INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Fuzzy Tree t -Spanners of Ratio Labelled Butterfly Network

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OPEN ACCESS

Received: 29-08-2023

Accepted: 05-03-2024

Published: 27-05-2024

Editor: ICDMMMDE-2023 Special Issue Editors: Dr. G. Mahadevan & Prof. Dr. P. Balasubramaniam

Citation: Pritha RM, Amutha A (2024) Fuzzy Tree t -Spanners of Ratio Labelled Butterfly Network. Indian Journal of Science and Technology 17(SP1): 58-63. https://doi.org/10.17485/IJST/v17sp1.164

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Background/Objectives: The reliability and optimality of a communication system depend on the network that has a minimum distance. The minimum distance is the shortest path between the vertices. Even in error correction codes, the minimum Hamming distance ensures data integrity. Hence to have efficient routing protocols, the network is preferred to be a spanning tree that connects all the nodes of the communication system. As tree t-spanner is the optimization technique in determining minimum spanning tree, finding tree t-spanners of a ratio labelled fuzzy graph for the Butterfly network BF(n) is notion of this study. **Methods:** This study has introduced a new labelling called Ratio Labelling (RL) for examining the fuzziness in a crisp graph. Using the RL method, the crisp graphs were examined for the admittance of fuzziness and were thereby found, to have a tree t-spanners of ratio labelled fuzzy graphs. **Findings:** The quality of a network can be measured using the stretch factor of the network. The issues of network survival or link failures in a communication networks can be rectified by approximating the best flow spanner. As the optimal connectivity between the nodes is ensured by interconnection networks, the fuzzy tree t- spanners of undirected fuzzy Butterfly Network BF(n) are examined by labelling the vertices and edges using ratio labelling. The Breadth First Search (BFS) algorithm enhances the process of minimizing the spanning tree for an unweighted graph. Since the edge weightage is unique for all the edges of BF(n) under RL, searching for an edge with minimum weight by Kruskal's algorithm or Prim's algorithm are less meaningful. Hence, the BFS algorithm is adopted in finding the spanning tree T of Ratio Labelled BF(n). The bounds for the stretch factor t of a fuzzy tree t-spanner of ratio labelled BF(n) were obtained. **Novelty:** The literature survey shows that various classes of graphs were examined for the admissibility of tree t-spanners. Finding a tree t-spanner for a fuzzy graph was not found in literature, as per our knowledge. Extending the problem of finding a tree t -spanner of a ratio labelled fuzzy graph is an advancement in the field of fuzzy graphs.

Keywords: Fuzzy Graph; Fuzzy distance; Ratio Labelling; Tree t- spanner; Butterfly Network

1 Introduction

In 2020, Archana A. Deshpande and Onkar K. Chandhari compared a minimal spanning tree problem using various algorithms[1]. A new genetic algorithm was formulated by Dey, A., Son, L.H., Pal, A et al., to find a fuzzy minimum spanning tree with interval type 2 fuzzy arc length in 2019[2]. An algorithmic approach to finding MST(Minimum Spanning Tree) in an intuitionistic fuzzy graph was done by Katick Mohanta, Arindam Dey, Narayan C. Debath and Anita Pal in 2019[3]. Amutha A. and Mathu Pritha R., defined RL and examined the admittance of fuzziness in basic crisp graphs in 2020. In 2022, Amutha A. and Mathu Pritha R., further studied the interconnection networks for the admittance of fuzziness using RL[4]. Renzo Gomez, Flavio K. Miyazawa and Yoshiko Wakabayshi derived an algorithm for the existence of tree t-spanner in an arbitrary graph[5]. Strategies for generating tree spanners were discussed by Fernanda Couto et al.[6]. Fernando Couto et al. studied about the characteristics of tree t- spanners of graphs with a few P_4's[7]. Francis Remigius Perpetua Mary et al. discussed the minimum spanning tree problem under the fermatean fuzzy environment[8]. Various dynamics of examining tree t- spanners exist in the history for crisp graphs, however, the tree t- spanner of a fuzzy graph has not been addressed yet. This creates an eagerness to extend the topic to fuzzy graphs. Instead of using arbitrary labels for vertices and edges in a fuzzy graph, it is considered to utilize descriptive terms or contextual identifiers that reflect the characteristics or relationships between vertices and edges. This could involve naming vertices based on their attributes of functions and labelling edges to denote the strength or degree of connection between vertices. The concept of identifying a crisp graph as a fuzzy graph by introducing a degree of fuzziness or uncertainty, subject to some limitations was introduced. The notion introduced the concept of ratio labelling (RL), which typically doesn't allow for fuzziness in all crisp graphs but permits a degree of fuzziness under particular circumstances. Hence, introduced ratio labelling and admittance of fuzziness was examined with RL. All regular graphs, degree bounded graphs and some arbitrary graphs permits fuzziness due to ratio labelling. The crisp graph that admits fuzziness was further investigated for tree t- spanner. The main idea is to examine the tree t- spanner of all ratio labelled fuzzy graphs. Here, the ratio labelled Butterfly network BF(n) was examined for tree t-spanner. The bounds of tree spanner of BF(n) will be discussed.

2 Methodology

Labelling the vertices and edges of a crisp graph using RL does not guarantee fuzziness in all graphs. All regular graphs, paths, interconnection networks such as Hypercube, Cube Connected Cycles, Butterfly Network, Benes Network etc., and some arbitrary graphs admit fuzziness under RL. The definition of Ratio Labelling is specified as below.

Definition: 2.1

Let $G(V,E)$ be a crisp graph. The labelling of vertices and edges of G using the functions, $\sigma : V \to [0,1]\, 0,1$, *and* $\mu : E \to [0,\ 1]$ respectively, are defined as

$$\sigma(v) = \frac{|N(v)|}{|E|} \tag{1}$$

$$\mu(u,\mathrm{v}) = \frac{\max_{(u,v)\in E}[\sigma(u),\sigma(v)]}{\sum_{v\in V}\sigma(v)} \tag{2}$$

is called ratio labelling of G .

2.1 Fuzzy tree t-spanner of a Fuzzy Graph

A subgraph G' of a given graph G is a t-spanner for G if for each pair of points $u,v \in V$, there exists a path in G' of weight atmost t times the weight of path between $u,v \in V$, in G. For a general crisp graph G, the spanning tree T of G is a tree t- spanner of G if

$d_T(v,w) \leq t \times d_G(v,w)$, for every pair of nodes of G.

The fuzzy distance between two nodes u and v is defined as

$d_f(u,v) = \Lambda\sum\{\wedge(\sigma(u),\sigma(v)) \times \mu(u,v)\}$, where $\wedge$ represents minimum.

The novel idea of tree t-spanner in fuzzy graphs is introduced with the above definition to find the bounds for stretch index t. Hence, a fuzzy tree t- spanner for a fuzzy graph structure is defined as follows.

Definition : 2.2

Let $G(V,E)$ be a fuzzy graph. $T\ (V,\ E')$ is a fuzzy tree t- spanner of G, $E' \subset E$, if for each pair, $s,t \in V$ there is a fuzzy path between u and v in T of length $d_f^T\,(u,v)$ atmost t times the fuzzy distance $d_f\,(u,v)$ between s and t in G.

$$i.e.\quad d_f^T\,(u,v) \leq t \times d_f\,(u,v)$$

In the following chapter, ratio labelled butterfly network is to be investigated for tree t- spanner.

Under RL, BF(n) has its vertex weightage $\sigma\,(v)$ as

$\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{2}{n2^{n+1}}, \text{ for } v \in V \text{ in level 0 and } n \\ \frac{4}{n2^{n+1}}, \text{ for } v \in V \text{ in level } i, \text{ for } 1 \leq i\ \leq n-1 \end{cases}$

and edge weightage as

$\mu\,(u,v) = \frac{max[\sigma(u),\sigma(v)]}{\sum_{v\in V}\sigma(v)} = \frac{1}{n2^n}$, for all, $(u,v) \in E$

As the weightage to edges of ratio labelled BF(n) is uniform for all the edges, BFS algorithm is applied to find a minimum spanning tree.

The stretch factor t of tree spanner has been analysed for ratio labelled fuzzy BF(n).

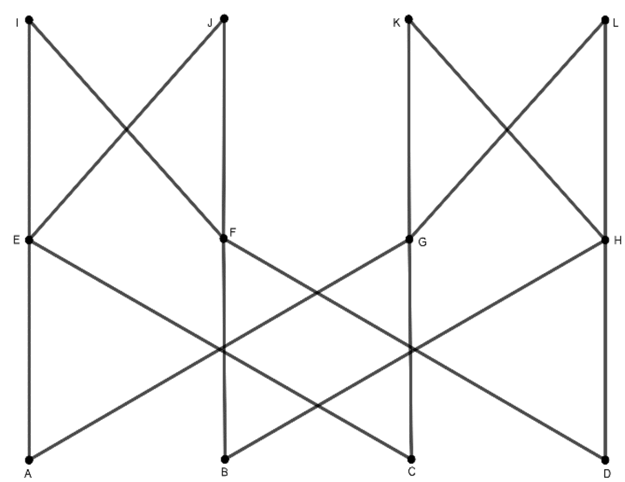

Fig 1. BF(2)

The ratio labelled BF(n) has its vertex and edge labelling as

$\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{1}{n\times 2^n}, \text{ for } v \in V \text{ in level 0 and } n \\ \frac{1}{n\times 2^{n-1}}, \text{ for } v \in V \text{ in level } i, 1 \leq i\ \leq n-1 \end{cases}$

For BF(2),

$$\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{1}{2\times 2^2}, \text{ for } v \in V \text{ in level 0 and 2} \\ \frac{1}{2\times 2^1}, \text{ for } v \in V \text{ in level 1} \end{cases}$$

$$\mu\,(u,v) = \frac{max[\sigma\,(u)\,,\sigma\,(v)]}{\sum_{v\in V}\sigma(v)} = \frac{1}{2\times 2^2}$$

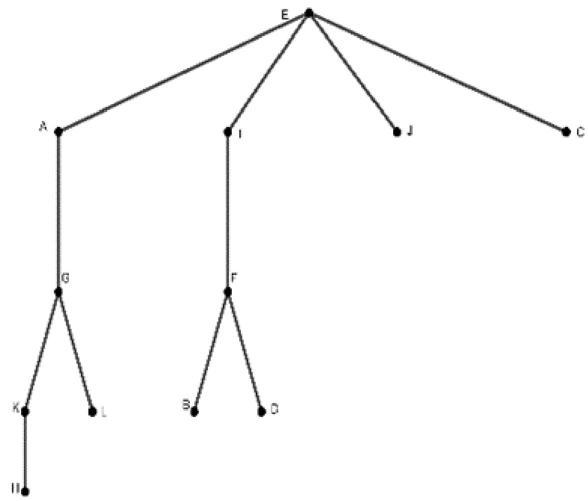

Fig 2. Spanning tree of BF(2) by BFS

Here, BFS algorithm generates a spanning tree in which the vertices H and B and H and D produce the maximum stretch which are adjacent in BF(2). Here, the stretch is 7.

The path between (H, B) and (H, D) pass through 7 edges and each edge has its one vertex in level either 0 or 2.

$$d_f^T(H,B) = 7 \times \frac{1}{2 \times 2^2} \times \frac{1}{2 \times 2^2} = \frac{7}{2^6}$$

$$d_f(H,B) = \frac{1}{2 \times 2^2} \times \frac{1}{2 \times 2^2} = \frac{1}{2^6}$$

$$d_f^T(H,B) = \frac{7}{2^6} = 7d_f(a,b)$$

Hence, for any two vertices u, v, of BF(2),

$$d_f^T(u,v) \leq 7d_f(u,v)$$

Hence, the minimum tree t- spanner of BF(2) has stretch factor 7.

3 Results and Discussion

This section deals with the novel approach of finding the stretch index t of the ratio labelled butterfly network BF(n). As the edge weightage of every edge is equal in BF(n) due to RL, the spanning tree is found by the conventional methodology: Breadth First Search Algorithm(BFS). Solomon and Elkin also studied the spanner with logarithmic lightness, logarithmic diameter, a linear number of edges, and bounded degree for a weighted trees, which results in a 1-spanner for the tree metric induced by the input tree. Brankovic M and et al., discussed a simple local routing algorithm for this tree metric spanner[9]. The edge weight of all the edges of the ratio labelled BF(n) is $\frac{1}{n2^n}$. The vertices of the ratio labelled BF(n) have two different weightages: the vertices in the extreme levels and vertices in intermediate levels. The vertices in extreme levels, (i.e) 0^{th} , $(n)^{th}$ level have minimum weightage $\frac{2}{n2^{n+1}}$. Let T be the spanning tree constructed by BFS algorithm. Due to the role played by vertex weightage in finding fuzzy distance $d_f(s,t)$, the path connecting the two vertices that cross through more number of vertices in extreme levels will have the minimum fuzzy distance. The choice of the root vertex for the BFS algorithm gives two different stretch factors in terms of fuzzy distance. The tree t – spanner of ratio labelled BF(n) is discussed separately for the root vertex taken from extreme levels and from intermediate levels in BFS. BF(n) has a stretch factor 4n-1 by BFS.

Theorem 3.1

Let G be a ratio labelled BF(n), $n > 1$. Then the tree t- spanner of G with root vertex from 0^{th} and n^{th} level is 8n-9.

Proof:

Consider a ratio labelled BF(n) whose vertex and edge weightages are

$$\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{1}{n2^n}, \text{ for } v \in V \text{ in level 0 and } n \\ \frac{1}{n2^{n-1}}, \text{ for } v \in V \text{ in level } i, \text{ for } 1 \leq i \leq n-1 \end{cases}$$

$$\mu(u,v) = \frac{\max[\sigma(u), \sigma(v)]}{\sum_{v \in V} \sigma(v)} = \frac{1}{n2^n}, \text{ for all, } (u,v) \in E$$

The stretch factor t, of BF(n) by BFS is 4n-1. The BFS algorithm is applied by taking the vertices from extreme levels as the root vertex to find a spanning tree T. There are two vertices in BF(n) that produces a maximum stretch in T which are adjacent in BF(n). Let a and b be the two vertices that generates a maximum stretch in T that is adjacent in BF(n) with either a or b in the extreme levels. The path between a and b in T passes through seven edges having end vertices either at 0 or nth level and the remaining

(4n-1-7) edges has their both end vertices in level i where $1 \leq i \leq n-1$.

$$\begin{aligned} d_f^T(a,b) &= \Lambda \sum \{\Lambda(\sigma(s), \sigma(t)) \times \mu(s,t)\} \\ &= 7 \times \frac{1}{n2^n} \times \frac{1}{n2^n} + (4n-8) \times \frac{1}{n2^{n-1}} \times \frac{1}{n2^n} \\ &= \frac{7}{n^2 2^{2n}} + \frac{8n-16}{n^2 2^{2n}} = \frac{(8n-9)}{n^2 2^{2n}} \end{aligned}$$

In G, $d_f(a,b) = \frac{1}{n2^n} \times \frac{1}{n2^n} = \frac{1}{n^2 2^{2n}}$

$$i.e. \quad d_f^T(a,b) = \frac{(8n-9)}{n^2 2^{2n}} = (8n-9) d_f(a,b)$$

Hence for any two vertices u, v,

$$i.e. \quad d_f^T(u,v) \leq (8n-9) d_f(u,v)$$

Thus, the stretch factor of ratio labelled BF(n) is 8n-9.

Hence the proof.

Theorem : 3.2

Let G be a ratio labelled BF(n), $n > 1$. Then the tree t- spanner of G with root vertex from 1th to (n-1)th level is 4n-5 or 8n-9.

Proof:

Consider a ratio labelled BF(n) whose vertex and edge weightages are

$$\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{1}{n2^n}, \text{ for } v \in V \text{ in level 0 and } n \\ \frac{1}{n2^{n-1}}, \text{ for } v \in V \text{ in level } i, \text{ for } 1 \leq i \leq n-1 \end{cases}$$

$$\mu(u,v) = \frac{\max[\sigma(u), \sigma(v)]}{\sum_{v \in V \sigma(v)}^{\sigma(1}} = \frac{1}{n2^n}, \text{ for all, } (u,v) \in E$$

The stretch factor t, of BF(n) by BFS is 4n-1. The BFS algorithm is applied by taking the vertices from level 1 to n-1 as root vertices to find a spanning tree T. There are two vertices in BF(n) that produce a maximum stretch in T, which are adjacent in BF(n). Let a and b be the two vertices that generates a maximum stretch in T, that is adjacent in BF(n) at intermediate levels.

In this case, the stretch between a and b depends on the order in which adjacent vertices of the root vertex are placed in spanning tree T. Depending on the order, the path between a and b may pass through either seven edges or eight edges with end vertices either at 0th level or at the nth level. The case of seven edges is discussed in theorem 3.1, which produces a stretch factor 8n-9. In the second case, the remaining (4n-1-8) edges have their end vertices at level i where $1 \leq i \leq n-1$.

$$\begin{aligned} d_f^T(a,b) &= \wedge \sum \{\wedge(\sigma(s), \sigma(t)) \times \mu(s,t)\} \\ &= 8 \times \frac{1}{n2^n} \times \frac{1}{n2^n} + (4n-9) \times \frac{1}{n2^{n-1}} \times \frac{1}{n2^n} \\ &= \frac{8}{n^2 2^{2n}} + \frac{8n-18}{n^2 2^{2n}} = \frac{(8n-10)}{n^2 2^{2n}} = \frac{4n-5}{n^2 2^{2n-1}} \end{aligned}$$

$$d_f(a,b) = \frac{1}{n2^n} \times \frac{1}{n2^{n-1}} = \frac{1}{n^2 2^{2n-1}}$$

$$i.e. \quad d_f^T(a,b) = \frac{(4n-5)}{n^2 2^{2n-1}} = (4n-5) d_f(a,b)$$

Hence for any two vertices u, v,

$$i.e. \quad d_f^T(u,v) \leq (4n-5) d_f(u,v)$$

Thus, the stretch factor of ratio labelled BF(n) is 4n-5 or 8n-9 for a root vertex in BFS from level i, where $1 \leq i \leq n-2$.

Hence the proof.

Theorem: 3.3

The stretch factor t of tree t- spanner of ratio labelled BF(n) has its upper bound 8n-9 and lower bound 4n-5.

Proof:

The proof follows from theorem 3.1 and 3.2.

4 Conclusion

The concept of fuzzy tree spanner is an essential way to create a hierarchical structure within a network that provides efficient access to nearby nodes or information. This concept helps in managing and accessing data within a network more effectively when dealing with uncertain or approximate distances between the vertices. The research on tree t- spanner of fuzzy graphs has a multidimensional scope of investigating the tree spanner in intuitionistic fuzzy graph, interval type 2 arc length, comparison of algorithms in finding tree spanner of G, proposing various algorithms in the application part, and so on. Here, we portrayed the tree t- spanner for ratio labelled Butterfly Network. Finding the tree t- spanner or finding the bounds of tree t- spanner of ratio labelled fuzzy of other inter connection networks like hypercube, Cube connected cycles is the scope in the near future.

Declaration

Presented in 9th INTERNATIONAL CONFERENCE ON DISCRETE MATHEMATICS AND MATHEMATICAL MODELLING IN DIGITAL ERA (ICDMMMDE-2023) during March 23-25, 2023, Organized by the Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram - 624302, Dindigul, Tamil Nadu, India. ICDMMMDE-23 was supported by: GRI-DTBU, CSIR

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