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Wijsman Ideal Convergence Sequence in Neutrosophic Metric Spaces

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Abstract

Objectives: An examination of convergence for a sequence of closed sets in Neutrosophic Metric Spaces is the main goal of this research. Particularly, the focus of this study is directed at Wijsman$I$ -convergence (WIC) and WijsmanI^*-convergence(WI^*C). **Methods:** The various properties of Neutrosophic Metric Spaces are investigated in this study. Using exact mathematical techniques, this study comprehends the features of convergent sequences in these domains. Furthermore, this study expresses particular views on Wijsman$I$ -Cauchy($WICa$)and WijsmanI^*-Cauchy(WI^*Ca) sequences. **Findings:** The convergence behaviour of closed sets in Neutrosophic Metric Spaces is a result of this research. it presents findings and remarks about the sequences of Wijsman$I$- convergence, WijsmanI^*- convergence, Wijsman$I$-Cauchy, and WijsmanI^*-Cauchy. These findings improve the current understanding of convergence in relation to NMS. **Novelty:** What makes this work unique is that it focuses on convergence notions in Neutrosophic Metric Spaces, specifically on Wijsman$I$ and WijsmanI^*- convergence enriches the current body of information in this domain.

Keywords: Neutrosophic Metric Spaces; Wijsman I convergent and Wijsman I ¿ convergent sequences; Wijsman I ¿Cauchy and Wijman I Cauchy sequences

1 Introduction

Kisi[1] started the hypothesis of factual combination. It is a very compelling device to concentrate on the combination of mathematical issues in grouping spaces by the possibility of thickness. Factual assembly among the series of sets was analysed by Granados et al.[2]. Granados[3] prevailed with regards to summing up the factual union and presented the idea of an ideal I -combination. Dundar et al.[4] dissected I- and I^*-union for twofold successions. Zadeh[5] first proposed the fuzzy sets concept in 1965, and therefore it has found varying uses in the realm of valid semantics. Atanassov[6], subsequent to summing up the fuzzy sets, likewise presented the idea of IFSs in 1986. Park[7] did an itemised assessment of the possibility of intuitively fuzzy metric spaces. Some of the many applications of fuzzy metric space[8] and fuzzy sets in the hard sciences include fixed-point hypothesis testing, image as well as signal processing (including clinical navigation and imaging), and more. In addition, measurable combination, optimal assembly, and various attributes of groupings in IFNSs were analysed by Mursaleen et al.[9] and Pancaroglu AK in et al.[10]. In the following, Jeyaraman et al. and Sowndararajan et al. proposed the Neutrosophic Metric Spaces concept and outlined several fixed-point solutions[11,12]. Researchers have since begun to work on distance spaces built over neutrosophic sets[13–15] The idea of the union of successions of focuses has been stretched out by a few creators to a combination of groupings of sets[16].

In this study, the Wijsman assembly concept[17,18] is one such extension that is examined. The union of set groups was extended to quantifiable mixing by M. Sen and M. Et.[19], who also provided a few key theories. Wijsman always looks into the regularity and features of sequences of sets that show triple ideal convergence. This helps us understand important aspects of convergence properties[20]. Moreover, Isi's[21] study on 'Ideal Convergence of Sequences in Neutrosophic Normed Spaces' contributes fresh insights into convergence within neutrosophic normed spaces, providing unique perspectives in this specialised mathematical field.

Verda G¨urdal's[22] research delves into the examination of generalised statistical limitpoints for triple sequences within random 2-normed 3 spaces, offering valuable insights within this particular mathematical realm. In a collaborative effort, Ayhan Esi, Vakeel A. Khan, Mobeen Ahmad, and Masood Alam[23] present significant findings on Wijsman Ideal Convergence in Intuitionistic Fuzzy Metric Spaces, enriching our understanding of Wijsman ideal convergence in the context of intuitionistic fuzzy metric spaces.

While some of the findings shown here are quite similar to the research emphasis at the right place, the confirmations often use a different method. Only when I *and* $I^{*}$are admissible Ideals some of the results are true. In order to support the main result, we have the required conditions that I is admissible ideal and the neutrosophic space must be separable.

2 Methodology

Definition 2.1. [12] Let M be nonempty set, η, φ and ρ be fuzzy sets on $M^2 \times (0,\infty)$, $*$ be a continuous t-norm, $\diamond$ be a continuous t-conorm and $\oplus$ be a continuous t-conorm. So, the 7-tuble$(M,\eta,\varphi,\rho,*,\diamond,\oplus)$is known as an Neutrosophic Metric Space (NMS) if the following requirement are met: for "every $\lambda, t > 0$ and $y,z,w \in \mathscr{M}$

(i) $\eta(y,z,\lambda)+\varphi(y,z,\lambda)+\rho(y,z,\lambda) \leq 3$,

(ii) $\eta(y,z,\lambda) > 0$,

(iii) $\eta\,(y,z,\lambda) = 1$, if and only if $y = z$,

(iv) $\eta(y,z,\lambda) = \eta(z,y,\lambda)$

(v) $\eta(y,z,\lambda)*\eta(z,w,t) \leq \eta(y,w,\lambda+t)$

(vi) $\eta\,(y,z,.) : (0,\infty) \to (0,1)$ is continuous and $\lim_{\lambda\to\infty} \eta\,(y,z,\lambda) = 1$ for every $\lambda > 0$

(vii) $\varphi(y,z,\lambda) < 1$

(viii) $\varphi\,(y,z,\lambda) = 0$ if and only if$y = z$,

(ix) $\varphi(y,z,\lambda) = \varphi(z,y,\lambda)$,

(x) $\varphi(y,z,\lambda)\diamond\varphi(z,w,t) \geq \varphi(y,w,\lambda+t)$

(xi) $\varphi\,(y,z,\lambda) : (0,\infty) \to (0,1)$ is continuous $\lim_{\lambda\to\infty} \varphi\,(y,z,\lambda) = 0 \; for\ every\ \lambda > 0$

(xii) $\rho(y,z,\lambda) < 1$,

(xiii) $\rho\,(y,z,\lambda) = 0$ if and only if $y = z$,

(xiv) $\rho(y,z,\lambda) = \rho(z,y,\lambda)$,

(xv) $\rho(y,z,\lambda)\oplus\rho(z,w,t) \geq \rho(y,w,\lambda+t)$

(xvi) $\rho\,(y,z,.) : (0,\infty) \to (0,1)$ is continuous $\lim_{\lambda\to\infty} \rho\,(y,z,\lambda) = 0 \; for\ every\ \lambda > 0$

(xvii) If $\lambda \leq 0$, then $\eta\,(y,z,\lambda) = 0$, $\varphi\,(y,z,\lambda) = 1$ and"$\rho\,(y,z,\lambda) = 1$

In such situation, (η,φ,ρ) is called the NM.

Example 2.2. [17] Suppose(M,d) is defined as metric space. Define

$a * b = ab, a \diamond b = min(a+b,1\} and\ a \oplus b = min(a+b,1\}$

$for\ every\ a,b \in [0,1]$.

Suppose η,φ and ρ are fuzzy sets on $M^2 \times (0,\infty)$ defined as

$\eta\,(y,z,\lambda) = \frac{\lambda}{\lambda+d(y,z)}$, $\varphi\,(y,z,s) = \frac{d(y,z)}{\lambda+d(y,z)}$and $\rho\,(y,z,\lambda) = \frac{d(y,z)}{\lambda}$.

Then $(M,\eta,\varphi,*,\diamond,\oplus)$ is an NMS.

Definition 2.3. [17] Let$(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be NMS where C is defined as M nonempty subset. For every$\lambda > 0$ and $x \in M$, we define $\eta\,(x,C,\lambda) = sup\,\{\eta\,(x,y,s) : y \in C\}$,

$\varphi(x,\mathscr{C},\lambda) = \inf\{\varphi(x,y,s) : y \in \mathscr{C}\}$ and $\rho(x,\mathscr{C},\lambda) = \inf\{\rho(x,y,s) : y \in \mathscr{C}\}$, where $\varphi(x,C,\lambda)$, is non nearness degree, $\eta(x,C,\lambda)$is nearness degree and$\rho(x,C,\lambda)$is in conclusiveness degree of x *to* C *at* λ,

Definition 2.4. [17] Let $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be NMS. A sequence $x=(x_k)$ is convergent to ξ if for any $0<\varepsilon<0$ and $s>0$ so $k_0 \in N$ in a form that $\eta(x_k,\xi,\lambda) > 1-\varepsilon, \varphi(x_k,\xi,\lambda) < \varepsilon$ and $\rho(x_k,\xi,\lambda) < \varepsilon$ for all $k \geq k_0$.

Definition 2.5.[17] A $NMS(M,\eta,\varphi,\rho,*,\diamond,\oplus)$is separable if and only if it has a dense subset, which can be counted $\{x_k\}$ as well as subsequent property: for every$\lambda>0$ and for every $\xi \in M$, there exists atleast one x_n so that $\eta(x_n,\xi,\lambda) \geq 1-\varepsilon$, $\varphi(x_n,\xi,\lambda) \leq \varepsilon$ and $\rho(x_n,\xi,\lambda) \leq \varepsilon$,for each $\varepsilon \in (0,1)$.

3 Results and Discussion

In the following section, I is denoted as admissible ideal in N. First, let's agree on some common terms.

Definition 3.1. Let $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be an NMS. A sequence of sets $(C_k\}$ is known to be Wijsman convergent to C if $\varepsilon>0$ and $s>0$ $k_0 \in N$ so that

$\lim_{k\to\infty}\eta(x,C_k,\lambda) = \eta(x,\mathscr{C},\lambda), \lim_{k\to\infty}\varphi(x,C_k,\lambda) = \varphi(x,C,\lambda)$

$\lim_{k\to\infty}\rho(x,C_k,\lambda) = \rho(x,C,\lambda)$ *for all* $k \geq k_0$.

In this context, $L_{(C_k)}$ refers to the collection of all Wijsman limit points for the sequence$\{C_k\}$

Definition 3.2. Let us consider $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$be NMS and I be proper ideal in N. A sequence $\{C_k\}$ of nonempty closed subsets of M is called as WIC to C with regards to $NM(\eta,\varphi,\rho)$, if for every $0<\varepsilon<1$, for each $x \in M$ and for all $\lambda>0$ so that

$$\left\{\begin{array}{c} k\in\mathbb{N} : |\eta(x,\mathscr{C}_k,\lambda) - \eta(x,\mathscr{C},\lambda)| \leq 1-\varepsilon \text{ or} \\ |\varphi(x,\mathscr{C}_k,\lambda) - \varphi(x,\mathscr{C},\lambda)| \geq \varepsilon \text{ and} \\ |\rho(x,\mathscr{C}_k,\lambda) - \rho(x,\mathscr{C},\lambda)| \geq \varepsilon \end{array}\right\} \in \mathfrak{I}$$

We write $(\eta,\varphi,\rho) - I_w - \lim_{k\to\infty} C_k = C$.

Example 3.3. Suppose $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ is an NMS and C, $\{C_k\}$ is an nonempty closed subsets of M. Assume $M=R^2$along with $\{C_k\}$ are sequence described by

$$C_k = \left(\begin{array}{ll} (x,y)\in R^2 \ : 0\leq x\leq k, 0\leq y\leq \frac{1}{k}.x, & if\ k\neq n^2 \\ (x,y)\in R^2 \ : x\geq 0,\ y=1, & if\ k=n^2, \end{array}\right. \quad C = \{(x,y)\in R^2 \ : x\geq 0, y=0\}$$

Since

$$\lim_{k\to\infty}\frac{1}{k}\left|\begin{array}{c} \{n\leq k : |\eta((x,y),\mathscr{C}_k,\lambda) - \eta((x,y),\mathscr{C},\lambda)| \leq 1-\varepsilon \\ \text{or } |\varphi((x,y),\mathscr{C}_k,\lambda) - \varphi((x,y),\mathscr{C},\lambda)| \geq \varepsilon\} \\ \text{and } |\rho((x,y),\mathscr{C}_k,\lambda) - \rho((x,y),\mathscr{C},\lambda)| \geq \varepsilon \end{array}\right| = 0$$

So, $\{C_k\}$ sets' sequence is Wijsman statistical convergent($WStC$) to set C.

$$S(\varepsilon) = \left\{\begin{array}{l} |\eta((x,y),\mathscr{C}_k,\lambda) - \eta((x,y),\mathscr{C},\lambda)| \leq 1-\varepsilon \\ k\in\mathbb{N} : \quad \text{or } |\varphi((x,y),\mathscr{C}_k,\lambda) - \varphi((x,y),\mathscr{C},\lambda)| \geq \varepsilon \\ \text{and } |\rho((x,y),\mathscr{C}_k,\lambda) - \rho((x,y),\mathscr{C},\lambda)| \geq \varepsilon \end{array}\right\}$$

If we consider$\mathfrak{I} = \mathfrak{I}_d$ then the $(WStC)$coincides with the Wijsman ideal convergence. Therefore

$$\left\{ k\in N : \begin{array}{c} (\eta((x,y),C_k,\lambda) - \eta((x,y),C,\lambda)| \leq 1-\varepsilon \\ or\ (\varphi((x,y),C_k,\lambda) - \varphi((x,y),C,\lambda)| \geq \varepsilon \\ and\ (\rho((x,y),C_k,\lambda) - \rho((x,y),C,\lambda)| \geq \varepsilon \end{array}\right\} = \left(k\in N \ : k=n^2\right\} \subset I_d.$$

Definition 3.4. Let $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable NMSwhile $\mathfrak{I}$ be admissible ideal in $\mathbb{N}$. A sequence $(C_k\}$ of nonempty closed subsets of M is called as $WICa$ with regards to $NM(\eta,\varphi,\rho)$, if for each $0<\varepsilon<1$, for every $x\in M$ and $\lambda>0$, there must be $l=l(\varepsilon)$such that $\left\{ k\in N : \begin{array}{c} (\eta((x,y),C_k,\lambda) - \eta((x,y),C,\lambda)| \leq 1-\varepsilon \\ or\ (\varphi((x,y),C_k,\lambda) - \varphi((x,y),C,\lambda)| \geq \varepsilon \\ and\ (\rho((x,y),C_k,\lambda) - \rho((x,y),C,\lambda)| \geq \varepsilon \end{array}\right\} \in I$

Definition 3.5. Let $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable NMSwhile $(C_k\}$ is nonempty closed subsets of M. A sequence $(C_k\}$ is called as WI^*Ca with regards to $NM(\eta,\varphi,\rho)$, if there exists

$P=\left\{p=(p_j) : p_j < p_{j+1}, j\in\mathbb{N}\right\} \subset \mathbb{N}$ and $P\in\mathscr{F}(\mathfrak{I})$ with the result that the subsequence $\mathscr{C}_p = \left\{\mathscr{C}_{p_k}\right\}$ is Wijsman Cauchy in M, that is,

$\lim_{k,l\to\infty}\left(\eta(x,C_{p_k},\lambda) - \eta(x,C_{p_l},\lambda)\right| = 1, \lim_{k,l\to\infty}\left(\varphi(x,C_{p_k},\lambda) - \varphi(x,C_{p_l},\lambda)\right| = 0$

and $\lim_{k,l\to\infty}\left(\rho(x,C_{p_k},\lambda) - \rho(x,C_{p_l},\lambda)\right| = 0$.

Definition 3.6. Let $(M,\eta,\varphi,*,\diamond,\oplus)$ be separable NMS while$I$ be proper ideal in N.Let us consider $(C_k\}$ be M.nonempty closed subsets. $(C_k\}$ sequence is called as WI^*C to C with regards to $NM(\eta,\varphi,\rho)$, if there exists$P\in F(I)$, *where*
$P=\{p=(p_j):p_j<p_{j+1}, j\in\mathbb{N}\}\subset\mathbb{N}$ therefore, for each $s>0$, we have
$\lim_{k\to\infty}\eta(x,C_{p_k},\lambda)=\eta(x,C,\lambda)$, $\lim_{k\to\infty}\varphi(x,C_{p_k},\lambda)=\varphi(x,C,\lambda)$and
$\lim_{k\to\infty}\rho(x,C_{p_k},\lambda)=\rho(x,C,\lambda)$.
In such case, we have to write $(\eta,\varphi,\rho)-I_w^*-\lim C_k=C$.
Following is a theorem that demonstrates this point, and we can show that each WIC denotes the $WICa$ condition in NMS.
Theorem 3.7. Let $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable NMS. Let I be arbitrary admissible ideal. Hence, each WIC sequence of closed sets $(C_k\}$ is $WICa$ Regarding $NM(\eta,\varphi,\rho)$.
Proof. Suppose $(\eta,\varphi,\rho)-I_w-\lim_{k\to\infty}C_k=C$.
Then, for every $0<\varepsilon<1$, *for all*$\lambda>0$ *and* $x\in M$, the set

$$U(\varepsilon,s)=\left\{k\in\mathbb{N}:\begin{array}{c}|\eta(x,\mathscr{C}_k,\lambda)-\eta(x,\mathscr{C},\lambda)|\le 1-\varepsilon\\ \text{or } |\varphi(x,\mathscr{C}_k,\lambda)-\varphi(x,\mathscr{C},\lambda)|\ge\varepsilon\varepsilon\\ \text{and } |\rho(x,\mathscr{C}_k,\lambda)-\rho(x,\mathscr{C},\lambda)|\ge\varepsilon\end{array}\right\}$$

belongs to I. Thus I is an admissible ideal, so $k_0\in N$ must bewith the result that
$k_0\notin U(\varepsilon,s)$. Now, suppose that

$$V(\varepsilon,s)=\left\{k\in\mathbb{N}:\begin{array}{c}\left|\eta(x,\mathscr{C}_k,\lambda)-\eta\left(x,\mathscr{C}_{k_0},\lambda\right)\right|\le 1-2\varepsilon\\ \text{or } \left|\varphi(x,\mathscr{C}_k,\lambda)-\varphi\left(x,\mathscr{C}_{k_0},\lambda\right)\right|\ge 2\varepsilon\\ \text{and } \left|\rho(x,\mathscr{C}_k,\lambda)-\rho\left(x,\mathscr{C}_{k_0},\lambda\right)\right|\ge 2\varepsilon\end{array}\right\}..$$

Considering the inequality

$$\left|\eta(x,\mathscr{C}_k,\lambda)-\eta\left(x,\mathscr{C}_{k_0},\lambda\right)\right|\le|\eta(x,\mathscr{C}_k,\lambda)-\eta(x,\mathscr{C},\lambda)|+\left|\eta\left(x,\mathscr{C}_{k_0},\lambda\right)-\eta(x,\mathscr{C},\lambda)\right|$$

$$\left|\varphi(x,\mathscr{C}_k,\lambda)-\varphi\left(x,\mathscr{C}_{k_0},\lambda\right)\right|\ge|\varphi(x,\mathscr{C}_k,\lambda)-\varphi(x,\mathscr{C},\lambda)|+\left|\varphi\left(x,\mathscr{C}_{k_0},\lambda\right)-\varphi(x,\mathscr{C},\lambda)\right|,$$

and

$$\left|\rho(x,\mathscr{C}_k,\lambda)-\rho\left(x,\mathscr{C}_{k_0},\lambda\right)\right|\ge|\rho(x,\mathscr{C}_k,\lambda)-\rho(x,\mathscr{C},\lambda)|+\left|\rho\left(x,\mathscr{C}_{k_0},\lambda\right)-\rho(x,\mathscr{C},\lambda)\right|$$

Observe that if $k\in V(\varepsilon,s)$, therefore

$$|\eta(x,\mathscr{C}_k,\lambda)-\eta(x,\mathscr{C},\lambda)|+\left|\eta\left(x,\mathscr{C}_{k_0},\lambda\right)-\eta(x,\mathscr{C},\lambda)\right|\le(1-2\varepsilon),$$

$$|\varphi(x,\mathscr{C}_k,\lambda)-\varphi(x,\mathscr{C},\lambda)|+\left|\varphi\left(x,\mathscr{C}_{k_0},\lambda\right)-\varphi(x,\mathscr{C},\lambda)\right|\ge 2\varepsilon,$$

and$|\rho(x,\mathscr{C}_k,\lambda)-\rho(x,\mathscr{C},\lambda)|+\left|\rho\left(x,\mathscr{C}_{k_0},\lambda\right)-\rho(x,\mathscr{C},\lambda)\right|\ge 2\varepsilon$
From another point of view, since $k_0\notin U(\varepsilon,s)$, we obtain

$$\left|\eta\left(x,C_{k_0},\lambda\right)-\eta(x,C,\lambda)\right|>1-\varepsilon,$$
$$|\varphi(x,C_{k0},\lambda)-\varphi(x,C,\lambda)|<\varepsilon$$
$$\textit{and } |\rho(x,C_{k0},\lambda)-\rho(x,C,\lambda)|<\varepsilon.$$

We get, $$|\eta(x,C_k,\lambda)-\eta(x,C,\lambda)|\le 1-\varepsilon,\ |\varphi(x,C_k,\lambda)-\varphi(x,C,\lambda)|\ge\varepsilon \\ \textit{and } |\rho(x,C_k,\lambda)-\rho(x,C,\lambda)|\ge\varepsilon.$$

Hence $k\in U(\varepsilon,s)$.
This implies that $U(\varepsilon,s)\subset V(\varepsilon,s)\in I$, for every $0<\varepsilon<1$ and for all $\lambda>0$ and $x\in M$. Therefore, $V(\varepsilon,s)\in I$,whereas sequence is $\{C_k\}$ that is Wijsman I- Cauchy.
Theorem 3.8. Let us consider $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable NMSwhile$I$ be an admissible ideal. Therefore, each closed sets sequence WI^*Ca is $WICa$.
Proof. If we assume that sequence $(C_k\}$ is WI^*Ca to $NM(\eta,\varphi,\rho)$. Then for each $x\in M$ and for every $0<\varepsilon<1$, there is $P\in F(I)$, *here*
$P=\{p=(p_j):p_j<p_{j+1}, j\in\mathbb{N}\}$ in a way that

$$\left|\eta\left(x,C_{p_k},\lambda\right)-\eta\left(x,C_{p_l},\lambda\right)\right|\le 1-\varepsilon,\ \left|\varphi\left(x,C_{p_k},\lambda\right)-\varphi\left(x,C_{p_l},\lambda\right)\right|\ge\varepsilon$$
$$\textit{and } \left|\rho\left(x,C_{p_k},\lambda\right)-\rho\left(x,C_{p_l},\lambda\right)\right|\ge\varepsilon,\ \textit{for all } k,l>k_0=k_0(\varepsilon)$$

Suppose $N = N(\varepsilon) = p_{k_0+1}$. Therefore, for each $\varepsilon > 0$, one obtains

$$\left|\eta\left(x, C_{p_k}, \lambda\right) - \eta\left(x, C_N, \lambda\right)\right| \le 1 - \varepsilon,$$
$$\left|\varphi\left(x, C_{p_k}, \lambda\right) - \varphi\left(x, C_N, \lambda\right)\right| \ge \varepsilon$$
$$and\ \left|\rho\left(x, C_{p_k}, \lambda\right) - \rho\left(x, C_N, \lambda\right)\right| \ge \varepsilon$$

Now let us assume that $K = N \backslash P$.Obviously, $K \in I$

$$Q(\varepsilon, \lambda) = \left\{ k \in N : \begin{array}{c} \left|\eta\left(x, C_k, \lambda\right) - \eta\left(x, C_N, \lambda\right)\right| \le 1 - \varepsilon \\ or\ \left|\varphi\left(x, C_k, \lambda\right) - \varphi\left(x, C_N, \lambda\right)\right| \ge \varepsilon \\ and\ \left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_N, \lambda\right)\right| \ge \varepsilon \end{array} \right\} \subset K \cup \{p_1, p_2, \ldots, p_{k_0}\}$$

Hence, for all $\lambda > 0$ and for each $0 < \varepsilon < 1$, one can determine $N = N(\varepsilon)$ so that $Q(\varepsilon, \lambda)$, that is sequence $\{C_k\}$ is$WICa$.

Theorem 3.9. Let us consider$I$to be an admissible ideal with(AP) $property$ as well as $(M, \eta, \varphi, \rho, *, \diamond, \oplus)$be separable NMS. Hence, WI^*Ca sequence of sets' idea matches with $WICa$ in that regard to $NM(\eta, \varphi, \rho)$.

Proof. In the preceding Theorem, we proved the direct part.

Let us assume that sequence $\{C_k\}$ is$WICa$ sequence in that regard $NM(\eta, \varphi, \rho)$. Then by definition, if for every $0 < \varepsilon < 1$, for every $x \in M$ and$\lambda > 0$, there is a $m = m(\varepsilon)$ such that

$$B(\varepsilon, \lambda) = \left\{ k \in N : \begin{array}{c} \left|\eta\left(x, C_k, \lambda\right) - \eta\left(x, C_n, \lambda\right)\right| \le 1 - \varepsilon \\ or\ \left|\varphi\left(x, C_k, \lambda\right) - \varphi\left(x, C_n, \lambda\right)\right| \ge \varepsilon \\ and\ \left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_n, \lambda\right)\right| \ge \varepsilon \end{array} \right\} \in I.$$

Now, suppose that

$$P_j(\varepsilon, \lambda) = \left\{ k \in N : \begin{array}{c} \left|\eta\left(x, C_k, s\right) - \eta\left(x, C_{m_j}, s\right)\right| > 1 - \frac{1}{j} \\ or\ \left|\varphi\left(x, C_k, \lambda\right) - \varphi\left(x, C_{m_j}, \lambda\right)\right| < \frac{i}{j} \\ and\ \left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_{m_j}, \lambda\right)\right| < \frac{i}{j} \end{array} \right\},$$

where $m_j = m\left(\frac{1}{j}\right), j = 1, 2, 3, \ldots,$ Obviously, for $j = 1, 2, 3 \ldots,$ $P_j(\varepsilon, s) \in F(I)$.

Utilizing Lemma,$P \subset N$, in this way that $P \in F(I)$while $P \backslash P_j$ represent a finite set for any j.

This is a case when we can finally demonstrate

$\lim_{k,l \to \infty} \left|\eta\left(x, C_k, \lambda\right) - \eta\left(x, C_l, \lambda\right)\right| = 1, \lim_{k,l \to \infty} \left|\varphi\left(x, C_k, \lambda\right) - \varphi\left(x, C_l, \lambda\right)\right| = 0$

and $\lim_{k,l \to \infty} \left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_l, \lambda\right)\right| = 0$.

Exemplifying the aforementioned equations, let $\varepsilon > 0$, $r \in N$In this way, $r > \frac{2}{\varepsilon}$.

If $k, l \in P$, so $P \backslash P_j$ is finite set; hence, $w = w(r)$ such that

$$\left|\eta\left(x, C_k, \lambda\right) - \eta\left(x, C_{l_r}, \lambda\right)\right| > 1 - \frac{1}{r}, \left|\eta\left(x, C_l, \lambda\right) - \eta\left(x, C_{l_r}, \lambda\right)\right| > 1 - \frac{1}{r},$$

$\left|\varphi\left(x, C_k, \lambda\right) - \varphi\left(x, C_{l_r}, \lambda\right)\right| < \frac{1}{r}$,$\left|\varphi\left(x, C_l, \lambda\right) - \varphi\left(x, C_{l_r}, \lambda\right)\right| < \frac{1}{r}$,

and$\left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_{l_r}, \lambda\right)\right| < \frac{1}{r}$,$\left|\rho\left(x, C_l, \lambda\right) - \rho\left(x, C_{l_r}, \lambda\right)\right| < \frac{1}{r}$, for all $k, l > w(r)$.

Then, the above inequalities follow that for $k, l > w(r)$

$\left|\eta\left(x, C_k, \lambda\right) - \eta\left(x, C_l, \lambda\right)\right| \le \left|\eta\left(x, C_k, \lambda\right) - \eta\left(x, C_{l_r}, \lambda\right)\right| + \left(\eta\left(x, C_l, \lambda\right) - \eta\left(x, C_{l_r}, \lambda\right)\right|$

$> \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \varepsilon,$

$\left|\varphi\left(x, \mathscr{C}_k, \lambda\right) - \varphi\left(x, \mathscr{C}_l, \lambda\right)\right| \le \left|\varphi\left(x, \mathscr{C}_k, \lambda\right) - \varphi\left(x, \mathscr{C}_{l_r}, \lambda\right)\right| + \left|\varphi\left(x, \mathscr{C}_l, \lambda\right) - \varphi\left(x, \mathscr{C}_{l_r}, \lambda\right)\right|$

$< \frac{1}{r} + \frac{1}{r} < \varepsilon$

and $\left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_l, \lambda\right)\right| \le \left|\rho\left(x, C_k, \lambda\right) - \rho\left(x, C_{l_r}, \lambda\right)\right| + \left|\rho\left(x, C_l, \lambda\right) - \rho\left(x, C_{l_r}, \lambda\right)\right|$

$< \frac{1}{r} + \frac{1}{r} < \varepsilon$.

Therefore, for each $\varepsilon > 0$, $therefor\ w = w(\varepsilon), k, l \in P \in F(I)$,

We get

$$\left\{ \begin{array}{c} k \in \mathbb{N} : \left|\eta\left(x, \mathscr{C}_k, \lambda\right) - \eta\left(x, \mathscr{C}_l, \lambda\right)\right| \le 1 - \varepsilon \text{ or} \\ \left|\varphi\left(x, \mathscr{C}_k, \lambda\right) - \varphi\left(x, \mathscr{C}_l, \lambda\right)\right| \ge \varepsilon \\ \text{and } \left|\rho\left(x, \mathscr{C}_k, \lambda\right) - \rho\left(x, \mathscr{C}_l, \lambda\right)\right| \ge \varepsilon \end{array} \right\} \in \mathfrak{I}.$$

This clearly demonstrates that sequence$\{C_k\}$ is WI^*Ca.

Theorem 3.10. Let us consider $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable *NMS* and I be an admissible ideal. Therefore $(\eta,\varphi,\rho)-I_w^*-\lim_{k\to\infty} C_k = C$ indicates that sequence $\{C_k\}$ is $WICa$ sequence regards to(η,φ,ρ).

Proof. If we suppose $(\eta,\varphi,\rho)-I_w^*-\lim_{k\to\infty} C_k = C$.

Therefore,

$P=\{p=(p_j): p_j<p_{j+1}, j\in\mathbb{N}\}\subset\mathbb{N}$ with $P\in\mathscr{F}(\mathfrak{J})$ so that $\mathscr{C}_p=\{\mathscr{C}_{p_k}\}$

$\lim_{k\to\infty}\eta\left(x,C_{p_k},\lambda\right)=\eta(x,C,\lambda), \lim_{k\to\infty}\varphi\left(x,C_{p_k},\lambda\right)=\varphi(x,C,\lambda)and\lim_{k\to\infty}\rho\left(x,C_{p_k},\lambda\right)=\rho(x,C,\lambda)$, for any $\varepsilon>0$ and $k,l>k_0$.

Let's assume $r\in N$ and $\varepsilon>0$in a form that $r>\frac{2}{\varepsilon}$.

If $k,l\in P$, then $P\backslash p_j$ is described as finite set; thus, $k(r)=k$ so that

$\left|\eta\left(x,\mathscr{C}_{p_k},\lambda\right)-\eta\left(x,\mathscr{C}_{p_l},\lambda\right)\right|\leq\left|\eta\left(x,\mathscr{C}_{p_k},\lambda\right)-\eta(x,\mathscr{C},\lambda)\right|+\left|\eta\left(x,\mathscr{C}_{p_l},\lambda\right)-\eta(x,\mathscr{C},\lambda)\right|$

$>\left(1-\frac{1}{r}\right)+\left(1-\frac{1}{r}\right)>1-\varepsilon,$

$\left|\varphi\left(x,C_{p_k},\lambda\right)-\varphi\left(x,C_{p_l},\lambda\right)\right|<\left|\varphi\left(x,C_{p_k},\lambda\right)-\varphi(x,C,\lambda)\right|+\left|\varphi\left(x,C_{p_l},\lambda\right)-\varphi(x,C,\lambda)\right|$

$<\frac{1}{r}+\frac{1}{r}<\varepsilon$

and$\left|\rho\left(x,C_{p_k},\lambda\right)-\rho\left(x,C_{p_l},\lambda\right)\right|<\left|\rho\left(x,C_{p_k},\lambda\right)-\rho(x,C,\lambda)\right|+\left|\rho\left(x,C_{p_l},\lambda\right)-\rho(x,C,\lambda)\right|$

$<\frac{1}{r}+\frac{1}{r}<\varepsilon.$

Therefore, $\lim_{k,l\to\infty}\left|\eta\left(x,C_{p_k},\lambda\right)-\eta\left(x,C_{p_l},\lambda\right)\right|=1,$

$\lim_{k,l\to\infty}\left(\varphi\left(x,C_{p_k},\lambda\right)-\varphi\left(x,C_{p_l},\lambda\right)\right|=0$ and $\lim_{k,l\to\infty}\left(\rho\left(x,C_{p_k},\lambda\right)-\rho\left(x,C_{p_l},\lambda\right)\right|=0$

Hence, sequence $\{C_k\}$ is$WICa$ with respect to$NMS(\eta,\varphi,\rho)$.

In the whole subsection, we refer to I as admissible ideal in N, describe Wijsman I- cluster along with I- limit points of the sequence of sets in *NMS*, and draw certain conclusions from these definitions.

Definition 3.11. Let $(M,\eta,\varphi,*,\diamond,\oplus)$ be separable *NMS*. So $C\in M$ is called as WI-cluster point of $\{C_k\}$ if and only if any $x\in M$and for all $\varepsilon>0$,

$$\left\{\begin{array}{c} k\in\mathbb{N}: \quad |\eta(x,\mathscr{C}_k,\lambda)-\eta(x,\mathscr{C},\lambda)|<1-\varepsilon \text{ or} \\ |\varphi(x,\mathscr{C}_k,\lambda)-\varphi(x,\mathscr{C},\lambda)|>\varepsilon \\ \text{and } |\rho(x,\mathscr{C}_k,\lambda)-\rho(x,\mathscr{C},\lambda)|>\varepsilon \end{array}\right\}\notin\mathfrak{J}.$$

We define $I_w^{(\eta,\varphi,\rho)}(\Gamma_{(C_k)})$ as the setof everyWI -cluster points.

Definition 3.12. Let us consider $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable *NMS*. $C\in M$ is called as WI -limit point of sequence$\{C_k\}$ of nonempty closed subsets of Mgiven that

$P=\left(p=(p_j): p_j<p_{j+1}, j\in N\right\}\subset N$ in form that $P\notin I$, as well as for every $x\in M$ along with $\lambda>0$, so we get $\lim_{k\to\infty}\eta(x,C_k,\lambda)=\eta(x,C,\lambda), \lim_{k\to\infty}\varphi(x,C_k,\lambda)=\varphi(x,C,\lambda)$and $\lim_{k\to\infty}\rho(x,C_k,\lambda)=\rho(x,C,\lambda)$. We denote $I_w^{(\eta,\varphi,\rho)}\left(\Lambda_{(C_k)}\right)$ as all WI-limit points collection.

Theorem 3.13. Let us consider $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be separable *NMS*.So, for all,

$$(C_k)\subset M, I_w^{(\eta,\varphi,\rho)}\left(\Gamma_{(C_k)}\right)\subset I_w^{(\eta,\varphi,\rho)}\left(\Gamma_{(C_k)}\right).$$

Proof. Suppose $C\in I_w^{(\eta,\varphi,\rho)}\left(\Gamma_{(C_k)}\right)$. Then there exists$P=(p_1<p_2<\ldots)\subset N$ such that

$P=\left(p=(p_j): p_j<p_{j+1}, j\in N\right\}\notin I$ and for all $\lambda>0$ and $x\in M$, we have

$\lim_{k\to\infty}\eta\left(x,C_{p_k},\lambda\right)=\eta(x,C,\lambda), \lim_{k\to\infty}\varphi\left(x,C_{p_k},\lambda\right)=\varphi(x,C,\lambda)$and

$$\lim_{k\to\infty}\rho\left(x,C_{p_k},\lambda\right)=\rho(x,C,\lambda) \tag{4.3.1}$$

According to Equations (4.3.1), there exists $k_0\in N$ so that for each $\varepsilon>0$ and for any $x\in M$ and $>k_0$.

$\left|\eta\left(x,C_{p_k},\lambda\right)-\eta(x,C,\lambda)\right|>1-\varepsilon,\left|\varphi\left(x,C_{p_k},\lambda\right)-\varphi(x,C,\lambda)\right|<\varepsilon$

and$\left|\rho\left(x,C_{p_k},\lambda\right)-\rho(x,C,\lambda)\right|<\varepsilon$.

Hence, $\{k\in N : \left|\eta\left(x,C_{p_k},s\right)-\eta(x,C,s)\right|>1-\varepsilon, \left|\varphi\left(x,C_{p_k},\lambda\right)-\varphi(x,C,\lambda)\right|<\varepsilon,$

$\left|\rho\left(x,C_{p_k},\lambda\right)-\rho(x,C,\lambda)\right|<\varepsilon\}\supseteq$

$$\left\{P\backslash p_1,p_2,\ldots p_{k_0}\right\}\cdot\left\{\begin{array}{c} k\in\mathbb{N}: \left|\eta\left(x,\mathscr{C}_{p_k},s\right)-\eta(x,\mathscr{C},s)\right|>1-\varepsilon, \\ \left|\varphi\left(x,\mathscr{C}_{p_k},\lambda\right)-\varphi(x,\mathscr{C},\lambda)\right|<\varepsilon \\ \left|\rho\left(x,\mathscr{C}_{p_k},\lambda\right)-\rho(x,\mathscr{C},\lambda)\right|<\varepsilon \end{array}\right\}\notin\mathfrak{J}$$

Which means that $C\in I_w^{(\eta,\varphi,\rho)}\left(\Gamma_{\{C_k\}}\right)$.

Theorem 3.14. Let us consider $(M,\eta,\varphi,\rho,*,\diamond,\oplus)$ be a separable NMS.In that case, for any sequence $\{C_k\} \subset M$, $I_w^{(\eta,\varphi,\rho)}\left(\Gamma_{\{C_k\}}\right) \subset L_{\{C_k\}}$.

Proof. Let $C \in I_w^{(\eta,\varphi,\rho)}\left(\Gamma_{(C_k)}\right)$. Then for each $\varepsilon > 0$ and for all $\lambda > 0$ and for each $x \in M$,

$$\left\{\begin{array}{l} k\in\mathbb{N}: |\eta\left(x,\mathscr{C}_k,\lambda\right)-\eta(x,\mathscr{C},\lambda)| < 1-\varepsilon, \\ |\varphi\left(x,\mathscr{C}_k,\lambda\right)-\varphi(x,\mathscr{C},\lambda)| > \varepsilon \\ |\rho\left(x,\mathscr{C}_k,\lambda\right)-\rho(x,\mathscr{C},\lambda)| > \varepsilon \end{array}\right\} \notin \mathscr{J}.$$

Suppose,

$$Q_k = \left\{ k\in\mathbb{N}: \begin{array}{l} |\eta\left(x,\mathscr{C}_k,\lambda\right)-\eta(x,\mathscr{C},\lambda)| > 1-\dfrac{1}{k}, \\ |\varphi\left(x,\mathscr{C}_k,\lambda\right)-\varphi(x,\mathscr{C},\lambda)| < \dfrac{1}{k} \\ |\rho\left(x,\mathscr{C}_k,\lambda\right)-\rho(x,\mathscr{C},\lambda)| < \dfrac{1}{k} \end{array}\right\}$$

for $k \in N$. $(Q_k\}_{k=1}^{\infty}$contains subsets list of N in decreasing order.

Therefore, $Q = \{k = (k_i < k_{i+1}, i \in N) \notin I$ so that

$\lim_{k\to\infty}\eta\left(x,\mathscr{C}_{k_i},\lambda\right) = \eta(x,\mathscr{C},\lambda), \lim_{k\to\infty}\varphi\left(x,\mathscr{C}_{k_i},\lambda\right) = \varphi(x,\mathscr{C},\lambda)$

and $\underset{k\to\infty}{lim}\,\rho\left(x,C_{k_i},\lambda\right) = \rho\left(x,C,\lambda\right)$, which means that $C \in L_{(C_k\}}$.

4 Conclusion

This study has investigated a variant of ideal union, named Wijsman I *and* I^*-convergent sequences of closed sets, in NMS. We examined new combination ideas for Wijsman$I$ *and* I^*-Cauchy sequences of closed sets in NMS. We analzed some results for admissible ideals in NMS and acquired a few significant outcomes. Furthermore, Wijsman I_2-limit focuses along with Wijsman I_2-cluster points of closed set sequences progressions in NMS have been described. We can apply all the results of the current study and introduce new theories in different spaces like neutrosophic normed linear space, locally solid Riesz space and so on. Once proved the completeness of the space, obtaining a fixed-point theory in the respective space will become easier.

Declaration

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