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Study of Matrix Units of the Group algebras KG_r and KSG_r

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Abstract

Objectives: The purpose of this work is to study p-Bratteli diagrams arising from a class of group algebras KG_r and KSG_r for every odd prime p, to define the irreducible representations using mutually orthogonal primitive idempotents and to index the complete set of irreducible representations for the group algebras KG_r and KSG_r. **Method:** We use primitive idempotents to define irreducible representations and systematically index the irreducible representations to establish vertices for the p-Bratteli diagram. Also, we compute the matrix units using the orthogonal primitive idempotents of the group algebras KG_r and KSG_r. **Findings:** The complete set of irreducible representations for the group algebras KG_r and KSG_r has been successfully indexed, and matrix units for the group algebras KG_r and KSG_r were determined using the orthogonal primitive idempotents. **Novelty:** This work provides a structured approach to constructing p-Bratteli diagrams for the group algebras related to odd primes, contributing new insights into their irreducible representations and matrix unit's structures.

Keywords: Primitive idempotent; Irreducible representation; Matrix units; Young diagram; Bratteli diagram

1 Introduction

The history of computing matrix units can be traced back to the work of Alfred Young, which was explained in detail in [(1)]. Young's research on the theory of invariance led to the semi-normal representation, further giving rise to semi-normal units, which are now known as matrix units. This was further revised by A. Ram and H. Wenzl in [(2)]. They computed the matrix units for Brauer's centralizer algebras and Hecke algebras of type A. This was used to construct the matrix units of the centralizer of tensor products of classical Lie groups and their quantum deformation. They used the Jones "basic construction" method to compute matrix units. This was further extended to another class of algebras such as G-Brauer algebras [(3)], Hecke algebras of type B_f and hyperoctahedral groups [(4)], signed Brauer's algebra [(5)], and the group algebra $k\left((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_f\right)$ [(6)]. Recently also, there have been attempts to compute matrix units for Hecke algebras of type D_n [(7)] and semi-simple group algebras, where the groups satisfy certain conditions [(8)] using different methods.

In this paper, we compute matrix units as a natural consequence of the orthogonal primitive idempotents for the group algebras KG_r and KSG_r. We also fill up the gaps for group algebras, since our groups and method do not come under any of the previous work. This computation of matrix units is significant because G_r is isomorphic to the Galois group of the polynomial $x^{p^r} - n$ over $\mathbb{Q}$, where n is a square-free integer. We explore the generators and relations of the group algebra KG_r and KSG_r in Section 2. In Section 3, we Index the orthogonal primitive idempotents by hook partitions, which will serve as vertices, and we also define the edges suitably to define the p-Bratteli diagram. We also explore the notions of restriction and induction for group algebras.

The study of Bratteli diagrams has many applications like combinatorics, Markov chains, random walks, iterated function systems, harmonic analysis on the path space of generalized Bratteli diagrams, etc. Young tableaux have applications in representation theory and geometry. We use this p-Bratteli diagram and matrix units computed above to get the Robinson-Schensted correspondence between elements of the groups and the pair of paths from the Bratteli diagram in our subsequent paper.

2 Methodology

In our paper, we consider group algebras over a field of characteristic zero that encompasses the p^r-th root of unity for all r $\geq$ 0 and p is an odd prime.

For any odd prime p, let $\mathbb{Z}_{p^r}$ denote the additive cyclic group of order p^r. Let $\mathbb{Z}^{\star}_{p^r} = \left\{\bar{t} \in Aut\left(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r}\right), \, (t, p^r) = 1\right\}$ represent the multiplicative cyclic group. Here $\phi : \ \mathbb{Z}^{\star}_{p^r} \to Aut\left(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r}\right)$ is a group homomorphism mapping $\bar{t}$ to $\phi_{\bar{t}}$, where $\phi_{\bar{t}}$ is an automorphism sending $\bar{1}$ to $\bar{t}$.

With the defined homomorphism, we establish a semi-direct product $\mathbb{Z}_{p^r} \rtimes \mathbb{Z}^{\star}_{p^r}$ denoted as the group G_r. Specifically, G_r is generated by elements τ and g. We denote the subgroup generated by τ as $H^{\star}_{p^r}$, which is isomorphic to $\mathbb{Z}^{\star}_{p^r}$. Additionally, let $H_{p^r} = \langle g, \, g^{p^r} = e\rangle$ represent the subgroup of G_r generate by g with order p^r.

Given that H_{p^r} is normal in G_r, the group $H^{\star}_{p^r}$ acts on H_{p^r} by conjugation $\tau.g = \tau^{-1} g \tau = g^t$, where $t^k \equiv 1 \, (mod \, p^r)$. Choose $t \ (mod \ p)$ such that its order is $p-1$ in the group of units of the ring $\mathbb{Z}_p$. The cyclic group $\mathbb{Z}_p$ is generated by g^t and as a result, $G_r \cong \mathbb{Z}_{p^r} \rtimes \mathbb{Z}^{\star}_{p^r}$. Moreover, we have $G_r = \{\tau^i g^j / \, 1 \leq i \, \leq \, p^{r-1} \, (p-1) \, , \, 1 \leq j \, \leq \, p^r\}$.

Let $SG_r = \{\tau^i g^{pj} / \, 1 \leq i \leq p^{r-1} \, (p-1) \, , \, 1 \leq j \leq p^{r-1}\}$ be a subgroup of G_r that is isomorphic to the group $\mathbb{Z}_{p^{r-1}} \rtimes \mathbb{Z}^{\star}_{p^r}$. Furthermore, let $\psi : \ \mathbb{Z}_{p^{r-1}} \rtimes \mathbb{Z}^{\star}_{p^r} \ \to \ SG_r$ be an isomorphism defined by $\psi(\tau) = \tau$ and $\psi(g) = g^p$.

2.1 Generators and Relations of the Group G_r and SG_r

The group G_r generated by $g, \, \tau$ subject to the following relations

- $g^{p^r} = e$
- $\tau^{p^{r-1}(p-1)} = e$
- $\tau^{-1} g \tau = \ g^{\bar{t}}$, where $o\left(\bar{t}\right) = \ p^{r-1}(p-1)$ in the group of units of the ring $\mathbb{Z}^{\star}_{p^r}$ and $\bar{t} \equiv t \ (mod \ p^r)$.

The group SG_r generated by $g, \, \tau$ subject to the following relations

- $(g^p)^{p^{r-1}} = e$
- $\tau^{p^{r-1}(p-1)} = e$
- $\tau^{-1} g \tau = \ g^{\bar{t}}$, where $o\left(\bar{t}\right) = \ p^{r-1} \, (p-1)$ in the group of units of the ring $\mathbb{Z}^{\star}_{p^r}$ and $\bar{t} \equiv t \ (mod \ p^r)$.

3 Results and Discussion

This work explains the construction of the p-Bratteli diagram of the group algebras KG_r and KSG_r associated with the odd prime powers. By using the primitive idempotents, we define the complete set of irreducible representations and index it with hook partitions, which gives an understanding of their structural properties. Additionally, the matrix units derived from these orthogonal primitive idempotents offer new insights into the algebraic structures of these group algebras, providing a novel perspective on their representation theory.

3.1 Representations and Bratteli diagram for the group $\mathbb{Z}_{p^r} \ \ \mathbb{Z}^{*}{}_{p^r}$ and the subgroup $\mathbb{Z}_{p^{r-1}} \ \ \mathbb{Z}^{*}{}_{p^r}$

In this section, we have explored the embedding of group algebras, the idempotents in group algebras, as well as the concept of irreducible representations within the group algebra KG_r and KSG_r. Moreover, we have analyzed the central idempotents of the group algebras KG_r and KSG_r. We introduced the notion of two indexing sets for the complete set of irreducible representations, one in terms of hook partition and the other in terms of minimal idempotents, and outlined the relationship between them. We have also discussed the concept of matrix units of the group algebras.

3.1.1 Irreducible representations and the idempotents for the group algebras KG$_r$ and KSG$_r$

Let $\tau \in H^{\star}_{p^r}$. Then, $\tau = \tau_1\tau_2$ and $H^{\star}_{p^r} = L_1L_2$, where $L_1 = \left\langle \tau_1, \tau_1^2, \ldots, \tau_1^{p^{r-1}} = e \right\rangle$, $L_1 = \left\langle \tau_2, \tau_2^2, \ldots, \tau_2^{p-1} = e \right\rangle$. Let ρ denote a primitive p^r-th root of unity, which is a generator of the group of the roots of the polynomial $x^{p^r} - 1$. Let K be the smallest field of characteristic zero which contains the roots of the polynomial $x^{p^r} - 1$ over $\mathbb{Q}$, for all $r \geq 1$. The homomorphisms φ_i, $1 \leq i \leq p^r$ from the group H_{p^r} into the nonzero elements of K defined by $g \mapsto \rho^i$ gives rise to idempotents denoted by

$$E_{\rho^i} = \frac{1}{p^r}\sum_{j=1}^{p^r} \rho^{ij} g^j, \;\; 1 \leq i \leq p^r$$

where ρ is a primitive p^r-th root of unity.

The above set of idempotents gets divided as follows

$$E^{(r,s)}_{\rho^i} = \frac{1}{p^r}\sum_{j=1}^{p^r} \rho^{ij} g^j \tag{1}$$

where ρ is a primitive p^s-th root of unity, $(i, p^s) = 1$ and $0 \leq s \leq r$. The idempotents $E^{(r,s)}_{\rho^i}$'s are conjugate to each other.

Define

$T^{(r,s)} = \left\{ \tau \in H^{\star}_{p^r} / \tau^{-1} E^{(r,s)}_{\rho^i} \tau = E^{(r,s)}_{\rho^i} \right\}$, $T^{(r,s)}$ is a subgroup of order p^{r-s}, since there are $p^{s-1}(p-1)$ distinct idempotents $E^{(r,s)}_{\rho^i}$, where $1 \leq s \leq r$. The idempotent $T^{(r,s)}_l$ is defined as

$$T^{(r,s)}_l = \frac{1}{p^{r-s}} \sum_{j=1}^{p^{r-s}} \rho^{lj} \tau^{jp^{s-1}(p-1)} \tag{2}$$

where ρ is a primitive p^{r-s}-th root of unity, $0 \leq l \leq p^{r-s} - 1$.

Since $E^{(r,s)}_{\rho^i} T^{(r,s)}_l = T^{(r,s)}_l E^{(r,s)}_{\rho^i}$ are idempotents.

Define

$$T^{(r)}_l = \frac{1}{p^{r-1}(p-1)} \sum_{j=1}^{p^{r-1}(p-1)} \zeta^{lj}\tau^j, \, 0 \leq l < p^{r-1}(p-1)$$

where ζ is a primitive $p^{r-1}(p-1)$-th root of unity.

We know that for any group G, $|G| = \sum_i m_i^G d_i^G$, where d_i^G denotes the degree of the representation and m_i^G denotes its multiplicity. We have the following identity;

$|G_r| = p^r \,.\, p^{r-1}\,(p-1)$
$= \left(p^{r-1}.\,(p-1)^2\right) + \cdots + p^{r-s}.\left(p^{s-1}\,(p-1)^2\right) + \cdots + p^{r-1}(p-1)$
$= \sum_i m_i^{G_r} d_i^{G_r}$
$|SG_r| = p^{r-1} \,.\, p^{r-1}\,(p-1)$
$= p^{r-1}\,(p-1)\left[p^{r-2}\,(p-1) + p^{r-3}\,(p-1) \cdots + p.\,(p-1) + (p-1) + 1\right]$
$= p.\left(p^{r-2}.\,(p-1)^2\right) + \cdots + p^{r-s}.\left(p^{s-1}\,(p-1)^2\right) + \cdots + p^{r-1}(p-1)$
$= \sum_i m_i^{SG_r} d_i^{SG_r}$
$|G_{r-1}| = p^{r-1}.p^{r-2}\,(p-1)$
$= p^{r-2}\,(p-1)\left[p^{r-2}\,(p-1) + p^{r-3}\,(p-1) + \cdots + p.\,(p-1) + (p-1) + 1\right] = \left(p^{r-2}.\,(p-1)^2\right) + \cdots + p^{r-s-1}.\left(p^{s-1}\,(p-1)^2\right) + \cdots + p^{r-2}(p-1) = \sum_i m_i^{G_{r-1}} d_i^{G_{r-1}}$

We now verify the above identities in the following

Theorem 3.1.1.

1. The idempotents $\{T_l^{(r,s)}E_{\rho^i}^{(r,s)},\ 1 \le i \le p^s \text{ and } (i,\ p^s) = 1\}_{0 \le l \le p^{r-s}-1},\ 1 \le s \le r$, form a set of mutually orthogonal primitive idempotents that are conjugate, giving rise to an irreducible representation of degree $p^{s-1}(p-1)$, for every l and for every s.

2. The idempotents $\{T_l^{(r)}E_1^{(r,0)}\}_{0 \le l < p^{r-1}(p-1)}$ form a set of primitive idempotents that give rise to an irreducible representation of degree one. These idempotents are central for the group algebra KG_r for every l.

3. The central idempotents of the group algebra KG_r are given as follows $\left\{T_l^{(r,s)}\widehat{E}^{(r,s)}\right\}$, where

$$\widehat{E}^{(r,s)} = \frac{1}{p^r} \sum_{(i,p^s)=1} i = 1^{p^r}\ E_{\rho^i}^{(r,s)} = \frac{p^{s-1}}{p^r}\left(p-1-\sum_{j=1}^{p-1} g^{p^{s-1}j}\right)\left(\sum_{\substack{k=1 \\ s\neq r}}^{p^{r-s}} g^{p^s k}\right)$$

Proof. 1. Since $T_l^{(r,s)}E_{\rho^i}^{(r,s)}\, g\, T_l^{(r,s)}E_{\rho^i}^{(r,s)} = \alpha T_l^{(r,s)}E_{\rho^i}^{(r,s)}$, for some scalar $\alpha \in K$, and $T_l^{(r,s)}E_{\rho^i}^{(r,s)}\,\tau\, T_l^{(r,s)}E_{\rho^i}^{(r,s)}$ is either 0 or $T_l^{(r,s)}E_{\rho^i}^{(r,s)}$, the product $T_l^{(r,s)}E_{\rho^i}^{(r,s)}(KG_r)T_l^{(r,s)}E_{\rho^i}^{(r,s)}$ form division algebras. Therefore, $T_l^{(r,s)}E_{\rho^i}^{(r,s)}$ represents the primitive idempotents. By using Equation (1)$T_l^{(r,s)}E_{\rho^i}^{(r,s)} = T_l^{(r,s)}\tau^{-j}E_{\rho^k}^{(r,s)}\tau^j = \tau^{-j}T_l^{(r,s)}E_{\rho^k}^{(r,s)}\tau^j$ for some j, hence any two idempotents are conjugate.

Proof for (2) and (3). It is easy to check that the idempotents defined in (2) and (3) commute with every element in G_r.

Note: Recall that SG_r denotes the subgroup of G_r and it is isomorphic to $\mathbb{Z}_{p^{r-1}} \rtimes \mathbb{Z}_{p^r}^{\star}$ for $r \ge 2$ and SG_1 is a subgroup of G_1 isomorphic to $Z_p^{\star}$. Let us define

$$\mathbb{E}_{\rho i}^{(r,s)} = \frac{1}{p^{r-1}} \sum_{j=1}^{p^{r-1}} \rho^{ij}\ (g^p)^j \tag{3}$$

Where ρ is a primitive p^s-th root of unity, $(i,p^s) = 1$ and $0 \le s \le r-1$, $\mathbb{E}_{\rho^i}^{(r,s)}$'s are conjugate to each other.

Theorem 3.1.2.

1. The idempotents $\left\{T_l^{(r,s)}\mathbb{E}_{\rho^i}^{(r,s)}, 1 \le i \le p^s \text{ and } (i,\ p^s) = 1\right\}_{0 \le l \le p^{r-s}-1},\ 1 \le s \le r-1$, form a set of mutually orthogonal primitive idempotents that are conjugate, giving rise to an irreducible representation of degree $p^{s-1}(p-1)$, for every l and for every s.

2. The idempotents $\left\{T_l^{(r)}\mathbb{E}_1^{(r,0)}\right\}_{0 \le l < p^{r-1}(p-1)}$ form a set of primitive idempotents that give rise to an irreducible representation of degree one. These idempotents are central for the group algebra KSG_r for every l.

3. The central idempotents of the group algebra KSG_r are given as follows $\left\{T_l^{(r,s)}\hat{\mathbb{E}}^{(r,s)}\right\}$, where

$$\hat{\mathbb{E}}^{(r,s)} = \frac{1}{p^{r-1}} \sum_{\substack{i=1 \\ (i,p^s)=1}}^{p^{r-1}} \mathbb{E}_{\rho^i}^{(r,s)} = \frac{p^{s-1}}{p^{r-1}}\left(p-1-\sum_{j=1}^{p-1} g^{p^s j}\right)\left(\sum_{\substack{k=1 \\ s\neq r}}^{p^{r-s-1}} g^{p^{s+1}k}\right)$$

Proof.

Proof is similar to the proof of the Theorem 3.1.

3.2 Relation between the centrally primitive idempotents of the group algebras

Remark 3.2.1 The group algebra KG_{r-1} is embedded as a sub algebra in KSG_r using idempotents, following the approach mentioned below.

Define

$$\Psi_r : KG_{r-1} \to KSG_r$$

$$\Psi_r(g) = g^p$$

$$\Psi_r(\tau^j) = \sum_{i=1}^{p} F_i(\eta^{i-1}\tau)^j$$

$$(\Psi_r(\tau))^j := \Psi_r(\tau^j)$$

Where η is a primitive p^{r-1}-th root of unity and $F_i = \frac{1}{p}\sum_{j=1}^{p} \alpha^{(p-i+1)j}\tau^{p^{r-2}(p-1)j},\ \alpha = \eta^{p^{r-2}}$.

Let us denote the centrally primitive idempotents $\widehat{E}^{(r,s)}T_l^{(r,s)}$ of the group algebra KG_r as $z_l^{(r,s)}$ and the centrally primitive idempotents $\widehat{E}^{(r,s)}T_l^{(r,s)}$of the group algebra KSG_r as $\overline{z}_l^{(r,s)}$.

Now explicitly establish the correspondence between the centrally primitive idempotents of the algebras KG_{r-1} and KSG_r.

1. $\Psi_r\left(z_l^{(r,s)}\right)$

$= \psi\left(\frac{p^{s-1}}{p^{r-1}}\left(p-1-\sum_{j=1}^{p-1} g^{p^{s-1}j}\right)\sum_{\substack{k=1 \\ s\neq r-1\, k=1}}^{p^{r-s-1}} g^{p^s k}\left(\frac{1}{p^{r-s-1}}\sum_{m=1}^{p^{r-s}-1}\rho^{lm}\tau^{mp^{s-1}(p-1)}\right)\right)$

$= \frac{p^{s-1}}{p^{r-1}}\left(p-1-\sum_{j=1}^{p-1} g^{p^s j}\right)\sum_{k=1}^{p^{r-s-1}} g^{p^{s+1}k}\left[\frac{1}{p^{r-s-1}}\sum_{i=1}^{p}\sum_{m=1}^{p^{r-s}}\rho^{lm}\eta^{p^{s-1}(p-1)i}\tau^{mp^{s-1}(p-1)}\right]$

$= \widehat{E}^{(r,s)}\left[T_{p^{s-1}(p-1)+pl}^{(r,s)} + T_{2p^{s-1}(p-1)+pl}^{(r,s)} + \cdots + T_{p^{s-1}(p-1)^2+pl}^{(r,s)} + T_{pl}^{(r,s)}\right]$

$= \sum_{i=1}^{p} \overline{z}_{p^{s-1}(p-1)i+pl}^{(r,s)}$

In other words, the centrally primitive idempotent $z_l^{(r-1,s)}$ in KG_{r-1} splits into p centrally primitive idempotents in KSG_r as $\left(\overline{z}_{p^{s-1}(p-1)+pl}^{(r,s)}, \overline{z}_{2p^{s-1}(p-1)+pl}^{(r,s)}, \ldots, \overline{z}_{p^{s-1}(p-1)^2+pl}^{(r,s)}, \overline{z}_{pl}^{(r,s)}\right)$.

Note that,

$$\Psi_r\left(T_l^{(r-1,s)}E_{\rho^j}^{(r-1,s)}\right) = \mathbb{E}_{\rho^j}^{(r,s)}\sum_{i=1}^{p} T_{ip^{s-1}(p-1)+pl}^{(r,s)}$$

2. The centrally primitive idempotent $z_{l'}^{(r-1,s)}$ is primitive in KG_r splits into p centrally primitive idempotents $\overline{z}_l^{(r,s-1)}$ where $l = l' + pj$ and $0 \leq j \leq p-1$ in KSG_r i.e. $\eta^l = \rho^{l'}\beta^j$, where $\beta^{jp} = 1$ (η is the p^{r-s}-th root of unity and $\rho^{p^{r-(s-1)}}$-th root of unity). We can write

$$z_{l'}^{(r,s)}\downarrow = \sum_{j=0}^{p-1}\overline{z}_{l'+pj}^{(r,s-1)}$$

which means that the central idempotent $z_{l'}^{(r,s)}$ is primitive in KG_r but it is not primitive in KSG_r and also it commutes with every element in KSG_r.

3. We now see how the idempotents are induced from KSG_r to KG_r

$$\overline{z}_l^{(r,s)}\uparrow = z_{pl}^{(r,s+1)}$$

In other words, the centrally primitive idempotent $\overline{z}_l^{(r,s)}$ in KSG_r induced to a centrally primitive idempotent in KSG_r as $z_{pl}^{(r,s)}$.

Definition 3.2 .1. A Young diagram is a hook $\lambda_{k,i}$ comprising k boxes, where the first row contains $k-i$ boxes and the first column from the second row contains i boxes.

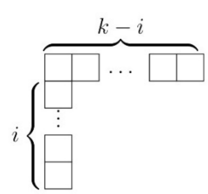

Fig 1. Young diagram of $\lambda = [k-i, 1^i]$

3.3 Indexing set for the irreducible representation by the hook partition

The following is a complete set of inequivalent irreducible representations of G_r, which will be indexed by the hook partition as follows

- $V_0^r = \{\lambda_{p^{r-1}(p-1),i}\}_{0\leq i<p^{r-1}(p-1)}$
- For $1 \leq s \leq r$, $V_s^r = \left\{\lambda_{p^{r-s}[2sp-(2s+1)],i}\right\}_{p^{r-s}[sp-(s+1)]\leq i<sp^{r-s}(p-1)}$

The following is a complete set of inequivalent irreducible representations of SG_r, which will be indexed by the hook partition as follows

- $W_0^r = \{\lambda_{p^{r-1}(p-1),i}\}_{0\leq i<p^{r-1}(p-1)}$
- For $1 \leq s \leq r-1$, $W_s^r = \left\{\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],i}\right\}_{p^{r-s-1}[sp-(s+1)]\leq i<(s+1)p^{r-s-1}(p-1)}$

Remark 3.3.1 The block $B_{m,n}$ containing m-nodes horizontally and n-nodes vertically.

Definition 3.3.1. A hook partition λ is obtained from a hook partition μ by adding a block $B_{m,n}$ of size $m+n$. This addition involves adding m-nodes at the end of the first row and n-nodes at the end of the first column. Similarly, the removal of the block $B_{m,n}$ from the partition entails removing of last m-nodes in the first row and last n-nodes in the first column. In this paper, we are considering only the blocks of size $p^k\,(p-1)$, $p^k(p-1)^2$, $p^k\,(p-2)$ where $k \geq 0$.

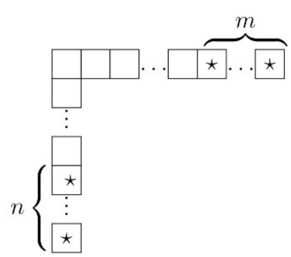

Fig 2. Adding the block $B_{m,n}$ to μ

Example 3.3.1. Let us take a partition $\lambda_{2,1}$ in $Z_3 \rtimes Z_3^\star$, add the block $B_{2-t,t}$, $0 \leq t \leq 2$ to this partition, we get the following partitions in $Z_3 \rtimes Z_{3^2}^\star$.

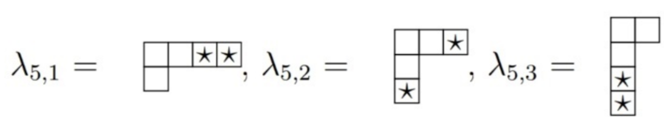

Fig 3. Adding the block $B_{2-t,t}$ to $\lambda_{2,1}$

Example 3.3.2. Let us take a partition $\lambda_{2,0}$ in $Z_3 \rtimes Z_3^\star$, add the block $B_{4-t,t}$, $0 \leq t \leq 2$ to this partition, we get the following partitions in $Z_3 \rtimes Z_{3^2}^\star$.

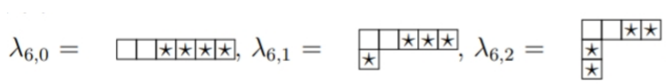

Fig 4. Adding the block $B_{4-t,t}$ to $\lambda_{2,0}$

Definition 3.3.2. We define the edges between the vertices as if there exists an edge from the vertex x on one floor to the vertex $y\,(x \hookrightarrow y)$ on the subsequent floor, obtained by adding a block suitably, then there is a corresponding edge from the vertex y to the vertex x $(y \hookleftarrow x)$ obtained by removing a block suitably.

3.4 Bijection between the hook partition and the minimal idempotents

We have so far given two sets of indexing sets for the complete set of irreducible representations of G_r and SG_r one in terms of minimal idempotents and the other in terms of hook partitions. We now set up a bijection between these two sets.

First, we define the induction rule for the hook partition $\lambda_{p^{r-s}(2p-3),k}$ to the hook partitions $\lambda_{p^{r-s}(3p-4),k_t}$, $0 \le t \le p-1$ in the following way:

$\lambda_{p^{r-s}(3p-4),k_t}$ is obtained from the hook partition $\lambda_{p^{r-s}(2p-3),k}$ where $k_t^{'} = pj + t$, $0 \le t \le p-1$, $0 \le j \le p^{r-s-1}$ and $k_t = p^{r-s}(p-2) + k_t^{'}$, if $k = p^{r-s}(p-2) + j$

Theorem 3.4.1

1. Let $\lambda_{p^{r-s}(2p-3),k'} \in V_1^{r-s+1}$, and $k = p^{r-s}(2p-3)$. Then

$\lambda_{k,\frac{k-1}{2} \pm p^{r-s-1}j_1 \pm p^{r-s-2}j_2 \pm \cdots \pm pj_{r-s-1} \pm j_{r-s}}$ will represent $T_l^{(r-s+2,1)}\mathbb{E}_\rho^{(r-s+2,1)}$

where $l = (p-1)\sum_{i=1}^{r-s} p^{r-s-i}\nu(j_i)$ and $\nu(j_i) = 2j_i$ if the coefficient of $p^{r-s-i}j_i$ is -1, $\nu(j_i) = 2j_i - 1$, if the coefficient of $p^{r-s-i}j_i$ is 1, $\nu(j_i) = 0$ if $j_i = 0$, $0 \le j_i \le \frac{p-1}{2}$.

2. Let $\lambda_{p^{r-s}(3p-4),k'} \in W_1^{r-s+2}$, and $k = p^{r-s}(3p-4)$. Then

$\lambda_{k,\frac{k-1}{2} \pm p^{r-s}j_1 \pm p^{r-s-1}j_2 \pm \cdots \pm pj_{r-s} \pm j_{r-s+1}}$ will represent $T_l^{(r-s+2,1)}E_{\rho^j}^{(r-s+2,1)}$,

where $l = (p-1)\sum_{i=1}^{r-s+1} p^{r-s+1-i}\nu(j_i)$ and $\nu(j_i) = 2j_i$ if the coefficient of $p^{r-s+1-i}j_i$ is -1, $\nu(j_i) = 2j_i - 1$, if the coefficient of $p^{r-s+1-i}j_i$ is 1, $\nu(j_i) = 0$ if $j_i = 0$, $0 \le j_i \le \frac{p-1}{2}$.

3. Let $\lambda_{p^{r-1}(p-1),k'}$ be a hook partition in V_0^r or W_0^r. We begin with the initial case $r = 1$. In this, the hook partition $\lambda_{p-1,i}$ is assigned to $T_i^{(1)}\widehat{E}^{(1,0)}$, where the coefficient of τ in $T_i^{(1)}$ is β^i. For the more general case, the coefficient of τ will be the product of ρ^k, $0 \le k < p^{r-1}$ and β^i, $0 \le i \le p-2$ where ρ^k is the p^{r-1}-th root of unity.

Now let's determine how to choose i and k for a given $k^{'}$. If $k^{'}$ satisfies the following inequality $mp^{r-1} \le k^{'} < (m+1)p^{r-1}$, $0 \le m \le p-2$ then we set $i = m$. Next, consider the irreducible representation assigned to the hook partition $\lambda_{p^{r-1}(2p-3),l'}$, where $l^{'} = k^{'} + p^{r-1}(p-2-t)$ and $0 \le t \le p-2$. If $\lambda_{p^{r-1}(2p-3),l'}$ represents the primitive idempotent $T_l^{(r,1)}E_1^{(r,1)}$ with the coefficient of τ^{p-1} is ρ^l then we set $k = l(p-1)$ which implies $\rho^k = (\rho^l)^{p-1}$. Finally, we fix $\eta^j = \rho^k\beta^i$, where η^j is a $p^{r-1}(p-1)$-th root of unity. Hence the hook partition $\lambda_{p^{r-1}(p-1),k'}$ will represent $T_j^{(r)}E_{\rho^i}^{(r,0)}$.

Remark 3.4.1 The map from $\lambda_{p^{r-s}(2p-3),k} \mapsto \lambda_{p^{r-s}[2mp-(2m+1)],k+mp^{r-s}(p-1)]}$ gives the bijection between $V_1^{r-s+1} \longrightarrow V_m^{r-s+1+m}$. Similarly, the map from $\lambda_{p^{r-s}(3p-4),k} \mapsto \lambda_{p^{r-s}[(2m+1)p-(2m+2),k+mp^{r-s}(p-1)]}$ gives the bijection between $W_1^{r-s+2} \longrightarrow W_m^{r-s+2+m}$. We now set up the bijection between V_s^r and $T_l^{(r,s)}E_{\rho^j}^{(r,s)}$ as follows

$$\lambda_{p^{r-s}[2sp-(2s+1)],k} \mapsto T_l^{(r,s)}E_{\rho^j}^{(r,s)}$$

where η^l is the coefficient of $\tau^{p^{s-1}(p-1)}$ and also the coefficient of τ^{p-1} for the minimal idempotent $T_l^{(r-s,1)}E_{\rho^j}^{(r-s,1)}$. Similarly, the bijection between W_s^{r+1} and $T_l^{(r+1,s)}E_{\rho^j}^{(r+1,s)}$ as follows

$$\lambda_{p^{r-s}[(2s+1)p-(2s+2)],k} \mapsto T_l^{(r+1,s)}\mathbb{E}_{\rho^j}^{(r+1,s)}$$

where η^l is the coefficient of $\tau^{p^{s-1}(p-1)}$ and also the coefficient of τ^{p-1} for the minimal idempotent $T_l^{(r-s+1,1)}E_{\rho^j}^{(r-s+1,1)}$.

3.5 Restriction and Induction on Vector space

In this section, we have delved into the topics of induction and restriction for groups SG_r and G_r, as well as for group algebras KG_{r-1} and KSG_r. Furthermore, we have examined the vertices and edges within the Bratteli diagram.

We now discuss the induction and restriction of the groups and group algebras using centrally primitive idempotents, as in Section 3.2 and Theorem 3.4.1.

1. **Restriction of G_r to SG_r**

a) Let $V_{\lambda_{p^{r-1}(p-1),k'}}$ denote the vector space of dimension 1 representing the hook $\lambda_{p^{r-1}(p-1),k'}$ in G_r. Then

$$V_{\lambda_{p^{r-1}(p-1),k'}} \downarrow = V_{\lambda_{p^{r-1}(p-1),k'}}$$

The vertices remain the same after the restriction and $V_{\lambda_{p^{r-1}(p-1),k'}}$ represent the vector space of dimension 1 representing the hook $\lambda_{p^{r-1}(p-1),k'}$ in SG_r.

b) For $2 \leq s \leq r$, and let $V_{\lambda_{p^{r-s}[2sp-(2s+1)],k}}$ be the vector space of dimension $p^{s-1}(p-1)$ representing the hook $\lambda_{p^{r-s}[2sp-(2s+1)],k}$ in G_r. Then

$$V_{\lambda_{p^{r-s}[2sp-(2s+1)],k}} \downarrow = \oplus_{t=0}^{p-1} V_{\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k'}}$$

where $k' = k - p^{r-(s-1)-1}(p-1-t)$. This restriction is obtained by removing the block $B_{p^{r-s}t,p^{r-s}(p-1-t)}$ of size $p^{r-s}(p-1)$. Let $V_{\lambda_{p^{r-s}[2sp-(2s+1)],k}} \downarrow = \oplus_{t=0}^{p-1} V_{\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k'}}$ be the vector space of dimension $p^{s-2}(p-1)$ representing the hook $\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k'}$ in SG_r.

c) For the case $s = 1$, let $V_{\lambda_{p^{r-1}(2p-3),k}}$ denote the vector space of dimension $p-1$ representing the hook $\lambda_{p^{r-1}(2p-3),k}$ in G_r. Then

$$V_{\lambda_{p^{r-1}(2p-3),k}} \downarrow = \oplus_{t=0}^{p-2} V_{\lambda_{p^{r-1}(p-1),k'}}$$

where $k' = k - p^{r-1}(p-2-t)$. This restriction is obtained by removing the block $B_{p^{r-1}t,p^{r-1}(p-2-t)}$ of size $p^{r-1}(p-2)$. Let $V_{\lambda_{p^{r-1}(p-1),k'}}$ be the vector space of dimension 1 representing the hook $\lambda_{p^{r-1}(p-1),k'}$ in SG_r.

2. **Induction of** SG_r **to** G_r

a) Let $V_{\lambda_{p^{r-1}(p-1),k'}}$ denote the vector space of dimension 1 representing the hook $\lambda_{p^{r-1}(p-1),k'}$ in SG_r. Then

$$V_{\lambda_{p^{r-1}(p-1),k'}} \uparrow = V_{\lambda_{p^{r-1}(p-1),k'}}$$

The vertices remain the same after the induction and $V_{\lambda_{p^{r-1}(p-1),k'}}$ represent the vector space of dimension 1 representing the hook $\lambda_{p^{r-1}(p-1),k'}$ in G_r.

b) For $2 \leq s \leq r$ and$0 \leq t \leq p-1$, let $V_{\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k'}}$ be the vector space of dimension $p^{s-2}(p-1)$ representing the hook $V_{\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k'}}$ in SG_r. Then

$$V_{\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k'}} \uparrow = V_{\lambda_{p^{r-s}[2sp-(2s+1)],k}}$$

where $k = k' + p^{r-(s-1)-1}(p-1-t)$ if $p^{r-s}[sp-(s+1)+t] \leq k' < p^{r-s}[sp-(s+1)+t+1]$. This induction is obtained by adding the block $B_{p^{r-s}t,p^{r-s}(p-1-t)}$ of size $p^{r-s}(p-1)$. Let $V_{\lambda_{p^{r-s}[2sp-(2s+1)],k}}$ be the vector space of dimension $p^{s-1}(p-1)$ representing the hook $V_{\lambda_{p^{r-s}[2sp-(2s+1)],k}}$ in G_r.

c) For $0 \leq t \leq p-2$ and $p^{r-1}(p-2) \leq k < p^{r-1}(p-1)$, let $V_{\lambda_{p^{r-1}(p-1),k'}}$ denote the vector space of dimension 1 representing the hook $\lambda_{p^{r-1}(p-1),k'}$ in SG_r. The induced representation is given by

$$V_{\lambda_{p^{r-1}(p-1),k'}} \uparrow = V_{\lambda_{p^{r-1}(2p-3),k}}$$

where $k = k' + p^{r-1}(p-2-t)$ if $p^{r-1}t \leq k' < p^{r-1}(t+1)$. This induction is obtained by adding the block $B_{p^{r-1}t,p^{r-1}(p-2-t)}$ of size $p^{r-1}(p-2)$. Let $V_{\lambda_{p^{r-1}(2p-3),k}}$ be the vector space of dimension $p-1$ representing the hook $\lambda_{p^{r-1}(2p-3),k}$ in G_r.

Example 3.5.1.

Restriction of G_3 to SG_3 when p=3 and s=1

$$V_{\lambda_{21,k}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,k'}}$$

where $9 \leq k < 12$.

For $k = 9$, we have $V_{\lambda_{21,9}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,9-3(2-t)}} = V_{\lambda_{15,3}} \oplus V_{\lambda_{15,6}} \oplus V_{\lambda_{15,9}}$

For $k = 10$, we have $V_{\lambda_{21,10}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,10-3(2-t)}} = V_{\lambda_{15,4}} \oplus V_{\lambda_{15,7}} \oplus V_{\lambda_{15,10}}$

For $k=11$, we have $V_{\lambda_{21,11}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,11-3(2-t)}} = V_{\lambda_{15,5}} \oplus V_{\lambda_{15,8}} \oplus V_{\lambda_{15,11}} =$

2. Induction of SG_3 to G_3 when p=3 and $s=1$

$$V_{\lambda_{15,k'}} \uparrow = V_{\lambda_{21,k}}$$

where $3 \leq k' < 12$.

If $3+3(0) \leq k' < 3+3(1)$ then $k = k' + 3(2-0) = k' + 6$

If $3+3(1) \leq k' < 3+3(2)$ then $k = k' + 3(2-1) = k' + 3$

If $3+3(2) \leq k' < 3+3(3)$ then $k = k' + 3(2-2) = k' + 0$

The induction (restriction) of KG_{r-1} to KSG_r (KSG_r) to (KG_{r-1}) are facilitated by embedding defined on the centrally primitive idempotents as in Section 3.2.

3. **Induction of KG_{r-1} to KSG_r**

a) Let $V_{\lambda_{p^{r-2}(p-1),k'}}$ denote the vector space of dimension 1 representing the hook $\lambda_{p^{r-2}(p-1),k'}$ in G_{r-1}. Then

$$V_{\lambda_{p^{r-2}(p-1),k'}} \uparrow = \oplus_{t=0}^{p-1} V_{\lambda_{p^{r-1}(p-1),k}}$$

where $k = pk' + t$. This induction is obtained by adding the block $B_{p^{r-2}(p-1)^2-\left(k'(p-1)+t\right),k'(p-1)+t}$ of size $p^{r-2}(p-1)^2$. Let $V_{\lambda_{p^{r-1}(p-1),k}}$ be the vector space of dimension 1, representing the hook $\lambda_{p^{r-1}(p-1),k}$ in SG_r.

b) For $1 \leq s \leq r-1$, and let $V_{\lambda_{p^{r-s-1}[2sp-(2s+1)],k'}}$ be the vector space of dimension $p^{s-1}(p-1)$ representing the hook $\lambda_{p^{r-s}[2sp-(2s+1)],k'}$ in G_{r-1}. Then

$$V_{\lambda_{p^{r-s-1}[2sp-(2s+1)],k'}} \uparrow = \oplus_{t=0}^{p-1} V_{\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}}$$

where $k' = p^{r-s-1}[sp-(s+1)] + j$ and $k = k' + j(p-1) + t$, $0 \leq j < p^{r-s-1}$ and $0 \leq t \leq p-1$. This induction is obtained by adding the block $B_{p^{r-s-1}(p-1)-((p-1)j+t],(p-1)j+t}$ of size $p^{r-s-1}(p-1)$. Let $V_{\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}}$ be the vector space of dimension $p^{s-1}(p-1)$ representing the hook $V_{\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}}$ in SG_r.

4. **Restriction of KG_{r-1} to KSG_r**

a) For a given k there is unique k' where $0 \leq k < p^{r-1}(p-1)$. Let $V_{\lambda_{p^{r-1}(p-1),k}}$ be the vector space of dimension 1, representing the hook $\lambda_{p^{r-1}(p-1),k}$ in SG_r. Then

$$V_{\lambda_{p^{r-1}(p-1),k}} \downarrow = V_{\lambda_{p^{r-2}(p-1),k'}}$$

where $k' = \lfloor \frac{k}{p} \rfloor$, $\lfloor \bullet \rfloor$ denotes the greatest integer function. This restriction is obtained by removing the block $B_{p^{r-2}(p-1)^2-\left(k'(p-1)+t\right),k'(p-1)+t}$ of size $p^{r-2}(p-1)^2$. Let $V_{\lambda_{p^{r-2}(p-1),k'}}$ be the vector space of dimension 1, representing the hook $\lambda_{p^{r-2}(p-1),k'}$ in G_{r-1}.

b) For $1 \leq s \leq r-1$, and let $\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}$ be the vector space of dimension $p^{s-1}(p-1)$, representing the hook $\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}$ in SG_r. Then

$$V_{\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}} \downarrow = V_{\lambda_{p^{r-s-1}[2sp-(2s+1)],k'}}$$

This restriction is obtained by removing the block $B_{p^{r-s-1}(p-1)-(k-k_0-j),k-k_0-j}$ of size $p^{r-s-1}(p-1)$, where $k' = k_0 + j$, $j = \left\lfloor \frac{k-k_0}{p} \right\rfloor$ and $k_0 = p^{r-s-1}[sp-(s+1)]$. Let $V_{\lambda_{p^{r-s-1}(2sp-(2s+1)],k_0+j}}$ be the vector space of dimension $p^{s-1}(p-1)$ representing the hook $\lambda_{p^{r-s-1}[2sp-(2s+1)],k_0+j}$ in G_{r-1}.

Example 3.5.2.

Induction of G_2 to SG_3 when p=3 and s=1

$$V_{\lambda_{9,k'}} \uparrow = \oplus_{t=0}^{2} V_{\lambda_{15,k}}$$

where $9 \leq k' < 12$ and $0 \leq j < 3$.

For $k^{'} = 3+0$, we have $V_{\lambda_{9,3}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,3+t}} = V_{\lambda_{15,3}} \oplus V_{\lambda_{15,4}} \oplus V_{\lambda_{15,5}}$

For $k^{'} = 3+1$, we have $V_{\lambda_{9,4}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,4+2+t}} = V_{\lambda_{15,6}} \oplus V_{\lambda_{15,7}} \oplus V_{\lambda_{15,8}}$

For $k^{'} = 3+2$, we have $V_{\lambda_{9,5}} \downarrow = \oplus_{t=0}^{2} V_{\lambda_{15,5+4+t}} = V_{\lambda_{15,9}} \oplus V_{\lambda_{15,10}} \oplus V_{\lambda_{15,11}}$.

2. Restriction of SG_2 to G_2 when p=3 and $s=1$

$$V_{\lambda_{15,k}} \downarrow = V_{\lambda_{9,k^{'}}}$$

where $3 \leq k < 12$. Here $k_0 = 3$.

For $k = 3,\ 4,\ 5$ we have $j = \lfloor \frac{k-3}{3} \rfloor = 0$ so $k^{'} = 3$

For $k = 6, 7, 8$ we have $j = 1$ so $k^{'} = 4$

For $k = 9,\ 10, 11$ we have $j = 2$ so $k^{'} = 5$

Edges from $2r$-th floor to $(2r-1)$-th floor

• There is an edge from the vertex $\lambda_{p^{r-1}(p-1),k^{'}}$ to $\lambda_{p^{r-1}(p-1),k^{'}}$ and also there is an edge from $\lambda_{p^{r-1}(2p-3),k} \hookleftarrow \lambda_{p^{r-1}(p-1),k^{'}}$ obtained by the restriction of G_r to SG_r as defined in the Section 3.5 1(a), 1(c).

• There is an edge from the vertex $\lambda_{p^{r-s}[2sp-(2s+1)],k} \hookleftarrow \lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k^{'}}$ obtained by the restriction of G_r to SG_r defined in the Section 3.5 1(b).

Edges from $(2r-1)$-th floor to $2r$-th floor

• There is an edge from the vertex $\lambda_{p^{r-1}(p-1),k^{'}}$ to $\lambda_{p^{r-1}(p-1),k^{'}}$ and also there is an edge from $\lambda_{p^{r-1}(p-1),k^{'}} \hookrightarrow \lambda_{p^{r-1}(2p-3),k}$ obtained by the induction of SG_r to G_r as defined in the Section 3.5 2(a), 2(c).

• There is an edge from the vertex $\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k^{'}} \hookrightarrow \lambda_{p^{r-s}[2sp-(2s+1)],k}$ obtained by the induction of SG_r to G_r defined in the Section 3.5 2(b).

Edges from $(2r-2)$-th floor to $(2r-1)$-th floor

• There is an edge from the vertex $\lambda_{p^{r-2}(p-1),k^{'}} \hookrightarrow \lambda_{p^{r-1}(p-1),k}$ obtained by the induction of G_{r-1} to SG_r defined in the Section 3.5 3(a).

• There is an edge from the vertex $\lambda_{p^{r-(s-1)-1}[(2s-1)p-2s],k^{'}} \hookrightarrow \lambda_{p^{r-s}[2sp-(2s+1)],k}$ obtained by the induction of G_{r-1} to SG_r defined in the Section 3.5 3(b).

Edges from $(2r-1)$-th floor to $(2r-2)$-th floor

• There is an edge from the vertex $\lambda_{p^{r-1}(p-1),k} \hookleftarrow \lambda_{p^{r-2}(p-1),k^{'}}$ obtained by the restriction of SG_r to G_{r-1} defined in the Section 3.5 4(a).

• There is an edge from the vertex $\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k} \hookleftarrow \lambda_{p^{r-s-1}[2sp-(2s+1)],k^{'}}$ obtained by the restriction from SG_r to G_{r-1} defined in the Section 3.5 4(b)

Note: The indexing set defined in section 3.3 and the edges defined above serves as the vertices and edges of the p-Bratteli diagram of our group algebras KG_r and KSG_r.

Remark 3.5.1. We begin with one dimensional partition of size $p^k\,(p-1)\,,\ k \geq 0$ then we add a block of size $p^k\,(p-2)$. We continue to add only the block of the same size $p^k\,(p-1)$, but the type of the block may differ.

$$\lambda_{p^k(p-1),l_1} \hookrightarrow \lambda_{p^k(2p-3),l_2} \hookrightarrow \lambda_{p^k(3p-4),l_3} \hookrightarrow \lambda_{p^k(4p-5),l_4} \cdots \hookrightarrow \lambda_{p^k(mp-(m+1)),l_m} \hookrightarrow \cdots$$

3.6 Matrix units for the irreducible representations of the group algebras

The mutually orthogonal primitive idempotents defined in Theorem 3.1.1 and 3.1.2 are utilized to compute the matrix units for each irreducible representation occurring in both G_r and SG_r.

Theorem 3.7.1. Matrix units for the irreducible representation $\lambda_{p^{r-s}[2sp-(2s+1)],k}$ defined in Theorem 3.1.1 of the group algebra KG_r are given by

$$\left(E_{ij}^{(s_r,l)}\right) = \tau^{-(i-1)} E_1^{(s_r,l)} \tau^{(i-1)} \tau^{j-i},\ 1 \leq i,j \leq p^{s-1}\,(p-1)$$

where $E_1^{(s_r,l)} = T_l^{(r,s)} E_\rho^{(r,s)}$ for $1 \leq s \leq r$ and $0 \leq l < p^{r-s}$.

Proof. By Theorem 3.10, each hook partition $\lambda_{p^{r-s}[2sp-(2s+1)],k}$ is corresponds to the minimal idempotent $T_l^{(r,s)} E_{\rho^j}^{(r,s)}$.

$$E_{ij}^{(s_r,l)} E_{mn}^{(s_r,l)} = \tau^{-(i-1)} E_1^{(s_r,l)} \tau^{(i-1)} \tau^{j-i} . \tau^{-(m-1)} E_1^{(s_r,l)} \tau^{(m-1)} \tau^{n-m}$$

$= \tau^{-(i-1)} E_1^{(s_r,l)} \tau^{(i-1)} \tau^{j-m} E_1^{(s_r,l)} \tau^{(n-1)}$
$= \delta_{jm} E_{in}^{(s_r,l)}$
and $\left\{ E_1^{(s_r,l)}, E_1^{(s_r,l)}\tau, E_1^{(s_r,l)}\tau^2, \cdots, E_1^{(s_r,l)}\tau^{p^{s-1}(p-1)-1} \right\}$ form a basis for the vector space.

Theorem 3.7.2. Matrix units for the irreducible representation $\lambda_{p^{r-s-1}[(2s+1)p-2(s+1)],k}$ defined in Theorem 3.1.2 of the group algebra KSG_r are given by

$$\left(\mathbb{E}_{ij}^{(s_r,l)}\right) = \tau^{-(i-1)} \mathbb{E}_1^{(s_r,l)} \tau^{(i-1)} \tau^{j-i}, 1 \le i, j \le p^{s-1}(p-1)$$

where $\mathbb{E}_1^{(s_r,l)} = T_l^{(r,s)} \mathbb{E}_{\rho j}^{(r,s)}$ for $1 \le s \le r-1$ and $0 \le l < p^{r-s}$.

Proof. The proof is similar to the proof of the Theorem 3.7.1.

4 Conclusion

We have successfully studied the p-Bratteli diagrams for each odd prime p that corresponds to the group algebras KG_r and KSG_r in this work. We defined and systematically indexed the full set of irreducible representations for these algebras by using primitive idempotents. Our method not only made these representations' structure more understandable, but it also made it easier to find the vertices in the corresponding p-Bratteli diagrams. We also obtained matrix units from the indexed irreducible representations, which further improved our understanding of KG_r and KSG_r algebraic structures. Our results provide a clear structure for examining the relationship between irreducible representations and the properties of these group algebras. Overall, this study provides important insight into representation theory and establishes a foundation for future research into group algebras KG_r and KSG_r.

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