+1$
$n = 3 \ mod\ 4$	$4t+3$	$4t+2$	$2t+1$	$2t+2$	$2t+1$	$2t+1$

For all n,

$$n_{N_{u_1}}(0) = n_{N_{u_n}}(0) = 1$$

$$n_{N_{u_1}}(1) = n_{N_{u_n}}(1) = 0$$

$$n_{N_{u_i}}(0) = n_{N_{u_i}}(1) = 1,\ 2 \leq i \leq n-1.$$

It is evident from the above cases that $|n_d(0) - n_d(1)| \leq 1, |n_e(0) - n_e(1)| \leq 1$ and $\left|n_{N_{u_i}}(0) - n_{N_{u_i}}(1)\right| \leq 1$ for all i. Hence Path P_n admits discrete labeling.

Theorem 3.1.2

Star $K_{1,n-1}$ is discrete.

Proof: Let $u, v_i(i = 1,2,\ldots,n-1)$ denote the central and the pendant vertices of the star graph. For Star $K_{1,n-1} |V| = n, |E| = n-1$

Define a function $f : V \to \{0,1\}$ by

$$f(v_i) = \begin{cases} f(u) = 1 \\ 0 \text{ when } i \text{ is odd} \\ 1 \quad \text{when } i \text{ is even} \end{cases}$$

Label the edges 1 if the vertex labels are distinct and 0 otherwise.

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed in Table 2 :

Table 2. Label Count in $K_{1,n-1}$

Case	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$	$n_{N_u}(0)$	$n_{N_u}(1)$	$n_{N_{v_i}}(0) 1 \leq i \leq n-1$	$n_{N_{v_i}}(1) 1 \leq i \leq n-1$
n is odd	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	0	1
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	0	1

It is obvious from the above table that $|n_d(0) - n_d(1)| \leq 1, |n_e(0) - n_e(1)| \leq 1$
$|n_{N_u}(0) - n_{N_u}(1)| \leq 1$ and $\left|n_{N_{v_i}}(0) - n_{N_{v_i}}(1)\right| = 1$ for all i . . Hence the star graph is discrete.

Theorem 3.1.3

Cycle graph $C_n$admits discrete labeling only when n is a multiple of 4.

Proof: Let the vertices of $C_n$be denoted by $u_1, u_2, \ldots, u_n$ [5]. Let us find a suitable function for vertex labeling as follows:

Case 1: When n is a multiple of 4 ($n = 4i, i = 1,2,\ldots$)

Define a function $f : V \to \{0,1\}$ by

$$f(u_{4i+1}) = f(u_{4i+2}) = 0\ ; i = 0,1,2,\ldots$$

$$f(u_{4i-1}) = f(u_{4i}) \ = \ 1;\ i = 1,2,3,\ldots$$

Table 3. Label Count in C_{4n}

Case	$\lvert V\rvert$	$\lvert E\rvert$	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$	$n_{N_{u_i}}(0) 1 \leq i \leq n$	$n_{N_{u_i}}(1) 1 \leq i \leq n$
$n = 0 \; mod \; 4$	$4t$	$4t$	$2t$	$2t$	$2t$	$2t$	1	1

Label the edges as 1 if the vertex labels are distinct and 0 otherwise.

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed below Table 3 :

Hence the Cycle graph admits , $|n_d(0) - n_d(1)| = 0, |n_e(0) - n_e(1)| = 0$ and $\left|n_{N_{u_i}}(0) - n_{N_{u_i}}(1)\right| \leq 1$ for all i . . Hence Cycle graph is discrete when n is a multiple of 4.

Case 2: When n is odd

In a cycle, every vertex has exactly two neighbours. When we try labeling the vertices optimally satisfying (i) and (ii) of the definition, we observe that there exists a vertex whose neighbours are either both 0 or 1 violating the condition (iii) of the definition. Hence C_n is not discrete when n is odd.

Case 3: When $n \equiv 2 \; mod \; 4$

In this case, labeling of vertices is not discrete since the cardinality of half the vertex set is odd. Hence it is not possible to label the vertices and edges satisfying conditions (i) and (ii). Therefore C_n is not discrete when $n \equiv 2 \; mod \; 4$.

The above cases clearly shows that the cycle graph $C_n$admits discrete labeling only when n is a multiple of 4.

3.2 Discrete Labeling of Complete graphs

Theorem 3.2.1

Complete bipartite graph $K_{r,s}$satisfies discrete labeling.

Proof: Let the vertices of $K_{r,s}$ be $u_1, u_2, ..., u_r, u_{r+1}, ..., u_{r+s}$. Here $|V| = r+s, |E| = rs$

Define a function $f : V \to \{0,1\}$ by

$$f(u_i) = \begin{cases} 0 \text{ when } i \text{ is odd} \\ 1 \quad \text{when } i \text{ is even} \end{cases}$$

Label the edges 1 if the vertex labels are distinct and 0 otherwise.

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed below Table 4 :

Table 4. Label Count in $K_{r,s}$

Case	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$	$n_{N_{u_i}}(0), \; 1 \; i \; r$	$n_{N_{u_i}}(1), \; 1 \; i \; r$
$r+s$ is odd	$\frac{r+s+1}{2}$	$\frac{r+s-1}{2}$	$\frac{rs}{2}$	$\frac{rs}{2}$	$\frac{s}{2}$	$\frac{s}{2}$
$r+s$ is even	$\frac{r+s}{2}$	$\frac{r+s}{2}$	$\frac{rs-1}{2}$	$\frac{rs+1}{2}$	$\frac{r+1}{2}$	$\frac{r-1}{2}$

Table 5.

$n_{N_{u_i}}(0), \quad r+1 \; i \; r+s$	$n_{N_{u_i}}(1) r+1 \; i \; r+s$
$\frac{r+1}{2}$	$\frac{r-1}{2}$
$\frac{s-1}{2}$	$\frac{s+1}{2}$

It is observed that

$$|n_d(0) - n_d(1)| = \begin{cases} 1 \text{ when } r+s \text{ is odd} \\ 0 \text{ when } r+s \text{ is even} \end{cases},$$

$$|n_e(0) - n_e(1)| = \begin{cases} 0 \text{ when } r+s \text{ is odd} \\ 1 \text{ when } r+s \text{ is even} \end{cases}$$

For every i, $\left|n_{N_{u_i}}(0) - n_{N_{u_i}}(1)\right| \leq 1$. . Hence we infer that $K_{r,s}$ is discrete.

Theorem 3.2.2

For $n \geq 3$, Complete graph K_n is not discrete.

Proof: It is obvious that K_n is discrete when $n < 3$. When $n = 3$, condition (iii) of the definition fails since K_3 is nothing but C_3. Assume $n \geq 4$. Since there are $\frac{n(n-1)}{2}$ number of edges which is even, equal number of 0's and 1's have to be shared for the edges. It is possible to obtain such an edge labeling, only if we assign $(n-1)$ zero or one labels to the vertices. This is not possible by the definition of the discrete labeling. Hence complete graph $(K_n, n \geq 3)$ is not discrete.

3.3 Discrete Labeling of Some Path and Star related graphs

Theorem 3.3.1

Comb graph is discrete.

Proof: Comb is a graph obtained by joining a single pendant edge to each vertex of a path P_n $(P_n \odot K_1)$. Let us denote the vertices of P_n by $u_1, u_2, \ldots, u_n$ and the pendant vertices by $v_1, v_2, \ldots, v_n$.

Case 1 : When $n \equiv 0 \; mod \; 4$

Define a function $f : V \to (0,1\}$ by

$$f(u_1) = 0$$

$$f(u_{2i}) = f(u_{2i+1}) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} f(v_5) = 1 \\ 0 \text{ when } i \text{ is odd, } i \neq 5 \\ 1 \text{ when } i \text{ is even, } i \neq n \\ f(v_n) = 0 \end{cases}$$

Case 2:

When $n \equiv 1$ mod 4, comb is discrete for we define a function $f : V \to (0,1\}$ by

$$f(u_1) = 0$$

$$f(u_{2i}) = f(u_{2i+1}) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

$$f(v_n) = 1$$

For the other cases of n (i.e) $n \equiv 2 \; mod \; 4$ and $n \equiv 3 \; mod \; 4$, we shall similarly define a function $f : V \to \{0,1\}$ by

$$f(u_1) = 0$$

$$f(u_{2i}) = f(u_{2i+1}) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 0 \text{ when } i \text{ is odd} \\ 1 \text{ when } i \text{ is even} \end{cases}$$

and label the edges 1 if the incident vertex labels are distinct and 0 otherwise.

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed below:

From the above table Table 6 it is very clear that $\begin{matrix} |n_d(0) - n_d(1)| = 0 \\ |n_e(0) - n_e(1)| = 1 \end{matrix}$,For $i = 1, n$; $|n_{N_{u_i}}(0) - n_{N_{u_i}}(1)| = 0$, for $2 \leq i \leq n-1$,$|n_{N_{u_i}}(0) - n_{N_{u_i}}(1)| = 1$ and for all i,$|n_{N_{v_i}}(0) - n_{N_{v_i}}(1)| = 1$.

Hence the proof.

Broom graph $B_{m,n}$ admits discrete labeling.

Proof: Theorem 3.3.2 : A broom graph $B_{m,n}$ is a graph with m vertices, which has a path P_n and $m-n$ pendant vertices. Let the vertices of P_n be $u_1, u_2, \ldots, u_n$ and the pendant vertices be $v_1, v_2, \ldots, v_{m-n}$.

Table 6. Labels of Comb graph

Case	$\lvert V\rvert$	$\lvert E\rvert$	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$	$n_{N_{u_i}}(0)i=1,n$	$n_{N_{u_i}}(1)i=1,n$
$n \equiv 3\ mod\ 4,$	$8t$	$8t-1$	$8t$	$8t$	$4t-1$	$4t$	1	1
$n \equiv 1\ mod\ 4$	$8t+2$	$8t+1$	$4t+1$	$4t+1$	$4t$	$4t+1$	1	1
$n \equiv 1\ mod\ 4$	$8t+4$	$8t+3$	$4t+2$	$4t+2$	$4t+2$	$4t+1$	1	1
$n = 3\ mod\ 4$	$8t+6$	$8t+5$	$4t+3$	$4t+3$	$4t+3$	$4t+2$	1	1

Table 7.

$n_{N_{u_i}}(0)2 \le i \le n-1$	$n_{N_{u_i}}(1)2 \le i \le n-1$	$n_{N_{u_5}}(0)$	$n_{N_{u_5}}(1)$	$n_{N_{v_1}}(0)$	$n_{N_{v_1}}(1)$	$n_{N_{v_{2i}}}(0)$	$n_{N_{v_{2i}}}(1)$
$1, i\ even 2, i\ odd (i \neq 5)$	$2, i\ even 1, i\ odd (i \neq 5)$	1	2	1	0	$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$
$1, i\ even 2, i\ odd$	$2, i\ even 1, i\ odd$	–	–	1	0	$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$
$1, i\ even 2, i\ odd$	$2, i\ even 1, i\ odd$	–	–	1	0	$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$
$1, i\ even 2, i\ odd$	$2, i\ even 1, i\ odd$	–	–	1	0	$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$

Table 8.

$n_{N_{v_{2i+1}}}(0)$	$n_{N_{v_{2i+1}}}(1)$
$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$
$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$
$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$
$0, i\ odd 1, i\ even$	$1, i\ odd 0, i\ even$

For the Broom graph $|V| = m, |E| = m-1$
Define a function $f : V \to \{0,1\}$ by

$$f(u_1) = 0$$

$$f(u_{2i+1}) = f(u_{2i}) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

Now label the pendant vertices as follows:
Case 1: When $m-n$ is even($n\ odd/n$ even)
Label $v_i(i = 1,2,\ldots,m-n)$ as $f(v_i) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$
Case 2: When $m-n$ is odd
Subcase (i):When $m-n$ is odd and n is odd
Label

$$v_i, \forall i \le m-n-1 \text{ as } \quad f(v_i) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

and $f(v_{m-n}) = f(\bar{u}_n)$
Subcase (ii): When $m-n$ is odd and n *is* even
Label

$$v_i, \forall i \le m-n-1 \text{ as } \quad f(v_i) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

and $f(v_{m-n}) = f(u_n)$.
Then Label the edges 1 if the incident vertex labels are distinct and 0 otherwise.

Table 9. Label Count in $B_{m,n}$

Case	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$	$n_{N_{u_1}}(0)$	$n_{N_{u_1}}(1)$
$m-n$ *is odd* &$n = 0 \bmod 4$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	0	1
$m-n$ *is odd* &$n = 2 \bmod 4$	$\frac{m-1}{2}$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	0	1
$m-n$ *is odd* & $n = 1 \bmod 4$	$\frac{m}{2}$	$\frac{m}{2}$	$\frac{m-2}{2}$	$\frac{m}{2}$	0	1
$m-n$ *is odd* & $n = 3 \bmod 4$	$\frac{m}{2}$	$\frac{m}{2}$	$\frac{m-2}{2}$	$\frac{m}{2}$	0	1
$m-n$ *is even* & $n = 0 \bmod 4$	$\frac{m}{2}$	$\frac{m}{2}$	$\frac{m-2}{2}$	$\frac{m}{2}$	0	1
$m-n$ *is even* & $n = 2 \bmod 4$	$\frac{m}{2}$	$\frac{m}{2}$	$\frac{m-2}{2}$	$\frac{m}{2}$	0	1
$m-n$ *is even* & $n = 1 \bmod 4$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	0	1
$m-n$ *is even* & $n = 3 \bmod 4$	$\frac{m-1}{2}$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	0	1

Table 10.

$n_{N_{u_i}}(0) 2 \le i \le n-1$	$n_{N_{u_i}}(1) 2 \le i \le n-1$	$n_{N_{u_n}}(0)$	$n_{N_{u_n}}(1)$	$n_{N_{v_i}}(0) 1 \le i \le n$	$n_{N_{v_i}}(1) 1 \le i \le n$
1	1	$\frac{m-n+1}{2}$	$\frac{m-n+1}{2}$	1	0
1	1	$\frac{m-n+1}{2}$	$\frac{m-n+1}{2}$	0	1
1	1	$\frac{m-n+1}{2}$	$\frac{m-n+1}{2}$	1	0
1	1	$\frac{m-n+1}{2}$	$\frac{m-n+1}{2}$	0	1
1	1	$\frac{m-n}{2}$	$\frac{m-n+2}{2}$	1	0
1	1	$\frac{m-n+2}{2}$	$\frac{m-n}{2}$	0	1
1	1	$\frac{m-n+2}{2}$	$\frac{m-n}{2}$	1	0
1	1	$\frac{m-n}{2}$	$\frac{m-n+2}{2}$	0	1

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed below Table 9 :

Hence we can conclude that $|n_d(0) - n_d(1)| \le 1, |n_e(0) - n_e(1)| \le 1$,,

$$\left| n_{N_{u_i}}(0) - n_{N_{u_i}}(1) \right| = 1 \text{ and } \left| n_{N_{v_i}}(0) - n_{N_{v_i}}(1) \right| = 1 \forall i \, .$$

Hence $B_{m,n}$ admits discrete labeling.

Theorem3.3.3

Bistar is discrete.

Proof: Bistais a graph obtained by joining the central vertices of two copies of the star graphs $K_{1,m}$ and $K_{1,n}$ respectively. Here, $|V| = m+n+2, |E| = m+n+1$

Without loss of generality assume, $m \ge n$ .Let u, v denote the central vertices and $u_i (i = 1, 2, \ldots, m)$ and $v_i (i = 1, 2, \ldots, n)$ be the pendant vertices of $K_{1,m}$ and $K_{1,n}$ respectively. Now, Let us define a function $f : V \to \{0, 1\}$ by

$$f(u) = 0, f(v) = 1$$

$$f(u_i) = \begin{cases} 0 \text{ when } i \text{ is odd} \\ 1 \text{ when is even} \end{cases}$$

$$f(v_i) = \begin{cases} 1 \text{ when } i \text{ is odd} \\ 0 \text{ when } i \text{ is even} \end{cases}$$

which gives the desired vertex labeling. Then assign labels 0 and 1 to the edges if the incident vertex labels are similar and distinct respectively.

The number of vertices and edges labeled 0 and 1 and for every vertex the number of neighbouring vertices labeled 0 and 1 are listed below:

Table 11. Label Counts of Bistar graph

Case	$n_d(0)$	$n_d(1)$	$n_e(0)$	$n_e(1)$	$n_{N_u}(0)$	$n_{N_u}(1)$
m even , n odd	$\frac{m+n+1}{2}$	$\frac{m+n+3}{2}$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$	$\frac{m}{2}$	$\frac{m+2}{2}$
m odd, n even	$\frac{m+n+3}{2}$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$	$\frac{m+1}{2}$	$\frac{m+1}{2}$
m odd, n odd	$\frac{m+n+2}{2}$	$\frac{m+n+2}{2}$	$\frac{m+n+2}{2}$	$\frac{m+n}{2}$	$\frac{m+1}{2}$	$\frac{m+1}{2}$
m even, n even	$\frac{m+n+2}{2}$	$\frac{m+n+2}{2}$	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$	$\frac{m}{2}$	$\frac{m+2}{2}$

Table 12.

$n_{N_v}(0)$	$n_{N_v}(1)$	$n_{N_{u_i}}(0) 1 \le i \le m$	$n_{N_{u_i}}(1) 1 \le i \le m$	$n_{N_{v_i}}(0) 1 \le i \le n$	$n_{N_{v_i}}(1) 1 \le i \le n$
$\frac{n}{2}$	$\frac{n}{2}$	1	0	0	1
$\frac{n+2}{2}$	$\frac{n}{2}$	1	0	0	1
$\frac{n+1}{2}$	$\frac{n+1}{2}$	1	0	0	1
$\frac{n+2}{2}$	$\frac{n}{2}$	1	0	0	1

Hence from all the above cases Table 11 it is obvious from the table that $|n_d(0) - n_d(1)| \le 1$,,
$|n_e(0) - n_e(1)| \le 1$,
$|n_{N_u}(0) - n_{N_u}(1)| \le 1, |n_{N_v}(0) - n_{N_v}(1)| \le 1, \left|n_{N_{u_i}}(0) - n_{N_{u_i}}(1)\right| = 1$ and $\left|n_{N_{v_i}}(0) - n_{N_{v_i}}(1)\right| =$
$1 \forall i$.
Therefore, we can conclude that Bistar is discrete.

3.4 Discrete Labeling of some Special graphs

3.4.1: The Diamond graph is a Hamiltonian,planar,unit distance graph with 4 vertices and 5 edges.It is a complete graph K_4minus one edge.It is often referred as double triangle graph. In diamond graph, $|n_d(0) - n_d(1)| = |2 - 2| = 0$ and $|n_e(0) - n_e(1)| = |2 - 3| = 1$. Also, for every vertex the cardinality of the neighbouring vertices labeled 0 and 1 differs by at most 1.Hence diamond graph is discrete.

3.4.2: The Moser Spindle is a planar, unitdistance, Laman graph with seven vertices and eleven edges.It is sometimes termed as Hajo'sgraph. It is obvious(Figure 2) that $|n_d(0) - n_d(1)| = |3 - 4| = 1$ and $|n_e(0) - n_e(1)| = |5 - 6| = 1$. Also, for every vertex the cardinality of the neighbouring vertices labeled 0 and 1 differs by at most 1 satisfying all the 3 conditions of the definition which draws a conclusion that Moser Spindle is discrete.

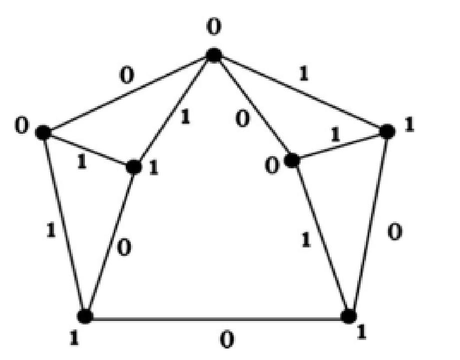

Fig 2. Moser Spindle

3.4.3: The Wagner graph is a 3-regular,Hamiltonian,planar graph with 8 vertices and 12 edges. It is the 8-vertex Mobius Ladder graph. Wagner graph is discrete for, $|n_d(0) - n_d(1)| = |4 - 4| = 0$, $|n_e(0) - n_e(1)| = |6 - 6| = 0$. (Figure 3) and for every vertex the (iii) condition of the definition holds.

3.4.4: The Herschel graph is a non-hamiltonian, bipartite, polyhedral, perfect,planar graph with 11 vertices and 18 edges. It is also known as Goldner-Harary graph. In this graph, (Figure 4) $|\, n_d(0) - n_d(1)| = |6-5| = 1$ and $|n_e(0) - n_e(1)| = |9-9| = 0$ which shows that the number of vertices labeled 0 and 1 and the number of edges labeled 0 and 1 have an absolute difference at most 1. We also observe that for every vertex the cardinality of the neighbouring vertices labeled 0 and 1 differs by at most 1.Therefore, Herschel graph admits Discrete Labeling.

3.4.5: The Petersen graph is a cubic, non-planar (10, 15) graph . Petersen graph admits Discrete labeling for, the absolute difference between the number of vertices labeled 0 and 1 =$|5-5|$ =0. The absolute difference between the number of edges labeled 0 and 1 = $|8-7|$ = 1.In either cases, the absolute difference is less than or equal to unity (Figure 5). Also, the cardinality of the neighbours of every vertex labeled 0 and 1 differs by at most 1. Hence, Petersen graph is discrete.

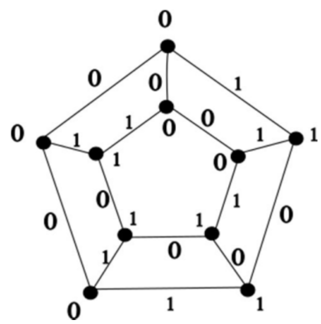

Fig 3. Wagner graph

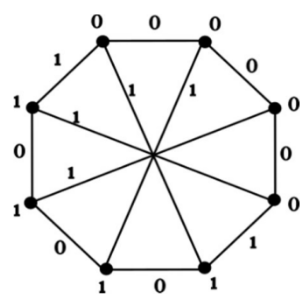

Fig 4. Herschel Graph

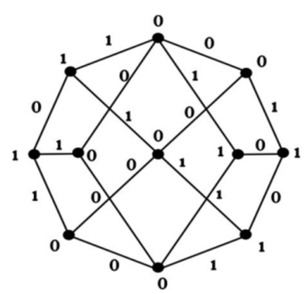

Fig 5. Petersen graph

4 Discussion

All Discrete graphs are cordial but not all Cordial graphs are discrete. This study has found that Cycle graph $C_n$admits discrete labeling only when n is a multiple of 4. But Cycle graph is cordial if and only if $n \not\equiv 2 \; mod \; 4$ [(6)]. Further studies are being done in classification of graphs that satisfy all the three conditions. It is observed that certain cycle related graphs [7] like tadpoles, wheels and friendship graph [8] admit discrete labeling for only certain cases; whereas, they are cordial for more cases [9].

5 Conclusion

This work represents the stronger form of cordial labeling which incorporates the cardinality of the neighbor vertex labels. In this paper we have discussed the Discrete labeling of some standard and special graphs. Further explorations are being done on star related graphs and double graphs. Also, this work can be extended by applying the adjacency condition of the vertices to the edges resulting in edge discrete labeling [10,11].

Declaration

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