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Solving Neutral Delay Differential Equations Using Galerkin Weighted Residual Method Based on Successive Integration Technique and its Residual Error Correction

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Abstract

Objectives: The main objectives of this work are to solve Neutral Delay Differential Equations (NDDEs) using Galerkin weighted residual method based on successive integration technique and to obtain the Estimation of Error using Residual function. **Methods:** The Galerkin weighted residual method based on successive integration technique is proposed to obtain approximate solutions of the NDDEs. In this study, the most widely used classical orthogonal polynomials, namely, the Bernoulli polynomials, the Chebyshev polynomials, the Hermite polynomials, and the Fibonacci polynomials are considered. **Findings:** Numerical examples of linear and nonlinear NDDEs have been considered to demonstrate the efficiency and accuracy of the method. Approximate solutions obtained by the proposed method are well comparable with exact solutions. **Novelty:** From the results it is observed that the accuracy of the numerical solutions by the proposed method increases as N increases. The proposed method is very effective, simple, and suitable for solving the linear and nonlinear NDDEs in real-world problems.

Keywords: Galerkin Weighted Residual method; Polynomials; Hermite; Bernoulli; Chebyshev; Fibonacci; Successive integration technique; Neutral Delay Differential Equations

1 Introduction

Delay Differential Equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. The terms involving previous times are called delay terms. The delay terms may be constant, state dependent and time dependent. Neutral delay differential equations (NDDEs) are another type of DDEs in which the highest-order derivative of the unknown function occurs with delay. DDEs arise in the field of signal processing, digital images, control system, epidemiology, chemical kinetics, etc.

Some notable applications of DDEs and NDDEs are in electrochemical biosensor [1], cancer cells growth [2] and population model [3], human balancing models [4], quasi-static piezoelectric beams [5].

DDEs and NDDEs have been studied by many authors and various analytical and numerical methods have been proposed. Some of the numerical methods are New One-Step Technique [6], Hybrid multistep block method [7], Legendre pseudo spectral method [8], Euler Wavelet method [9], matrix method based on Clique polynomial [10], Collocation Method based on successive integration technique [11].

Kun Jiang et al. [12] have applied discontinuous Galerkin method for solving multi-pantograph DDEs. Saray and Lakestani [13] have proposed wavelet Galerkin method for solving the time varying delay systems. Suayip Yuzbas and Murat Karacayir [14] have presented a weighted residual Galerkin method to solve linear functional differential equations.

The above-mentioned methods use the conventional approach which is based on differentiation and operational matrices. When the order of polynomials increases, the dimension of the operational matrices increases. Hence, the computational effort is also increases in these methods. In this study, we propose a new way of approaching Galerkin method and its residual error correction with various polynomials using successive integration technique for solving NDDEs. This technique does not involve operational matrices and hence the computational will be easier.

This paper is organized as follows: In Section 2, the basic definitions of different polynomials are given. In Section 3, the description of the method for solving NDDEs is provided. In Section 4, the description of the residual correction method for improving the errors and error analysis of the proposed method are given. In Section 5, illustrative examples are provided.

2 Basic Definition of Polynomials

2.1 Hermite Polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty, \infty)$. There are different ways to define for Hermite polynomial, one of them is the so-called Rodrigues' formula.

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \tag{1}$$

From Equation (1), the recurrence relation for the polynomials can be derived as

$$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t) \tag{2}$$

$H_0(t)$ can be obtained from Equation (1) and the remaining terms are determined by using the recursion relation Equation (2).

Thus, we have the following sequence of polynomials:

$$H_0(t) = 1$$

$$H_1(t) = 2t$$

$$H_2(t) = 4t^2 - 2$$

$$H_3(t) = 8t^3 - 12t$$

$$H_4(t) = 16t^4 - 48t^2 + 12$$

and so on.

The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

2.2 Bernoulli Polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function for the Bernoulli polynomial of order n is defined by,

$$\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} = \frac{xe^{xt}}{e^x - 1} \tag{3}$$

The recursion formula for Bernoulli polynomial is:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(t) = nt^{n-1},\ n = 2,\ 3,\ \ldots \tag{4}$$

$B_0(t)$ can be obtained from Equation (3) and the remaining terms are determined by using the recursion relation Equation (4). Thus, we have few terms of the Bernoulli polynomials as:

$$B_0(t) = 1$$

$$B_1(t) = t - \frac{1}{2}$$

$$B_2(t) = t^2 - t + \frac{1}{6}$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$$

$$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$$

2.3 Chebyshev Polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1,\ 1]$ of order n is defined as

$$T_n(cost) = cos(nt) \tag{5}$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\ T_n(t) - T_{n-1}(t) \tag{6}$$

$T_0(t)$ and $T_1(t)$ can be obtained from Equation (5). Then the remaining terms are determined from Equation (6). Thus, we have the following sequence of polynomials:

$$T_0(t) = 1$$

$$T_1(t) = t$$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 3t$$

$$T_4(t) = 8t^4 - 8t^2 + 1$$

2.4 Fibonacci Polynomial

The Fibonacci polynomials are a polynomial sequence which can be considered of Fibonacci numbers. The Fibonacci polynomials are defined by a recurrence relation.

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{(n-1)}(t) + F_{(n-2)}(t), & if\ n \geq 2 \end{cases}$$

The first few Fibonacci polynomials are:

$$F_0(t) = 0$$

$$F_1(t) = 1$$

$$F_2(t) = t$$

$$F_3(t) = t^2 + 1$$

$$F_4(t) = t^3 + 2t$$

3 Description of the Proposed Galerkin Method

Consider the nth order NDDE of the form,

$$y^{(n)}(t) = f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y(t-\tau_0), y'(t-\tau_1), \ldots, y^{(n)}(t-\tau_n)\right), \quad t > t_0 \tag{7}$$

with initial conditions

$$y^{(i)}(t_0) = \varnothing(t),\ \ i = 1,\ 2,\ 3, \ldots,\ (n-1)\ for\, t \leq t_0 \tag{8}$$

Here $\varnothing(t)$ is the initial function and τ is the delay term.

For the successive integration technique, we assume that,

$$y^{(n)}(t) \approx B^T P(t)^T = \sum_{j=0}^{N} c_j P_j(t) \tag{9}$$

where N being any positive integer,

$$B^T = (c_0, c_1, \ldots c_N)$$

$$P(t) = (P_0(t), P_1(t) \ldots P_N(t))$$

Our aim is to determine the polynomial coefficients $c_j^{'}s.$ For this, we integrate Equation (9) with respect to t from $t_0\ to\ t$,

$$\left.\begin{array}{c} y^{(n-1)}(t) = y(t_0) + \int_{t_0}^{t} B^T\ P_j(t)\ dt \\ y^{(n-2)}(t) = y(t_0) + y'(t_0) = \int_{t_0}^{t} \int_{t_0}^{t} B^T\ P_j(t)\ dt \\ \cdots \\ y'(t) = \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t} \int_{t_0}^{t} \cdots \int_{t_0}^{t} B^T\ P_j(t)\ dt \\ y(t) = \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t} \int_{t_0}^{t} \cdots \int_{t_0}^{t} B^T\ P_j(t)\ dt \end{array}\right\} \tag{10}$$

Now, for delay terms

$$\left.\begin{array}{r} y^{(n)}(t-\tau_n) = B^T\ P(t-\tau_n) \\ y^{(n-1)}(t-\tau_{n-1}) = y(t_0) + \int_{t_0}^{t} B^T\ P(t-\tau_{n-1})\ dt \\ y^{(n-2)}(t-\tau_{n-2}) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T\ P(t-\tau_{n-2})\ dt \\ \cdots \\ y'(t-\tau_1) = \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} B^T\ P(t-\tau_1)\ dt \\ y(t-\tau_0) = \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} B^T\ P(t-\tau_0)\ dt \end{array}\right\} \tag{11}$$

In order to adopt Galerkin method, consider the residue function,

$R(t) = y^{(n)}(t) - f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y(t-\tau_0), y'(t-\tau_1), \ldots, y^{(n)}(t-\tau_n)\right)$

The weight functions W_i's are the coefficients of c_i in y(t).

Hence, the weighted residual function is given by,

$\int_a^b W_i R(t)\ dt = 0$ i = 1, 2…n.

This would yield system of linear or non-linear system of equations subject to the linear and non-linear terms in Equation (7). On solving this system of equations, we get the respective polynomial coefficients c_i's.

4 Error Analysis and Residual Error Correction

In this section, the estimation of error for the proposed method is carried out based on the residual function of the nth order NDDE given by Equation (7). It shows how this error estimation is used to obtain an improved approximate solution called the corrected solution for the equation. Finally, Taylor theorem is used to get an upper bound for the error of the corrected solution.

4.1 Estimation of Error using Residual function

Let us consider the residual function,

$$R(t) = y^n(t) - f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y(t-\tau_0), y'(t-\tau_1), \ldots, y^{(n)}(t-\tau_n)\right) = 0 \tag{12}$$

Now, replacing y(t) by the approximate solution $y_N(t)$.

$$\begin{array}{l} R_N(t) = y_N^n(t) - f(t,\ y_N(t), y_N'(t), \ldots, y_N^{(n-1)}(t), \\ y_N(t-\tau_0), y_N'(t-\tau_1), \ldots, y_N^{(n-1)}(t-\tau_{n-1}), y_N^{(n)}(t-\tau_{n-1})) = 0 \end{array} \tag{13}$$

Subtracting Equation (12) from Equation (13) we obtain

$$\begin{array}{l} e_N'(t) = -R_N(t) + f(t, e_N(t), e_N'(t), \ldots, e_N^{(n-1)}(t), \\ e_N(t-\tau_0), e_N'(t-\tau_1), \ldots, e_N^{(n-1)}(t-\tau_{n-1}), e_N^{(n)}(t-\tau_{n-1})) \end{array} \tag{14}$$

which is just as Equation (7) with non-homogeneous term $-R_N(t)$ and $y(t) - y_N(t)$ is replaced by $e_N(t)$. Since the approximate solution $y_N(t)$ also satisfies the initial condition, we get,

$$e_N(a) = y(a) - y_N(a) = y_a - y_a = 0.$$

This is the initial condition of Equation (14). Then applying the described in Section 3 upon two different cases and an approximate solution $e_{N,M}(t)$ for Equation (14), where M is any arbitrary constant. Finally, we employ this approximation to get an improved solution for Equation (7) as

$$y_{N,M}(t) = y_N(t) + e_{N,M}(t).$$

Here $E_{N,M}(t)$ represents the actual error of $y_{N,M}(t)$ which is given by,

$$E_{N,M}(t) = y(t) - y_{N,M}(t)$$

4.2 Estimation of Upper Bound for the Error

For any natural number M, the Taylor truncation error can be used to give an upper bound for the absolute error $|E_{N,M}(t)|$of the improved solution, which depends on first $M+1$ derivatives of the exact solution $y(t)$ and first M derivatives of $E_{N,M}(t)$.

Let $y_{T,M}(t)$ denote the M^{th}degree of the truncated Taylor polynomial of $y(t)$ in a neighbouring point $c\varepsilon[a,b]$ and $R_{T,M}(t)$ denote its truncation error given by $\frac{(t-c)^{M+1}}{(M+1)!}y^{M+1}(\xi)$ for some $\xi\,\varepsilon\,(c,t)$.

Theorem 4.2.1

If $y(t)$ is the exact solution of NDDE **(7)**, then the upper bound for the absolute error of the improved solution $y_{N,M}(t)$ is given by,

$$|E_{N,M}(t)| = |y(t) - y_{N,M}(t)| \leq |R_{T,M}(t)| + |y_{T,M}(t) - y_{N,M}(t)| \tag{15}$$

Proof:

According to Taylor's theorem with Lagrange-type remainder term

$$\begin{aligned} y(t) &= y(c) + y'(c)(t-c) + \cdots + \frac{y^M(c)}{M!}(t-c)^M + \frac{(t-c)^{M+1}}{(M+1)!}y^{M+1}(\xi) \\ &= y_{T,M}(t) + R_{T,M}(t) \end{aligned}$$

for some $\xi\,\varepsilon(a,b)$. Using the triangle inequality for $|E_{N,M}(t)|$, we get

$|E_{N,M}(t)| = |y(t) - y_{N,M}(t)|$

$= |y(t) - y_{T,M}(t) + y_{T,M}(t) - y_{N,M}(t)|$

$\leq |R_{T,M}(t)| + |y_{T,M}(t) - y_{N,M}(t)|$.

which completes the proof.

The above result can be explained as follows:

Since $y_{N,M}(t)$ is a polynomial of degree M, it can be expressed in terms of orthogonal polynomials P(t) as

$$y_{N,M}(t) = \sum_{j=0}^{M} a_{N,j} P_j(t)$$

where $a_{N,j}$ are constants. Alternatively, it can also be written as

$$y_{N,M}(t) = \sum_{j=0}^{M} \bar{a}_{N,j} P_j(t-c)$$

Now, Equation (15) can be expressed as

$$|y(t) - y_{N,M}(t)| \leq \left| \frac{(t-c)^{M+1}}{(M+1)!} y^{(M+1)}(\xi) \right| + \left| \sum_{k=0}^{M} \left(\frac{y^{(k)}(c)}{k!} - \bar{a}_{N,k} \right) (t-c)^{(k)} \right| \tag{16}$$

To express this bound in a compact form, let us denote the maximum absolute value of $y^k(t)$

$$S_k = \underset{a\leq t\leq b}{Sup} \left| y^{(k)}(t) \right|$$

Since $c \in [a,b]$, assuming $t \in [a,b]$ we also have $(t-c)^k \leq (b-a)^k$. On combining these Equation (16) takes the form as in the following Corollary:

Corollary 4.2.2

The absolute error $|E_{N,M}(t)|$ of $y_{N,M}(t)$ satisfies

$$|y(t) - y_{N,M}(t)| \leq \frac{S_{M+1}}{(M+1)!}(b-a)^{M+1} + \sum_{k=0}^{M} \left(\frac{S_k}{k!} - \left| \bar{a}_{N,k} \right| \right) (b-a)^k.$$

Thus, the error of the proposed method is closely related to the magnitude of the first M derivatives of the exact solution $y(t)$. Therefore, Equation (16) is the upper bound to be used in computations.

The aim of Corollary 4.2.2 is just to emphasize the dependence of the error on first $M+1$ derivatives of exact solution $y(t)$.

5 Numerical Simulations

In this section, three numerical examples are given to demonstrate the accuracy and effectiveness of the proposed method. We solved these examples by using Galerkin Weighted Residual Method with four different polynomials, namely Hermite, Chebyshev, Bernoulli and Fibonacci polynomial.

Example 1

Consider the second order linear NDDE with constant delay and variable coefficient

$ty''(t) + ty(t) + y''(t-1) + y'(t-1) = 2cos(t-1)$

with initial conditions

$y(0) = -1$ and $y'(0) = 1$.

Exact solution is $y(t) = sin(t) - cos(t)$.

For this example, the numerical results are obtained by Galerkin method with successive integration technique and its residual correction for different N. The actual error results and the corrected error results by using Hermite polynomial are presented in Table 1. The comparative results by Galerkin method using different polynomials at $t = 1$ are presented in Table 2. The solution graph obtained by using the proposed method with N = 3 is presented in Figure 1.

Table 1. Actual Errors and Corrected Errors for Example 1 (Hermite Polynomial)

Time t	Actual Error for y_3(t) N = 3	Corrected Error $E_{3,5}$ N = 3, M = 5	Actual Error for y_7(t) N = 7	Corrected Error $E_{7,9}$ N = 7, M = 9
0.2	1.892 e-04	1.136 e-05	6.306 e-07	2.772 e-08
0.4	5.753 e-04	3.660 e-05	2.045 e-06	3.902 e-08
0.6	8.327 e-04	5.850 e-05	3.306 e-06	3.849 e-07
0.8	6.600 e-04	6.043 e-05	3.500 e-06	1.073 e-06
1.0	7.448 e-05	3.365 e-05	2.131 e-06	2.013 e-06

Table 2. Comparison of Absolute Errors for Example 1

Polynomials	Proposed Method	Galerkin
	N = 3	N = 7
Bernoulli	8.462 e-04	8.103 e-07
Chebyshev	7.448 e-05	2.198 e-06
Hermite	7.448 e-05	2.131 e-06
Fibonacci	9.756 e-05	1.967 e-06

Example 2

Consider the following non-linear NDDE,

$\frac{1}{2}y''\left(\frac{t}{2}\right) + \frac{1}{t}y'\left(\frac{t}{2}\right) + y^2(t) = -\frac{1}{2}sin\left(\frac{t}{2}\right) + \frac{1}{t}cos\left(\frac{t}{2}\right) + sin^2(t)$

with initial condition as

$y(0) = 0$ and $y'(0) = 1$.

Exact solution is $y(t) = sin(t)$.

For this example, the numerical results are obtained by Galerkin method with successive integration technique and its residual correction for different N. The actual error results and the corrected error results by using Hermite polynomial are presented in Table 3. The solution graph obtained by using the proposed method with N = 5 is presented in Figure 2.

Table 3. Actual Errors and Corrected Errors for Example 2 (Hermite Polynomial)

Time t	Actual Error y_3(t) N = 3	Corrected Error $E_{3,5}$ N = 3, M = 5
0.2	6.497 e-07	6.639 e-09
0.4	3.012 e-07	1.089 e-08
0.6	8.816 e-07	1.247 e-06
0.8	4.191 e-06	1.452 e-05
1.0	2.762 e-05	6.640 e-05

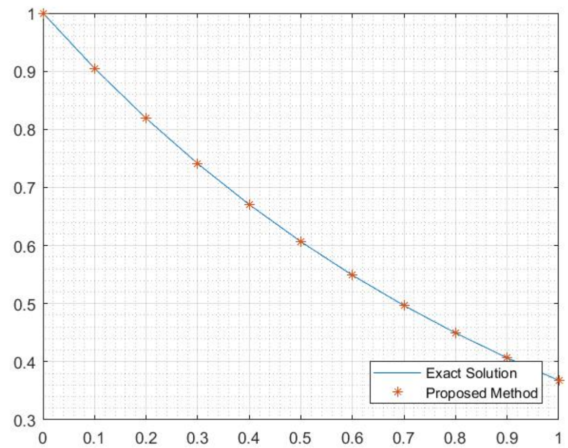

Fig 1. Solution Graph at N = 3 for Example 1

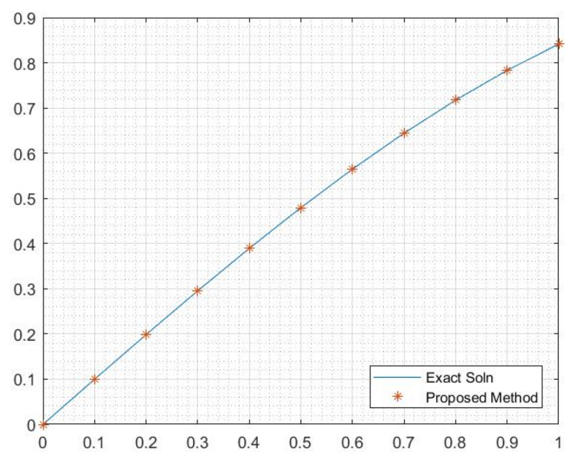

Fig 2. Solution Graph at N = 5 for Example 2

6 Conclusion

In this paper, a new way of approaching Galerkin method and its residual error correction based on the successive integration technique is proposed for solving neutral delay differential equations. This method uses various polynomials such as Bernoulli, Chebyshev, Hermite and Fibonacci.

Numerical examples are considered to demonstrate the efficiency of the method. From the Tables, it is observed that the accuracy of the results by the proposed method increases as N (order of the polynomial) increases. The estimation of residual correction improves the accuracy of the results. This technique does not involve operational matrices and hence the computational will be easier. It is concluded that the proposed Galerkin method based on successive integration technique with residual error correction is very effective, simple and suitable for solving linear and nonlinear NDDEs.

References

1) Martsenyuk V, Klos-Witkowska A, Dzyadevych S, Sverstiuk A. Nonlinear Analytics for Electrochemical Biosensor Design Using Enzyme Aggregates and Delayed Mass Action. *Sensors*. 2022;22(3):1–17. Available from: https://doi.org/10.3390/s22030980.
2) Das A, Dehingia K, Sarmah HK, Hosseini K, Sadri K, Salahshour S. Analysis of a delay-induced mathematical model of cancer. *Advances in Continuous and Discrete Models*. 2022;2022:1–20. Available from: https://doi.org/10.1186/s13662-022-03688-7.

3) Chen-Charpentier B. Delays and Exposed Populations in Infection Models. *Mathematics*. 2023;11(8):1–22. Available from: https://doi.org/10.3390/math11081919.
4) Domoshnitsky A, Levi S, Kappel RH, Litsyn E, Yavich R. Stability of neutral delay differential equations with applications in a model of human balancing. *Mathematical Modelling of Natural Phenomena*. 2021;16(21):1–18. Available from: https://doi.org/10.1051/mmnp/2021008.
5) Loucif S, Guefaifia R, Zitouni S, Khochemane HE. Global well-posedness and exponential decay of fully dynamic and electrostatic or quasi-static piezoelectric beams subject to a neutral delay. *Zeitschrift für angewandte Mathematik und Physik*. 2023;74(3):1–22. Available from: https://doi.org/10.1007/s00033-023-01972-4.
6) Shaalini JV, Pushpam AEK. Analysis of Composite Runge Kutta Methods and New One-Step Technique for Stiff Delay Differential Equations. *IAENG International Journal of Applied Mathematics*. 2019;49(3):1–10. Available from: https://www.iaeng.org/IJAM/issues_v49/issue_3/IJAM_49_3_14.pdf.
7) Ismail NIN, Majid ZA, Senu N. Hybrid Multistep Block Method for Solving Neutral Delay Differential Equations. *Sains Malaysiana*. 2020;49(4):929–940. Available from: https://www.ukm.my/jsm/pdf_files/SM-PDF-49-4-2020/22.pdf.
8) Jafari H, Mahmoudi M, Noori skandari MH. A new numerical method to solve pantograph delay differential equations with convergence analysis. *Advances in Difference Equations*. 2021;2021(1):1–12. Available from: https://doi.org/10.1186/s13662-021-03293-0.
9) Mohammad M, Trounev A. A New Technique for Solving Neutral Delay Differential Equations Based on Euler Wavelets. *Complexity*. 2022;2022:1–8. Available from: https://doi.org/10.1155/2022/1753992.
10) Yuzbasi S, Tamar ME. A new approaching method for linear neutral delay differential equations by using Clique polynomials. *Turkish Journal of Mathematics*. 2023;47(7):2098–2121. Available from: https://journals.tubitak.gov.tr/cgi/viewcontent.cgi?article=3481&context=math#:~:text=using%20a%20truncated%20series%20expansion,system%20of%20linear%20algebraic%20equations.
11) Kayelvizhi C, Pushpam AEK. Application of Polynomial Collocation Method Based on Successive Integration Technique for Solving Delay Differential Equation in Parkinson's Disease. *Indian Journal Of Science And Technology*. 2024;17(2):112–119. Available from: https://doi.org/10.17485/IJST/v17i2.2573.
12) Jiang K, Huang Q, Xu X. Discontinuous Galerkin Methods for Multi-Pantograph Delay Differential Equations. *Advances in Applied Mathematics and Mechanics*. 2020;12(1):189–211. Available from: https://doc.global-sci.org/uploads/Issue/AAMM/v12n1/121_189.pdf?code=1IBB99GZdzOztNjb2EMVmQ%3D%3D#:~:text=The%20discontinuous%20Galerkin%20(DG)%20method,and%20integral%20dif%2D%20ferential%20equations.
13) Saray BN, Lakestani M. On the sparse multi-scale solution of the delay differential equations by an efficient algorithm. *Applied Mathematics and Computation*. 2020;381:125291. Available from: https://doi.org/10.1016/j.amc.2020.125291.
14) Yuzbasi S, Karacayir M. Galerkin-like method for solving linear functional differential equations under initial conditions. *Turkish Journal of Mathematics*. 2020;44(1):85–97. Available from: https://journals.tubitak.gov.tr/cgi/viewcontent.cgi?article=1350&context=math#:~:text=We%20consider%20the%20problem%20with,degree%20N%20of%20our%20choice.