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On Wilker's and Huygen's Type Inequalities for Generalized Trigonometric and Hyperbolic Functions

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Abstract

Objectives: The Trigonometric inequalities, generalized trigonometric inequalities which have been obtained by Wilker and Cusa Huygens have attracted attention of so many researchers. Generalized trigonometric functions are simple generalization of the classical trigonometric functions. It is related to the r- Laplacian, which is known as a non-linear differential operator. **Method:** For the establishment of inequalities involving generalized trigonometric and hyperbolic functions convexity plays the important role in many aspects, also Monotonicity rule is used for sharpness of inequalities. This technique is used to refine and sharpness of inequalities. **Findings:** Our main result of this paper focus on generalization of Wilker and Cusa Huygens type inequalities for generalized trigonometric and hyperbolic functions with one parameter. **Novelty:** The inequalities with generalized trigonometric and hyperbolic functions proved in this research paper are Wilker's and Cusa Huygens generalization. It can be used for further refinement and sharpness.

Keywords: Trigonometric function; Hyperbolic function; Generalized Trigonometric; Hyperbolic functions; Wilker Inequality and Huygen's Inequality

1 Introduction

Inequalities have various applications in various branches of mathematics, among which are numerical analysis, probability, and statistics as well as in the other sciences such as information theory, fractional calculus, engineering, and technology. From the basic calculus we know that,

$$sin^{-1}(\zeta) = \int_0^{\zeta} \frac{1}{(1-s^2)^{\frac{1}{2}}} ds,\ 0 \le \zeta \le 1 \ and \ \frac{\pi}{2} = sin^{-1}(1) = \int_0^1 \frac{1}{(1-s^2)^{\frac{1}{2}}} ds.$$

Now we define the function sin on $\left[0, \frac{\pi}{2}\right]$ as the inverse of sin^{-1} and extend it to $(-\infty, \infty)$. For $r > 1$, the generalized inverse trigonometric sine function with one parameter $sin_r^{-1}(\zeta)$ is defined as,

$$sin_r^{-1}(\zeta) = \int_0^{\zeta} \frac{1}{(1-s^r)^{\frac{1}{r}}} ds,\ 0 \le \zeta \le 1 \text{ and } \frac{\pi_r}{2} = sin_r^{-1}(1) = \int_0^1 \frac{1}{(1-s^r)^{\frac{1}{r}}} ds.$$

$sin_r^{-1}(\zeta)$ is the inverse of generalized sin function defined on $\left[0, \frac{\pi_r}{2}\right]$. We can also define generalized cosine function as[1], $cos_r(\zeta) = \frac{d}{d\zeta} sin_r(\zeta)$. Clearly, $cos_r(\zeta) = (1 - |sin_r(\zeta)|^r)^{\frac{1}{r}}, 0 \leq \zeta \leq \frac{\pi_r}{2}$ and

$$|\sin_r(\zeta)|^r + |\cos_r(\zeta)|^r = 1,\ \zeta \in R. \tag{1.1}$$

We can easily prove that $\frac{d}{d\zeta} cos_r(\zeta) = -cos_r^{r-2}(\zeta) sin_r^{r-1}(\zeta), 0 \leq \zeta \leq \frac{\pi_r}{2}$.

The generalized tangent function is defined as[1] $tan_r(\zeta) = \frac{sin_r(\zeta)}{cos_r(\zeta)}, \zeta \in R \setminus \left\{k\pi_r + \frac{\pi_r}{2} : k \in Z\right\}$.

From (Equation (1.1)) we have,

$$\frac{d}{d\zeta} tan_r(\zeta) = 1 + |tan_r(\zeta)|^r, \frac{-\pi_r}{2} < \zeta < \frac{\pi_r}{2}$$

Now we can define generalized inverse hyperbolic function as, $sinh_r^{-1}(\zeta) = \begin{cases} \int_0^{\zeta} \frac{1}{(1+s^r)^{\frac{1}{r}}}, & 0 \leq \zeta < \infty \\ -sinh_r^{-1}(\zeta), & -\infty < \zeta \leq 0 \end{cases}$.

The inverse of $sinh_r^{-1}(\zeta)$ is known as the generalized hyperbolic sine function and it is denoted by $sinh_r(\zeta)$. The generalized hyperbolic cosine function is defined as, $cosh_r(\zeta) = \frac{d}{d\zeta} sinh_r(\zeta)$.

These definitions show that,

$cosh_r(\zeta)^r - |sinh_r(\zeta)|^r = 1, \zeta \in R$ and $\frac{d}{d\zeta} cosh_r(\zeta) = cosh_r^{2-r}(\zeta) sin_r^{r-1}(\zeta), \zeta \geq 0$.

The generalized hyperbolic tangent function is defined as,

$tanh_r(\zeta) = \frac{sinh_r(\zeta)}{cosh_r(\zeta)}$ and $cosh_r(\zeta)^r - |sinh_r(\zeta)|^r = 1, \zeta \in R$.

It is clear that, all these generalized functions coincide with the classical ones when $r = 2$.[1]

Some known inequalities for generalized trigonometric and hyperbolic functions was already established which is helpful for proving main result of this paper. Our main results in this paper focused on generalization of Wilker and Huygens type inequalities using trigonometric and hyperbolic functions. These inequalities were sharpened and proved by many researchers using generalized trigonometric and hyperbolic functions of two parameters[2,3]. The study establishes a generalized inequality for higher class of the hyperbolic and trigonometric functions extended beyond the initial scope. This finding is expressed in terms of a one parameter family of hyperbolic and trigonometric functions provide a unified framework for understanding their relations. The study establishes a unified formulation that connects the inequalities involving generalized functions with classical trigonometric inequalities. These results provide insights into the relationships between classical and generalized functions, further enriching our understanding of these mathematical constructs.

2 Methodology

The main findings in this article include a comparative analysis with other existing generalizations of trigonometric and hyperbolic functions. By highlighting similarities and differences, this analysis contributes to the ongoing dialogue within the mathematical community and aids in refining the understanding of these generalizations. To establish the inequalities and for more refine results we the results as listed.

Lemma 2.1.[4] Let p and q be different numbers which satisfy the inequality $pq > 1$, then

$$p + q > \frac{1}{p} + \frac{1}{q}.$$

Lemma 2.2.[4] Let $p > 0, q > 0$ be different numbers and let m and n be any positive numbers such that $m + n = 1$. If $mp + nq > \frac{m}{p} + \frac{n}{q} > 1$ then, $mp^r + nq^r > \frac{m}{p^r} + \frac{n}{q^r} > 1$. The first part of inequality proved if $r > 0$ and second part holds when $r \geq 1$.

Lemma 2.3.[4] Let $p > 0, q > 0$ be different numbers and let m and n be any positive numbers such that $m + n = 1$. If $q^m q^n > 1$ then, $mp + nq > m\left(\frac{p}{q}\right)^n + n\left(\frac{q}{p}\right)^m$.

With the addition of these results Arithmetic and Geometric mean and Monotonicity Rules were used for clarification of our main findings. Following two inequalities were established and used for the proof of inequalities listed in result and discussion section[4,5],

$$\left(\cos_r\left(\frac{\zeta}{2}\right)\right)^{\frac{4}{3}} < \frac{\sin_r(\zeta)}{\zeta} < \frac{\cos_r^2\left(\frac{\zeta}{2}\right) + 2\cos_r\left(\frac{\zeta}{2}\right)}{3}, 0 < \zeta < \frac{\pi_r}{2} \tag{2.1}$$

$$\left(\cos h_r\left(\frac{\zeta}{2}\right)\right)^{\frac{4}{3}} < \frac{\sin h_r(\zeta)}{\zeta} < \frac{\cos h_r^2\left(\frac{\zeta}{2}\right)+2\cos h_r\left(\frac{\zeta}{2}\right)}{3},\ \zeta \neq 0 \tag{2.2}$$

3 Results and Discussion

The following inequalities of Wilker type was established by researchers[5],

$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^2+\frac{tan_r(\zeta)}{\zeta} > 2, 0<\zeta<\frac{\pi_r}{2}\ and\ \left(\frac{sinh_r(\zeta)}{\zeta}\right)^2+\frac{tanh_r(\zeta)}{\zeta} > 2, \zeta\neq 0.$$

The generalization of these inequalities of generalized trigonometric and hyperbolic functions are defined as,

$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^{2r}+\left(\frac{tan_r(\zeta)}{\zeta}\right)^{r} > \left(\frac{\zeta}{sin_r(\zeta)}\right)^{2r}+\left(\frac{\zeta}{tan_r(\zeta)}\right)^{r} > 2, 0<\zeta<\frac{\pi_r}{2} \tag{3.1}$$

and

$$\left(\frac{\sin h_r(\zeta)}{\zeta}\right)^{2r}+\left(\frac{\tan h_r(\zeta)}{\zeta}\right)^{r} > \left(\frac{\zeta}{\sin h_r(\zeta)}\right)^{2r}+\left(\frac{\zeta}{\tan h_r(\zeta)}\right)^{r} > 2, \zeta\neq 0 \tag{3.2}$$

To establish the result of these inequalities, here we consider the particular case of inequalities and combinedly define the results as,

Theorem 3.1. Let, $0<\zeta<\frac{\pi_r}{2}$ then,

$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^2+\frac{tan_r(\zeta)}{\zeta} > \left(\frac{\zeta}{sin_r(\zeta)}\right)^2+\frac{\zeta}{tan_r(\zeta)} > \frac{sin_r(\zeta)}{\zeta}+\left(\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^2 > \frac{\zeta}{sin_r(\zeta)}+\left(\frac{\frac{\zeta}{2}}{tan_r\left(\frac{\zeta}{2}\right)}\right)^2 > 2. \tag{3.3}$$

Proof: To prove the first part of inequality (Equation (3.3)), put $r=1$ in Equation (3.1), which gives us,

$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^2+\frac{tan_r(\zeta)}{\zeta} > \left(\frac{\zeta}{sin_r(\zeta)}\right)^2+\frac{\zeta}{tan_r(\zeta)} > 2$$

Using the Bernoulli's inequality[6] which states that, for a >1, $s>0, (1+s)^a > 1+as$by considering, $s=\frac{sin_r(\zeta)}{\zeta}-1$ and $r=a=2$, we get,

$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^2 > 1+2\left(\frac{sin_r(\zeta)}{\zeta}-1\right) = 1+2\frac{sin_r(\zeta)}{\zeta}-2$$

Since, $r\left(\frac{sin_r(\zeta)}{\zeta}\right)+\frac{tan_r(\zeta)}{\zeta} > 1+r$, for $r=2$[6]

$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^2 > -1+3-\frac{tan_r(\zeta)}{\zeta} \Rightarrow \left(\frac{sin_r(\zeta)}{\zeta}\right)^2 > 2-\frac{tan_r(\zeta)}{\zeta}$$
$$\left(\frac{sin_r(\zeta)}{\zeta}\right)^2+\frac{tan_r(\zeta)}{\zeta} = \left(\frac{\zeta}{sin_r(\zeta)}\right)^2+\frac{\zeta}{tan_r(\zeta)} > 2.$$

Now by using lemma 2.1, setting $p=\left(\frac{sin_r(\zeta)}{\zeta}\right)^2$ and $q=\frac{tan_r(\zeta)}{\zeta}$ first inequality holds.

In order to prove the second inequality in Equation (3.3), let us assume that $a = \frac{sin_r(\zeta)}{\zeta}$ and $d = \frac{2}{1+cos_r(\zeta)}$ and using the half angle formula of tangent function, $tan_r\left(\frac{\zeta}{2}\right) = \frac{sin_r(\zeta)}{1+cos_r(\zeta)}$, the second inequality in (Equation (3.3)) can be rewritten in simple form as,

$$1 + acos_r(\zeta) > a^3 + a^4d^2$$

Let,

$$f(\zeta) = 1 + acos_r(\zeta) - a^3 + a^4d^2 = g(\zeta) + h(\zeta) \tag{3.4}$$

where, $g(\zeta) = 1 + acos_r(\zeta)$ and $h(\zeta) = -\left(a^3 + a^4d^2\right)$.

It is easy to verify that $g(\zeta)$ is strictly decreasing and strictly concave for $0 < \zeta < \frac{\pi_r}{2}$.

From left inequality in (Equation (3.3)), we get

$$d^2 > a^{-3} \Rightarrow d^2a^3 > 1$$

Multiplying by a and then adding a^3 to both sides, we have,

$$a^3 + a^4d^2 > a^3 + a, -\left(a^3 + a^4d^2\right) > -\left(a^3 + a\right)$$

Since, $a = a(\zeta) = \frac{sin_r(\zeta)}{\zeta}$ is strictly decreasing and strictly concave on $0 < \zeta < \frac{\pi_r}{2}$. we have

$$h(\zeta) < -\left(a^3 + a\right)$$

which is strictly increasing and strictly convex function, such that, $-2 < h(\zeta) < -\left[\left(\frac{\pi_r}{2}\right)^3 + \frac{\pi_r}{2}\right]$

Using properties of functions, we can write $f(\zeta)$ from (Equation (2.2)) as, $f(\zeta) > 0$ for $0 < \zeta < \frac{\pi_r}{2}$.

Hence, the second inequality of (Equation (3.3)) holds.

For proving the third inequality in (Equation (3.3)) we use lemma 2.1.

Now substitute, $p = \frac{sin_r(\zeta)}{\zeta}$ and $q = \left(\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^2$ in lemma 2.1, we get,

$$\frac{sin_r(\zeta)}{\zeta} + \left(\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^2 > \frac{1}{\frac{sin_r(\zeta)}{\zeta}} + \frac{1}{\left(\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^2}$$

$$\therefore \frac{sin_r(\zeta)}{\zeta} + \left(\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^2 > \frac{\zeta}{sin_r(\zeta)} + \left(\frac{\frac{\zeta}{2}}{tan_r\left(\frac{\zeta}{2}\right)}\right)^2.$$

For proving fourth inequality, put $\frac{\zeta}{2} = s$, therefore the inequality in (Equation (2.1)) can be written as,

$$\frac{sin_r(\zeta)}{\zeta} < \frac{1}{2}\left(1 + cos_r^3(s)\frac{s}{sin_r(s)}\right). \tag{3.5}$$

Using second inequality of (Equation (2.1)), we get,

$$\begin{aligned} &\frac{\sin_r(\zeta)}{\zeta} < \frac{\cos_r^2(s)+2\cos_r(s)}{3} \\ &\Rightarrow \frac{\cos_r^2(s) + 2\cos_r(s)}{3} < \frac{1}{2}\left(1 + \cos_r^3(s)\frac{s}{\sin_r(s)}\right) \end{aligned} \tag{3.6}$$

$$\Rightarrow 2cos_r^2(s) + 4cos_r(s) < 3 + 3cos_r^3(s)\frac{s}{sin_r(s)} \tag{3.7}$$

Since, $\frac{s}{sin_r(s)} < (cos_r(s))^{\frac{-1}{3}}$

Therefore inequality (Equation (3.7)) can be written as,

$$2cos_r^2(s) + 4cos_r(s) - 3\,(cos_r(s))\frac{8}{3} - 3 < 0. \tag{3.8}$$

Let, $f(s) = 2cos_r^2(s) + 4cos_r(s) - 3(cos_r(s))^{\frac{8}{3}} - 3$

$$\therefore f'(s) = -4sin_r(s)\left(\frac{c+1}{2} - c^{\frac{5}{3}}\right) < 0,\ where\ c = cos_r(s).$$

Using Arithmetic and Geometric mean and Monotonicity Rules in Calculus[4], the previous inequality can be written as, $\frac{c+1}{2} > c^{\frac{1}{2}} > c^{\frac{5}{3}} \Rightarrow \frac{cos_r(s)+1}{2} > cos_r(s)^{\frac{1}{2}} > cos_r(s)^{\frac{5}{3}}$.

Since $f(s)$ is strictly decreasing, we conclude that, $f(s) < 0$ on $\left(0, \frac{\pi_r}{2}\right)$ which gives us,

$$\frac{\zeta}{sin_r(\zeta)} + \left(\frac{\frac{\zeta}{2}}{tan_r\left(\frac{\zeta}{2}\right)}\right)^2 > 2$$

Hence, all the inequalities of theorem (Equation (3.3)) are proved.

The inequality proved in this theorem is generalized as follows which is established in similar manner.

Corollary 3.1.1. If, $0 < \zeta < \frac{\pi_r}{2}$ then,

$$\left(\frac{\sin_r(\zeta)}{\zeta}\right)^r + \left(\frac{\tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^{2r} > \left(\frac{\zeta}{\sin_r(\zeta)}\right)^r + \left(\frac{\frac{\zeta}{2}}{\tan_r\left(\frac{\zeta}{2}\right)}\right)^{2r} > 2 \tag{3.9}$$

The first inequality holds for all $r > 0$, while the second one is valid for $r \geq 1$.

Some more inequalities were established well before as[4,7],

$$2\frac{sin_r(\zeta)}{\zeta} + \frac{tan_r(\zeta)}{\zeta} > 3, 0 < \zeta < \frac{\pi_r}{2}\ and\ 2\frac{sinh_r(\zeta)}{\zeta} + \frac{tanh_r(\zeta)}{\zeta} > 3, \zeta \neq 0$$

The generalization of these inequalities of generalized trigonometric and hyperbolic functions are defined as,

$$2\frac{sin_r(\zeta)}{\zeta} + \frac{tan_r(\zeta)}{\zeta} > 2\frac{\zeta}{sin_r(\zeta)} + \frac{\zeta}{tan_r(\zeta)} > 3, 0 < \zeta < \frac{\pi_r}{2}\ and$$
$$2\frac{sinh_r(\zeta)}{\zeta} + \frac{tanh_r(\zeta)}{\zeta} > 2\frac{\zeta}{sinh_r(\zeta)} + \frac{\zeta}{tanh_r(\zeta)} > 3, \zeta \neq 0.$$

To establish the result of these inequalities, here we consider the particular case of inequalities and combinedly define the results as,

Theorem 3.2. Let, $0 < \zeta < \frac{\pi_r}{2}$ then,

$$2\frac{sin_r(\zeta)}{\zeta} + \frac{tan_r(\zeta)}{\zeta} > \frac{sin_r(\zeta)}{\zeta} + 2\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}} > 2\frac{\zeta}{sin_r(\zeta)} + \frac{\zeta}{tan_r(\zeta)} > 3. \tag{3.10}$$

Proof: For the proof of first inequality in (Equation (3.10)), we assume that,

$$1 + \frac{1}{cos_r(\zeta)} = \frac{4}{1+cos_r(\zeta)}, 0 < \zeta < \frac{\pi_r}{2}. \tag{3.11}$$

$\Rightarrow (1 - cos_r(\zeta))^2 > 0$. Multiplying both sides of Equation (3.11) by $\frac{sin_r(\zeta)}{\zeta}$, we have,

$$\frac{sin_r(\zeta)}{\zeta}\left(1 + \frac{1}{cos_r(\zeta)}\right) = \frac{sin_r(\zeta)}{\zeta}\left(\frac{4}{1+cos_r(\zeta)}\right) \Rightarrow \frac{sin_r(\zeta)}{\zeta} + \frac{tan_r(\zeta)}{\zeta} > \frac{2sin_r(\zeta)}{1+cos_r(\zeta)}\frac{1}{\left(\frac{\zeta}{2}\right)}.$$

Adding $\frac{sin_r(\zeta)}{\zeta}$ to both sides of previous inequality we get,

$$2\frac{sin_r(\zeta)}{\zeta}+\frac{tan_r(\zeta)}{\zeta}>\frac{sin_r(\zeta)}{\zeta}+\frac{2sin_r(\zeta)}{1+cos_r(\zeta)}\frac{1}{\left(\frac{\zeta}{2}\right)}\ Since,\ tan_r\left(\frac{\zeta}{2}\right)=\frac{sin_r(\zeta)}{1+cos_r(\zeta)},$$

we can write previous inequality as, $2\frac{sin_r(\zeta)}{\zeta}+\frac{tan_r(\zeta)}{\zeta}>\frac{sin_r(\zeta)}{\zeta}+2\frac{tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}$.

Hence the first inequality in (Equation (3.10)) is proved.

For the second inequality in (Equation (3.10)), we know that [8,9],

$$\frac{\zeta}{\sin_r(\zeta)}+2\frac{\frac{\zeta}{2}}{\tan_r\left(\frac{\zeta}{2}\right)}=2\frac{\zeta}{\sin_r(\zeta)}+\frac{\zeta}{\tan_r(\zeta)} \tag{3.12}$$

Now we can prove the following inequality,

$$\frac{\sin_r(\zeta)}{\zeta}+2\frac{\tan_r\frac{\zeta}{2}}{\frac{\zeta}{2}}>\frac{\zeta}{\sin_r(\zeta)}+2\frac{\frac{\zeta}{2}}{\tan_r\left(\frac{\zeta}{2}\right)} \tag{3.13}$$

Let us assume that $a=\frac{sin_r(\zeta)}{\zeta}, d=cos_r(\zeta), \zeta=\frac{s}{2}$ in Equation (3.13), it can be written as,

$$a^2>\frac{2d^2+1}{d^2+2} \tag{3.14}$$

Now, the first inequality in (Equation (2.1)) can be written as, $\frac{sin_r(\zeta)}{\zeta}>\left(cos_r\left(\frac{\zeta}{2}\right)\right)^{\frac{4}{3}}\Rightarrow a^2>\left(\frac{1+d}{2}\right)^{\frac{4}{3}}$

Now, Equation (3.14) can be written as,

$$f(d)=\frac{(1-d)^2}{16(d^2+2)^3}\left(d^8+6d^7+23d^6+68d^5+34d^4+72d^3+4d^2+16d-8\right)$$

As, $0<\zeta<\frac{\pi_r}{2}, 0<s<\frac{\pi}{4}$. We have, $\frac{1}{\sqrt{2}}<d<1\Rightarrow f(d)>0$.

So the inequalities in the Equations (3.12), (3.13) and (3.14) holds true.

Hence, $2\frac{\zeta}{sin_r(\zeta)}+\frac{\zeta}{tan_r(\zeta)}>3$. So, the inequality of Theorem 3.2 is proved.

Corollary 3.2.1. If $0<\zeta<\frac{\pi_r}{2}$ then,

$$\left(\frac{\sin_r(\zeta)}{\zeta}\right)^r+2\left(\frac{\tan_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}\right)^r>\left(\frac{\zeta}{\sin_r(\zeta)}\right)^r+2\left(\frac{\frac{\zeta}{2}}{\tan_r\left(\frac{\zeta}{2}\right)}\right)>3. \tag{3.15}$$

first inequality holds true for $r>0$ and second holds for $r\geq 1$.

With the addition of this result for generalized trigonometric function same results were define for generalized hyperbolic function also as,

Theorem 3.3. If $\zeta\neq 0$ then,

$$2\frac{\sin h_r(\zeta)}{\zeta}+\frac{\tan h_r(\zeta)}{(\zeta)}>\frac{\sin h_r(\zeta)}{\zeta}+2\frac{\tan h_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}>\frac{\zeta}{\sin h_r(\zeta)}+\frac{\zeta}{\tan h_r(\zeta)}>3. \tag{3.16}$$

Proof: For first inequality in (Equation (3.16)) let us assume,

$$1+\frac{1}{\cos h_r(\zeta)}=\frac{4}{1+\cos h_r(\zeta)} \tag{3.17}$$

Multiplying both sides of Equation (3.17) by $\frac{sinh_r(\zeta)}{\zeta}$ and adding $\frac{sinh_r(\zeta)}{\zeta}$ on both sides, which gives us,

$$2\frac{sinh_r(\zeta)}{\zeta}+\frac{tanh_r(\zeta)}{\zeta}>\frac{sinh_r(\zeta)}{\zeta}+\frac{2sinh_r(\zeta)}{1+cosh_r(\zeta)}\frac{1}{\left(\frac{\zeta}{2}\right)}$$
$$\Rightarrow 2\frac{sinh_r(\zeta)}{\zeta}+\frac{tanh_r(\zeta)}{\zeta}>\frac{sinh_r(\zeta)}{\zeta}+2\frac{tanh_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}$$

For proving second inequality of (Equation (3.16)), we know that,

$$\frac{\zeta}{sinh_r(\zeta)}+2\frac{\frac{\zeta}{2}}{tanh_r\left(\frac{\zeta}{2}\right)}=2\frac{\zeta}{sinh_r(\zeta)}+\frac{\zeta}{tanh_r(\zeta)}$$

which gives, $\frac{sinh_r(\zeta)}{\zeta}+2\frac{tanh_r\left(\frac{\zeta}{2}\right)}{\frac{\zeta}{2}}>\frac{\zeta}{sinh_r(\zeta)}+2\frac{\frac{\zeta}{2}}{tanh_r\left(\frac{\zeta}{2}\right)}$.

Proof of remaining inequalities of Equation (3.16) follows from theorem 3.1.

4 Conclusion

Generalized trigonometric and hyperbolic function of one parameter verified and refined new inequalities of Wilker and Huygens type. Building upon the initial one parameter results, the research extends inequalities to functions involving multiple parameters. The multi-parameter inequalities showcase the versatility of proposed framework and its applicability to a wider range of mathematics.

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