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A Pair of Fractional Power of Generalized Hankel-Clifford Type Transformations and Their Differential Operators

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Abstract

Objectives: This research aims to introduce a pair of fractional powers of the generalized Hankel-Clifford transforms and their associated differential operators. The study also seeks to explore these differential operators' characteristics in Zemanian type spaces. **Methods:** This study used a mathematical approach to establish and examine the fractional power of generalized Hankel-Clifford transformations. Apart from this we also study the behavior and properties of differential operators on Zemanian type spaces. **Findings:** The study reveals some properties and behaviors of the differential operators with respect to the fractional power of generalized Hankel-Clifford transforms. These results provide an understanding of how these operators work within Zemanian-type spaces. **Novelty:** The novelty of this research lies in introducing the fractional power of generalized Hankel-Clifford transforms and the detailed studies of their differential operators. This work extends the existing knowledge by investigating these operators on Zemanian-type spaces, which has not been extensively studied before.

Keywords: Fractional Hankel-Clifford Transforms; Generalized Fractional Hankel-Clifford Transforms; Differential Operators; Zemanian Type Spaces; Isomorphism

1 Introduction

The study of integral transformations and their associated differential operators has been for decades an essential component of mathematical analysis. It covers hands-on methods for solving problems in differential equations, signal processing, and science and engineering. Hankel-Clifford transform is one of them, and it got attention due to its utility in solving various boundary value problems and their connections to higher-dimensional spaces. Later, this transformation was generalized and studied by many researchers[1–3] and our study is motivated by the same.

The primary goal of our study is twofold: first, we introduced and characterized a pair of fractional power generalized Hankel-Clifford transforms, explaining their mathematical properties and relationships to existing Hankel-Clifford transform; second, we investigate the associated differential operators that naturally arise from these transforms, with a specific focus on their behavior and properties when applied to Zemanian type spaces which are defined in this paper.

Let us begin with the traditional definition of a pair of Hankel-Clifford transforms of order $\mu \geq 0$ was initially introduced by Betancor[4]and Pérez et.al.[5]which is defined as

$$(h_{1,\mu}\,\varphi)(y) = y^{\mu}\int_0^{\infty} C_{\mu}\,(xy)\varphi(x)dx, \tag{1}$$

$$(h_{2,\mu}\,\varphi)(y) = \int_0^{\infty} x^{\mu}\, C_{\mu}\,(xy)\varphi(x)dx, \tag{2}$$

where $C_{\mu}(xy) = (xy)^{-\mu/2} J_{\mu}(2\sqrt{xy})$ is the Bessel-Clifford function of the first kind of order μ. Later, Malgonde et.al.[6] introduced generalized Hankel-Clifford transformations of order $(\alpha-\beta) \geq -1/2$ which has real parameters α and β, and it is further investigated by Pansare et.al.[7] and is defined by

$$\widehat{\varphi}(y) = (h_{1,\alpha,\beta}\,\varphi)(y) = y^{\alpha-\beta}\int_0^{\infty} C_{\alpha,\beta}\,(xy)\varphi(x)dx, \tag{3}$$

$$\widetilde{\psi}(y) = (h_{2,\alpha,\beta}\,\Psi)(y) = \int_0^{\infty} x^{\alpha-\beta}\, C_{\alpha,\beta}\,(xy)\Psi(x)dx, \tag{4}$$

where $C_{\alpha,\beta}(xy)$ is first kind generalized Bessel-Clifford function of the of order $\alpha-\beta$ defined by

$$C_{\alpha,\beta}\,(x) = (x)^{\dfrac{-\alpha+\beta}{2}}\, J_{\alpha-\beta}\,(2\sqrt{x}). \tag{5}$$

Here, we use these generalized Hankel-Clifford transformations to introduce the fractional power of generalized Hankel-Clifford transformations of order $a-b$ which depends on α, β

$$(h^{\theta}{}_{1,\alpha,\beta,a,b}\,\varphi)(y) = \widehat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}\,(y) = \int_0^{\infty} K_1^{\theta}\,(x,y)\varphi(x)dx, \tag{6}$$

where,

$$K_1^{\theta}(x,y) = \begin{cases} \gamma_{a,b,\theta}\, y^{\alpha-\beta} C_{\alpha,\beta,a,b}(xy csc\,\theta) e^{\left(\frac{i}{2}(x^2+y^2)cot\,\theta\right)} & \theta \neq n\pi \\ y^{\alpha-\beta} c_{\alpha,\beta,a,b}(xy) & \theta = \dfrac{\pi}{2} \\ \delta(x-y) & \theta = n\pi \end{cases} \tag{7}$$

And $C_{\alpha,\beta,a,b}(t) = (t)^{(-\alpha+\beta)/2} J_{a-b}(2\sqrt{t})$; $\gamma_{a,b,\theta} = exp[i(a-b+1)(\pi/2-\theta)]$. Similarly,

$$\left(h^{\theta}_{2,\alpha,\beta,a,b}\Psi\right)(y) = \widetilde{\psi}^{\theta}_{2,\alpha,\beta,a,b}\,(y) \;=\; \int_0^{\infty} K_2^{\theta}\,(x,y)\Psi(x)dx, \tag{8}$$

where,

$$K_2^{\theta}(x,y) = \begin{cases} \gamma_{a,b,\theta}\, y^{\alpha-\beta} C_{\alpha,\beta,a,b}(xy csc\,\theta) e^{\left(\frac{i}{2}(x^2+y^2)cot\,\theta\right)} & \theta \neq n\pi \\ x^{\alpha-\beta} c_{\alpha,\beta,a,b}(xy) & \theta = \dfrac{\pi}{2} \\ \delta(x-y) & \theta = n\pi \end{cases} \tag{9}$$

Note that

$$\left(h^{\theta}_{2,\alpha,\beta,a,b}\psi\right)(y) = y^{\alpha-\beta}\left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi\right)(y).$$

The inverse is defined as,

$$\varphi = \left(\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{-1}\widehat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}\right)(x) = \int_0^{\infty} \overline{K}^{\theta}_1\,(y,x)\,\widehat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}\,(y)\,dy. \tag{10}$$

$$\psi = \left(\left(h^{\theta}_{2,\alpha,\beta,a,b} \right)^{-1} \widetilde{\psi}^{\theta}_{2,\alpha,\beta,a,b} \right)(x) = \int_0^{\infty} \overline{K}^{\theta}_{2}(x,y)\, \widetilde{\psi}^{\theta}_{2,\alpha,\beta,a,b}(y)\, dy. \tag{11}$$

where, $\overline{K}^{\theta}_1$ and $\overline{K}^{\theta}_2$ are conjugates of K^{θ}_1 and K^{θ}_2 respectively. Observe that,

$$\varphi(y) = \left(h^{0}_{1,\alpha,\beta,a,b}\varphi \right)(y) = \left(h^{\pi}_{1,\alpha,\beta,a,b}\varphi \right)(y) = \left(h^{2\pi}_{1,\alpha,\beta,a,b}\varphi \right)(y),$$

$$\psi(y) = \left(h^{0}_{2,\alpha,\beta,a,b}\psi \right)(y) = \left(h^{\pi}_{2,\alpha,\beta,a,b}\psi \right)(y) = \left(h^{2\pi}_{2,\alpha,\beta,a,b}\psi \right)(y).$$

If $\alpha = a, \beta = b$ and $\theta = \pi/2$, then it reduces to Equation (3) and Equation (4). The fractional power of generalized Hankel-Clifford transforms are periodic of period 2π with respect to θ.

$$\left(h^{\theta+2\pi}_{1,\alpha,\beta,a,b}\varphi \right)(y) = \left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi \right)(y),$$

$$\left(h^{\theta+2\pi}_{2,\alpha,\beta,a,b}\varphi \right)(y) = \left(h^{\theta}_{2,\alpha,\beta,a,b}\varphi \right)(y).$$

The relation between the generalized Hankel-Clifford transforms and the fractional generalized Hankel-Clifford transform is as

$$\begin{aligned} \left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi \right)(y) &= \gamma_{a,b,\theta}\, y^{\alpha-\beta} \int_0^{\infty} C_{\alpha,\beta,a,b}(xycsc\theta) e^{\left(\frac{i}{2}(x^2+y^2)cot\theta\right)} \varphi(x) dx \\ &= \gamma_{a,b,\theta}\, e^{\left(\frac{i}{2}y^2 cot\theta\right)} (sin\theta)^{\alpha-\beta} h_{1,\alpha,\beta}\left(e^{\left(\frac{i}{2}x^2 cot\theta\right)} \varphi(x) \right)(ycsc\theta). \end{aligned}$$

Put, $e^{\left(\frac{i}{2}x^2 cot\theta\right)}\varphi(x) = \psi(x)$, then we have,

$$\left(h^{\theta}_{1,\alpha,\beta,a,b}\left[e^{\left(-\frac{i}{2}x^2 cot\theta\right)}\varphi \right] \right)(y) = \gamma_{a,b,\theta}\, e^{\left(\frac{i}{2}y^2 cot\theta\right)} (sin\theta)^{\alpha-\beta} \left(h_{1,\alpha,\beta}\psi \right)(ycsc\theta).$$

Similarly,

$$\left(h^{\theta}_{2,\alpha,\beta,a,b}\left[e^{\left(-\frac{i}{2}x^2 cot\theta\right)}\varphi \right] \right)(y) = \gamma_{a,b,\theta}\, e^{\left(\frac{i}{2}y^2 cot\theta\right)} (sin\theta)^{\alpha-\beta} \left(h_{2,\alpha,\beta}\psi \right)(ycsc\theta).$$

- **Parseval's Identity**

$$\int_0^{\infty} x^{-\alpha+\beta} \varphi(x) \overline{\psi}(x)\, dx = \int_0^{\infty} y^{-\alpha+\beta} \left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi \right)(y) \left(\overline{h}^{\theta}{}_{1,\alpha,\beta,a,b}\varphi \right)(y)\, dx,$$

is Parseval's identity for Equation (3) and for Equation (5) we have,

$$\int_0^{\infty} x^{\alpha-\beta} \varphi(x) \overline{\psi}(x)\, dx = \int_0^{\infty} y^{\alpha-\beta} \left(h^{\theta}_{2,\alpha,\beta,a,b}\varphi \right)(y) \left(\overline{h}_{2,\alpha,\beta,a,b}\varphi \right)(y)\, dx.$$

We can also express mixed Parseval's identity by Fubuni's theorem, as

$$\int_0^{\infty} \varphi(x) \overline{\psi}(x)\, dx = \int_0^{\infty} \left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi \right)(y) \left(\overline{h}^{\theta}{}_{2,\alpha,\beta,a,b}\varphi \right)(y)\, dx.$$

Also,

$$D^r_x\, C_{\alpha,\beta}(x) = (-1)^r\, C_{\alpha,\beta+r}(x), \tag{12}$$

$$D^r_x[x^{\alpha-\beta+r} C_{\alpha,\beta+r}(x)] = x^{\alpha-\beta}\, C_{\alpha,\beta+r}(x)\, \forall r \in \mathbb{N}_0 \tag{13}$$

2 Methodology

2.1 Material and Methods

Let us introduce some differential operators correspond to fractional power of generalized Hankel-Clifford transforms followed by Torre [8] as

$$R_{1,\alpha,\beta,a,b,\theta} = e^{\left(\frac{i}{2}x^2 cot\theta\right)} x^{\frac{\alpha-\beta-a+b+1}{2}} D_x x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \tag{14}$$

$$S_{1,\alpha,\beta,a,b,\theta} = e^{\left(\frac{i}{2}x^2 cot\theta\right)} x^{\frac{\alpha-\beta-a+b}{2}} D_x x^{\frac{-\alpha+\beta-a+b+1}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \tag{15}$$

Also, the Kepinski-type differential operator is defined as

$$\Delta_{1,\alpha,\beta,a,b,\theta} = S_{1,\alpha,\beta,a,b,\theta} R_{1,\alpha,\beta,a,b,\theta}$$

$$= e^{\left(\frac{i}{2}x^2 cot\theta\right)} x^{\frac{\alpha-\beta-a+b}{2}} D_x x^{a-b+1} D_x x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \tag{16}$$

$$\begin{aligned} &= xD_x^2 + \left[(1-\alpha+\beta) - 2ix^2 cot\theta\right] D_x - \Big[(2-\alpha+\beta)\, ixcot\theta + x^3 cot^2\theta \\ &\qquad + \frac{a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab)}{4x}\Big] \end{aligned} \tag{17}$$

Similarly,

$$R_{2,\alpha,\beta,a,b,\theta} = -e^{\left(\frac{i}{2}x^2 cot\theta\right)} x^{\frac{-\alpha+\beta-a+b}{2}} D_x x^{\frac{\alpha-\beta+a-b+1}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \tag{18}$$

$$S_{2,\alpha,\beta,a,b,\theta} = -e^{\left(\frac{i}{2}x^2 cot\theta\right)} x^{\frac{-\alpha+\beta-a+b}{2}} D_x x^{\frac{\alpha-\beta+a-b+1}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \tag{19}$$

$$\Delta_{2,\alpha,\beta,a,b,\theta} = R_{2,\alpha,\beta,a,b,\theta} S_{2,\alpha,\beta,a,b,\theta}$$

$$= e^{\left(\frac{i}{2}x^2 cot\theta\right)} x^{\frac{-\alpha+\beta-a+b}{2}} D_x x^{a-b+1} D_x x^{\frac{\alpha-\beta-a+b}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \tag{20}$$

$$\begin{aligned} &= xD_x^2 + \left[(1+\alpha-\beta) - 2ix^2 cot\theta\right] D_x - \Big[(2+\alpha-\beta)\, ixcot\theta + x^3 cot^2\theta \\ &\qquad + \frac{a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab)}{4x}\Big] \end{aligned} \tag{21}$$

Note that if $\alpha = \beta = 0, a = b = 0, \theta = \pi/2$ in $\Delta_{1,\alpha,\beta,a,b,\theta}$ and $\Delta_{2,\alpha,\beta,a,b,\theta}$ then, in both cases we get

$$\Delta_{1,0,0,0,0,\pi/2} = xD_x^2 + D_x,$$

$$\Delta_{2,0,0,0,0,\pi/2} = xD_x^2 + D_x.$$

Thus, $\Delta_1 = \Delta_2$.

And for $\alpha = \beta, a = b, \theta = \pi/2$ we have,

$$\Delta_{1,\alpha,\alpha,a,a,\pi/2} = xD_x^2 + D_x,$$

$$\Delta_{2,\alpha,\alpha,a,a,\pi/2} = xD_x^2 + D_x.$$

Thus, $\Delta_1 = \Delta_2$.

Further, one can verify that $y = (xcsc\theta)^{\frac{\beta-\alpha}{2}} J_{a-b}(2\sqrt{xcsc\theta})e^{\left(\frac{i}{2}x^2cot\theta\right)}$ is solution of the differential equation

$$x^2D_x^2y + x\left[(1+\alpha-\beta) - 2ix^2cot\theta\right]D_xy - \left[xcsc\theta - (2-\alpha+\beta)ix^2cot\theta - x^4cot^2\theta + \frac{a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab)}{4x}\right] y = 0,$$

and $y = (xcsc\theta)^{\frac{\alpha-\beta}{2}} J_{a-b}(2\sqrt{xcsc\theta})e^{\left(\frac{i}{2}x^2cot\theta\right)}$ is a solution of the differential equation

$$x^2D_x^2y + x\left[(1-\alpha+\beta) - 2ix^2cot\theta\right]D_xy + \left[xcsc\theta - (2-\alpha+\beta)ix^2cot\theta - x^4cot^2\theta - \frac{a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab)}{4x}\right] y = 0.$$

- **The Spaces $\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}$ (I) and $\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}$ (I)**

In this section, we introduce the spaces analogues to the Zemanian type of spaces[9] viz. $H^{\theta}_{1,\alpha,\beta,a,b}(I)$ and $H^{\theta}_{2,\alpha,\beta,a,b}(I)$ with their duals.

$H^{\theta}_{1,\alpha,\beta,a,b}(I)$ is space consisting of all the complexed valued smooth function φ defined on I such that, for $\forall q,k \in N_0$, it satisfies

$$\begin{aligned} \rho^{1,\alpha,\beta,a,b}_{q,k,\theta}(\varphi) &= \sup_{x\in I}\left| x^qD_x^k x^{-\dfrac{\alpha+a-\beta-b}{2}} e^{\left(-\frac{i}{2}x^2cot\theta\right)}\varphi(x)\right| \\ &= \sup_{x\in I}\left| x^qD_x^k x^{-\dfrac{\alpha+a-\beta-b}{2}} e^{\left(\frac{i}{2}x^2cot\theta\right)}\varphi(x)\right| < \infty. \end{aligned} \tag{22}$$

Also, the family of seminors $\left\{\rho^{1,\alpha,\beta,a,b}_{q,k,\theta}\right\}_{q,k\in N_0}$ generate topology over $H^{\theta}_{1,\alpha,\beta,a,b}(I)$.

$H^{\theta}_{2,\alpha,\beta,a,b}(I)$ is space consisting of all the complexed valued smooth function φ defined on I such that, for $\forall q,k \in N_0$, it satisfies

$$\begin{aligned} \rho^{2,\alpha,\beta,a,b}_{q,k,\theta}(\varphi) &= \sup_{x\in I}\left| x^qD_x^k x^{\dfrac{\alpha-\beta-a+b}{2}} e^{\left(-\frac{i}{2}x^2cot\theta\right)}\varphi(x)\right| \\ &= \sup_{x\in I}\left| x^qD_x^k x^{\dfrac{\alpha-\beta-a+b}{2}} e^{\left(\frac{i}{2}x^2cot\theta\right)}\varphi(x)\right| < \infty. \end{aligned} \tag{23}$$

Also, the family of seminors $\left\{\rho^{2,\alpha,\beta,a,b}_{q,k,\theta}\right\}_{q,k\in N_0}$ generate topology over $H^{\theta}_{1,\alpha,\beta,a,b}(I)$.

Now, consider a locally integrable function ψ of a slow growth as $x \to \infty$ and $x^{\frac{\alpha-\beta+a-b}{2}}e^{\left(\frac{i}{2}x^2cot\theta\right)}\psi(x)$ is absolutely integrable on I. Then ψ generates regular generalized function in dual of $H^{\theta}_{1,\alpha,\beta,a,b} viz, \psi \in \left(H^{\theta}_{1,\alpha,\beta,a,b}\right)'(I)$ by

$$<\psi,\varphi> = \int_0^{\infty} \psi(x)\varphi(x)\,dx,\ \varphi \in H^{\theta}_{1,\alpha,\beta,a,b}(I)$$

$$= \int_0^1 x^{\frac{\alpha-\beta+a-b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)} \psi(x)\, x^{-\frac{\alpha-\beta+a-b}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \psi(x)\, dx + \int_1^\infty x^{-m+\frac{\alpha-\beta+a-b}{2}}$$

$$\times e^{\left(\frac{i}{2}x^2 cot\theta\right)} \psi(x)\, x^{m-\frac{\alpha-\beta+a-b}{2}} e^{\left(-\frac{i}{2}x^2 cot\theta\right)} \psi(x)\, dx.$$

As $\psi(x)$ is of slow growth, let us choose m enough large so that $x^{-m+\frac{\alpha-\beta+a-b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)} \psi(x)$ is absolutely convergent on I. Then,

$$|<\psi,\varphi>| \leq \rho_{0,0,\theta}^{1,\alpha,\beta,a,b}(\varphi) \int_0^1 \left| x^{\frac{\alpha-\beta+a-b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)} \psi(x) \right| dx$$

$$+\rho_{m,0,\theta}^{1,\alpha,\beta,a,b}(\varphi) \int_1^\infty \left| x^{-m+\frac{\alpha-\beta+a-b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)} \psi(x) \right| dx.$$

Hence, $H_{2,\alpha,\beta,a,b}^{\theta}(I) \subset \left(H_{1,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$. Following a similar way by considering a locally integrable function ψ of a slow growth as $x \to \infty$ and $x^{\frac{-\alpha+\beta+a-b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)} \psi(x)$ is absolutely integrable on I. Then ψ generates regular generalized function ψ in $\left(H_{2,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$ by

$$\langle \psi, \varphi \rangle = \int_0^\infty \psi(x)\, \varphi(x)\, dx, \quad \varphi \in H_{2,\alpha,\beta,a,b}^{\theta}(I).$$

Hence, we conclude that $H_{1,\alpha,\beta,a,b}^{\theta}(I) \subset \left(H_{2,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$. Also, observe that $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$ and $\left(H_{2,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$ are complete.

3 Results and Discussion

- **Properties of Spaces $H_{1,\alpha,\beta,a,b}^{\theta}$ (I) and $H_{2,\alpha,\beta,a,b}^{\theta}$ (I)**

Here we state some properties of spaces $H_{1,\alpha,\beta,a,b}^{\theta}(I)$ and it's dual that can be observed easily. All the properties below are simultaneously true for $H_{2,\alpha,\beta,a,b}^{\theta}(I)$ and it's dual.

1. Consider an isomorphism $\varphi \to x^{\frac{1}{2}}\varphi$ from $H_{1,\alpha,\beta,a,b}^{\theta}(I)$ and $H_{1,\alpha+1,\beta,a,b}^{\theta}(I)$

Then we define an operator $\psi \to x^{1/2}\psi$ where $\psi \in \left(H_{1,\alpha+1,\beta,a,b}^{\theta}\right)^{'}(I)$ and

$$(x^{1/2}\psi, \varphi) = (\psi, x^{1/2}\varphi), \varphi \in \left(H_{1,\alpha,\beta,a,b}^{\theta}\right)(I)$$

which implies $\left(H_{1,\alpha+1,\beta,a,b}^{\theta}\right)^{'}(I)$ and $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$ is isomorphic. Similarly,

2. $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$ is isomorphic to $\left(H_{1,\alpha,\beta-1,a,b}^{\theta}\right)^{'}(I)$, $\left(H_{1,\alpha,\beta,a+1,b}^{\theta}\right)^{'}(I)$ and $\left(H_{1,\alpha,\beta,a,b-1}^{\theta}\right)^{'}(I)$.

3. For an even positive integer r, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)^{'}(I)$ is a subspace of $\left(H_{1,\alpha+r,\beta,a,b}^{\theta}\right)^{'}(I)$.

Proof: For even a positive integer of $r, H_{1,\alpha+r,\beta,a,b}^{\theta}(I)$ is a subspace of $H_{1,\alpha,\beta,a,b}^{\theta}(I)$ and topology generates on $H_{1,\alpha+r,\beta,a,b}^{\theta}(I)$ is stronger than the induced topology of $H_{1,\alpha,\beta,a,b}^{\theta}(I)$.
Clearly,

$$\rho_{q,0,\theta}^{1,\alpha,\beta,a,b}(\varphi) = \rho_{q+1,0,\theta}^{1,\alpha+2,\beta,a,b}(\varphi),\ \forall q.$$

Also, for $\forall q, k$, let $f(x) = x^{-\frac{\alpha+a-\beta+b+2}{2}} exp(-\frac{i}{2}x^2 cot\theta)\, \varphi$.

$$\begin{aligned} D_x^k x^{-\frac{\alpha+a-\beta-b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}\varphi(x) &= D_x^k x f(x) \\ &= D_x^{k-1}[xD_x f(x) + f(x)] \\ &= D_x^{k-1}f(x) + D_x^{k-1}xD_x f(x) \\ &= D_x^{k-1}f(x) + D_x^{k-2}[xD_x^2 f(x) + D_x f(x)] \\ &= 2D_x^{k-1}f(x) + D_x^{k-2}xD_x^2 f(x) \\ &= \cdots \\ &= kD_x^{k-1}f(x) + xD_x^k f(x). \end{aligned}$$

Hence,

$$\rho_{q,k,\theta}^{1,\alpha,\beta,a,b}(\varphi) \le k\rho_{q,k-1,\theta}^{1,\alpha+2,\beta,a,b}(\varphi) + \rho_{q+1,k,\theta}^{1,\alpha+2,\beta,a,b}(\varphi).$$

As this holds for $r=2$, by induction on r we can follow the general case. Observing the above property, we would like to state similar properties of the space.

4. For an even positive integer r, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)'(I)$ is a subspace of $\left(H_{1,\alpha,\beta-r,a,b}^{\theta}\right)'(I)$, $\left(H_{1,\alpha,\beta,a+r,b}^{\theta}\right)'(I)$, and $\left(H_{1,\alpha,\beta,a,b-r}^{\theta}\right)'(I)$.

5. For an even positive integer r, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)'(I)$ is a subspace of $\left(H_{1,\alpha+r,\beta-r,a+r,b}^{\theta}\right)'(I)$, $\left(H_{1,\alpha+r,\beta-r,a,b-r}^{\theta}\right)'(I)$, $\left(H_{1,\alpha+r,\beta,a+r,b-r}^{\theta}\right)'(I)$, and $\left(H_{1,\alpha,\beta-r,a+r,b-r}^{\theta}\right)'(I)$.

6. For an arbitrary positive integer r, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)'(I)$ is a subspace of $\left(H_{1,\alpha+r,\beta-r,a,b}^{\theta}\right)'(I)$.

Proof: For an arbitrary positive integer r, $H_{1,\alpha+r,\beta-r,a,b}^{\theta}(I)$ is a subspace of $H_{1,\alpha,\beta,a,b}^{\theta}(I)$ and topology generates on $H_{1,\alpha+r,\beta-r,a,b}^{\theta}(I)$ is stronger than the induced topology of $H_{1,\alpha,\beta,a,b}^{\theta}(I)$.

Therefore, by using the technique in the above proof observe that

$$\rho_{q,0,\theta}^{1,\alpha,\beta,a,b}(\varphi) = \rho_{q+1,0,\theta}^{1,\alpha+1,\beta-1,a,b}(\varphi),\ \forall q.$$

And for $k = 1,2,3,\ldots,$

$$\rho_{q,k,\theta}^{1,\alpha,\beta,a,b}(\varphi) \le k\rho_{q,k-1,\theta}^{1,\alpha+1,\beta-1,a,b}(\varphi) + \rho_{q+2,k,\theta}^{1,\alpha+1,\beta-1,a,b}(\varphi).$$

Hence, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)'(I)$ is a subspace of $\left(H_{1,\alpha+r,\beta-r,a,b}^{\theta}\right)'(I)$. Similarly,

7. For an arbitrary positive integer r, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)'(I)$ is a subspace of $\left(H_{1,\alpha+r,\beta,a+r,b}^{\theta}\right)'(I)$, $\left(H_{1,\alpha+r,\beta,a,b-r}^{\theta}\right)'(I)$, $\left(H_{1,\alpha,\beta-r,a+r,b}^{\theta}\right)'(I)$, $\left(H_{1,\alpha,\beta-r,a,b-r}^{\theta}\right)'(I)$, and $\left(H_{1,\alpha,\beta,a+r,b-r}^{\theta}\right)'(I)$.

8. For an arbitrary positive integer r, $\left(H_{1,\alpha,\beta,a,b}^{\theta}\right)'(I)$ is a subspace of $\left(H_{1,\alpha+r,\beta-r,a+r,b-r}^{\theta}\right)'(I)$.

Theorem 1. Let $\varphi \in H_{1,\alpha,\beta,a,b}^{\theta}$, then
$\left(\bar{\Delta}_{1,\alpha,\beta,a,b,\theta}\right)^r \varphi(x) = \sum_{i=0}^{r} a_i e^{\left(-\frac{i}{2}x^2\cot\theta\right)} x^{\frac{\alpha+a-\beta-b}{2}+i} D_x^{r+i} x^{-\frac{\alpha+a-\beta-b}{2}} e^{\left(\frac{i}{2}x^2\cot\theta\right)}\varphi(x)$,
where $\bar{\Delta}_{1,\alpha,\beta,a,b,\theta}$ is a complex conjugate of $\Delta_{1,\alpha,\beta,a,b,\theta}$ and a_i are constant depends on a, b.

Proof. From Equation (16), consider

$$\begin{aligned} \overline{\Delta}_{1,\alpha,\beta,a,b,\theta} = &\quad e^{\left(-\frac{i}{2}x^2 cot\theta\right)} x^{\frac{\alpha-\beta-a+b}{2}} D_x x^{a-b+1} D_x x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)}\varphi(x) \\ = &\quad e^{\left(-\frac{i}{2}x^2 cot\theta\right)} x^{\frac{\alpha-\beta-a+b}{2}} \Big[(a-b+1)x^{a-b} D_x x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)}\varphi(x) \\ &\quad + x^{a-b+1} D_x^2 x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)}\varphi(x)\Big] \\ = &\quad \sum_{i=0}^{1} a_i e^{\left(-\frac{i}{2}x^2 cot\theta\right)} x^{\frac{\alpha-\beta+a-b}{2}+i} D_x^{1+i} x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(\frac{i}{2}x^2 cot\theta\right)}\varphi(x). \end{aligned}$$

Next, Consider

$$\begin{aligned}\left(\overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\right)^2\varphi(x) = \; & \overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\left(\overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(x)\\ & = (a-b+1)e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}}D_x x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\\ & \times\Big[(a-b+1)e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}}D_x x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)\\ & +e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}+1}D_x x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)\Big]+e^{(-\frac{i}{2}x^2cot\theta)}\\ & \times x^{\frac{\alpha-\beta+a-b}{2}+1}D_x^2 x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\Big[(a-b+1)e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}}\\ & \times D_x x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)+e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}+1}D_x x^{\frac{-\alpha-\beta+a-b}{2}}\\ & \times e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)\Big]\\ & = e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}+2}D_x^4 x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)+2(a-b+1)\\ & \times e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}+1}D_x^3 x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)+(a-b+1)\\ & \times(a-b+2)e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}}D_x^2 x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x)\\ & = \textstyle\sum_{i=0}^2 a_i e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}+i}D_x^{2+i}x^{\frac{-\alpha-\beta+a-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x).\end{aligned}$$

Continuing this way, we get,

$$\left(\bar{\Delta}_{1,\alpha,\beta,a,b,\theta}\right)^r\varphi(x)=\sum_{i=0}^r a_i e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha+a-\beta-b}{2}+i}D_x^{r+i}x^{-\frac{\alpha+a-\beta-b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x).$$

Hence proved.

Similarly, we can state that for $\varphi\in H^\theta_{2,\alpha,\beta,a,b}$, then

$$\left(\bar{\Delta}_{2,\alpha,\beta,a,b,\theta}\right)^r\varphi(x)=\sum_{i=0}^r a_i e^{(-\frac{i}{2}x^2cot\theta)}x^{\frac{a-\alpha-b+\beta}{2}+i}D_x^{r+i}x^{\frac{\alpha-a-\beta+b}{2}}e^{(\frac{i}{2}x^2cot\theta)}\varphi(x),$$

where $\bar{\Delta}_{2,\alpha,\beta,a,b,\theta}$ is a complex conjugate of $\Delta_{2,\alpha,\beta,a,b,\theta}$ and a_i are constant depends on a,b.

Theorem 2. For all φ and ψ contain in $H^\theta_{1,\alpha,\beta,a,b}(I)$ then

$$\left\langle\left(\Delta_{1,\alpha,\beta,a,b,\theta}\right)^r\varphi(x),\,\psi(x)\right\rangle=\left\langle\varphi(x),\,(\overline{\triangle}_{1,\alpha,\beta,a,b,\theta})^r\psi(x)\right\rangle,$$

where $\bar{\Delta}_{2,\alpha,\beta,a,b,\theta}$ is a complex conjugate of $\Delta_{2,\alpha,\beta,a,b,\theta}$.

Proof. It is enough to show that

$$\left\langle\left(\Delta_{1,\alpha,\beta,a,b,\theta}\right)\varphi(x),\,\psi(x))\right\rangle=\left\langle(\varphi(x),\,(\overline{\triangle}_{2,\alpha,\beta,a,b,\theta})\psi(x)\right\rangle,$$

Consider,

$$\left\langle\left(\Delta_{1,\alpha,\beta,a,b,\theta}\right)\varphi(x),\,\psi(x)\right\rangle=\int_0^\infty\Delta_{1,\alpha,\beta,a,b,\theta}\varphi(x)\psi(x)dx$$

$$\begin{aligned}& =\textstyle\int_0^\infty x(D_x^2\varphi(x)\psi(x)dx+\int_0^\infty(1-\alpha+\beta)\left(D_x\varphi(x)\psi(x)dx-\int_0^\infty 2ix^2cot\theta\,D_x\varphi(x)\psi(x)dx\right.\\ & -\textstyle\int_0^\infty[(2-\alpha+\beta)ixcot\theta+x^3\cot^2\theta+(a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab))\,(4x)^{-1}]\varphi(x)\psi(x)dx\\ & =\textstyle\int_0^\infty(xD_x^2\psi+2D_x\psi)\,(x)\,\varphi\,(x)\,dx+\int_0^\infty((\alpha-\beta-1)+2ix^2cot\,\theta)\,(D_x\psi(x)\varphi(x)dx\\ & -\textstyle\int_0^\infty[(-2-\alpha+\beta)ixcot\,\theta+x^3cot^2\,\theta+(a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab))\,(4x)^{-1}]\varphi(x)\psi(x)dx\\ & =\textstyle\int_0^\infty(xD_x^2\psi+2D_x\psi)\,(x)\,\varphi\,(x)\,dx+\int_0^\infty((\alpha-\beta-1)+2ix^2cot\,\theta)\,(D_x\psi(x)\varphi(x)dx\\ & -\textstyle\int_0^\infty[-(2+\alpha-\beta)ixcot\,\theta+x^3cot^2\,\theta+(a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab))\,(4x)^{-1}]\varphi(x)\psi(x)dx\\ & =\left\langle\varphi(x),\,\overline{\triangle}_{2,\alpha,\beta,a,b,\theta})\psi(x)\right\rangle\end{aligned}$$

By generalization, one can obtain the result.

Similarly, for all φ and ψ contain in $H^\theta_{2,\alpha,\beta,a,b}(I)$ then

$$\left\langle\left(\Delta_{2,\alpha,\beta,a,b,\theta}\right)^r\varphi(x),\,\psi(x))\right\rangle=\left\langle(\varphi(x),\,(\overline{\triangle}_{1,\alpha,\beta,a,b,\theta})^r\psi(x)\right\rangle$$

where $\bar{\Delta}_{1,\alpha,\beta,a,b,\theta}$ is complex conjugate of $\Delta_{1,\alpha,\beta,a,b,\theta}$. Here one can observe that operator $\Delta_{1,\alpha,\beta,a,b,\theta}$ and $\Delta_{2,\alpha,\beta,a,b,\theta}$ also satisfies,

$$\begin{aligned} \left(\Delta_{1,\alpha,\beta,a,b,\theta}\right)^r\left[K_2^\theta(x,y)\right] &= (-ycsc\theta)^r K_2^\theta(x,y), \\ \left(\Delta_{2,\alpha,\beta,a,b,\theta}\right)^r\left[K_1^\theta(x,y)\right] &= (-ycsc\theta)^r K_1^\theta(x,y). \end{aligned}$$

Proposition. i. The operator $\Delta_{1,\alpha,\beta,a,b,\theta}$ is a continuous and linear map from $H^\theta_{1,\alpha,\beta,a,b}(I)$ and $\left(H^\theta_{2,\alpha,\beta,a,b}\right)'(I)$ into themselves.

ii. The operator $\Delta_{2,\alpha,\beta,a,b,\theta}$ is a continuous and linear map from $H^\theta_{2,\alpha,\beta,a,b}(I)$ and $\left(H^\theta_{1,\alpha,\beta,a,b}\right)'(I)$ into themselves. This can be verified by reading the next theorems.

Theorem 3. There exists an isomorphism $R_{1,\alpha,\beta,a,b,\theta}$ between $H^\theta_{1,\alpha,\beta,a,b}(I)$ and $H^\theta_{1,\alpha+1,\beta,a,b}(I)$. Furthermore,

$$\left(R^{-1}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(x) = e^{(\frac{i}{2}x^2cot\theta)}x^{\frac{\alpha-\beta+a-b}{2}}\int_0^\infty s^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(\frac{i}{2}s^2cot\theta)}\varphi(s)ds \tag{24}$$

Proof. Let $\varphi\in H^\theta_{1,\alpha,\beta,a,b}(I)$, then consider

$$\rho^{1,\alpha+1,\beta,a,b}_{q,k,\theta}\left(R_{1,\alpha,\beta,a,b,\theta}\varphi\right) = \sup_{x\in I}\left|x^qD^k_x x^{-\frac{\alpha+a-\beta-b+1}{2}}e^{(-\frac{i}{2}x^2cot\theta)}R_{1,\alpha,\beta,a,b,\theta}\varphi(x)\right|$$

$$= \sup_{x\in I}\left|x^qD^{k+1}_x x^{-\frac{\alpha+a-\beta-b}{2}}e^{(-\frac{i}{2}x^2cot\theta)}\varphi(x)\right|$$

$$= \rho^{1,\alpha,\beta,a,b}_{q,k+1,\theta}(\varphi), \forall q,k. \tag{25}$$

Next for $\varphi\in H^\theta_{1,\alpha+1,\beta,a,b}(I)$, consider,

$$\rho^{1,\alpha,\beta,a,b}_{q,k,\theta}\left(R^{-1}_{1,\alpha,\beta,a,b,\theta}\varphi\right) = \sup_{x\in I}\left|x^qD^k_x x^{-\frac{\alpha+a-\beta-b}{2}}e^{(-\frac{i}{2}x^2cot\theta)}R^{-1}_{1,\alpha,\beta,a,b,\theta}\varphi(x)\right|$$

$$= \sup_{x\in I}\left|x^qD^{k-1}_x x^{-\frac{\alpha+a-\beta-b+1}{2}}e^{(-\frac{i}{2}x^2cot\theta)}\varphi(x)\right|$$

$$= \rho^{1,\alpha,\beta,a,b}_{q,k+1,\theta}(\varphi), \forall q;\ k=1,2,\ldots \tag{26}$$

Next, we proceed with the case $k=0$,

$$\begin{aligned} \rho^{1,\alpha,\beta,a,b}_{q,0,\theta}\left(R^{-1}_{1,\alpha,\beta,a,b,\theta}\varphi\right) = & \sup_{x\in I}\left|x^qx^{-\frac{\alpha+a-\beta-b}{2}}e^{(-\frac{i}{2}x^2cot\theta)}R^{-1}_{1,\alpha,\beta,a,b,\theta}\varphi(x)\right| \\ = & \sup_{x\in I}\left|x^q\int_0^\infty s^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)ds\right| \\ = & \sup_{x\in I}\left|\int_0^\infty s^qs^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)ds\right| \\ \le & \int_0^\infty \sup_{x\in I}\left|s^qs^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)\right|ds \\ \le & \sup_{x\in I}\left|s^q(1+s)^2s^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)\right|\int_0^\infty\frac{1}{(1+s)^2}ds \\ \le & \left|\sup_{x\in I}\left|s^qs^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)\right|+2\sup_{x\in I}\right|s^{q+1}s^{\frac{-\alpha+\beta-a+b-1}{2}} \\ & \times e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)\left|+\sup_{x\in I}\right|s^{q+2}s^{\frac{-\alpha+\beta-a+b-1}{2}}e^{(-\frac{i}{2}s^2cot\theta)}\varphi(s)\Bigg|, \end{aligned}$$

$$\rho_{q,0,\theta}^{1,\alpha,\beta,a,b}\left(R_{1,\alpha,\beta,a,b,\theta}^{-1}\varphi\right) \leq \rho_{q,0,\theta}^{1,\alpha+1,\beta,a,b}(\varphi)+2\rho_{q+1,0,\theta}^{1,\alpha+1,\beta,a,b}(\varphi)+\rho_{q+2,0,\theta}^{1,\alpha+1,\beta,a,b}(\varphi). \tag{27}$$

Therefore result also holds for $k=0$, Hence from Equations (25), (26) and (27), we conclude the proof of result.

Similarly, we can conclude that There exists an isomorphism $S_{2,\alpha,\beta,a,b,\theta}$ between $H_{2,\alpha,\beta,a,b}^{\theta}(I)$ and $H_{2,\alpha+1,\beta,a,b}^{\theta}(I)$. Furthermore,

$$S_{2\ \alpha,\beta,a,b,\theta}^{-1}\varphi(x) = e^{(\frac{i}{2}x^2 cot\theta)} x^{\frac{-\alpha+\beta+a-b}{2}} \int_0^\infty s^{\frac{\alpha-\beta-a+b-1}{2}} e^{(\frac{i}{2}s^2 cot\theta)}\varphi(s)ds. \tag{28}$$

Proof can be easily verified using the above technique.

Corollary 1. i. There exists an isomorphism $R_{1,\alpha,\beta,a,b,\theta}$ between $H_{1,\alpha,\beta,a,b}^{\theta}(I)$ and $H_{2,\alpha,\beta-1,a,b}^{\theta}(I)$ or $H_{2,\alpha,\beta,a+1,b}^{\theta}(I)$ or $H_{2,\alpha,\beta,a,b-1}^{\theta}(I)$.

ii. There exists an isomorphism $S_{2,\alpha,\beta,a,b,\theta}$ between $H_{2,\alpha,\beta,a,b}^{\theta}(I)$ and $H_{2,\alpha,\beta-1,a,b}^{\theta}(I)$ or $H_{2,\alpha,\beta,a+1,b}^{\theta}(I)$ or $H_{2,\alpha,\beta,a,b-1}^{\theta}(I)$.

Theorem 4. The differential operator $S_{1,\alpha,\beta,a,b,\theta}$ is a continuous and linear map from $H_{1,\alpha+1,\beta,a,b}^{\theta}(I)$ into $H_{1,\alpha,\beta,a,b}^{\theta}(I)$.

Proof. Let $\varphi \in H_{1,\alpha+1,\beta,a,b}^{\theta}(I)$, then consider,

$$\begin{gathered}
x^q D_x^k x^{-\frac{\alpha+a-\beta-b}{2}} e^{(-\frac{i}{2}x^2 cot\theta)} S_{1,\alpha,\beta,a,b,\theta}\varphi(x) \\
= x^q D_x^k x^{b-a} D_x x^{\frac{-\alpha+a+\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) \\
= x^q D_x^k x^{b-a} D_x x^{a-b+1} x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) \\
= x^q D_x^k x^{b-a}(a-b+1)x^{a-b} x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) + x^q D_x^k x^{b-a} x^{a-b+1} \\
\times D_x x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) \\
= (a-b+1)x^q D_x^k x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) + x^q D_x^k x D_x x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) \\
= (a-b+1)x^q D_x^k x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) + x^q D_x^k x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) \\
+ x^q D_x^{k-1} x D_x^2 x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) \\
= (a-b+2)x^q D_x^k x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) + x^q D_x^{k-1} x D_x^2 x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x).
\end{gathered}$$

Continuing in this way, we get,

$$x^q D_x^k x^{-\frac{\alpha+a-\beta-b}{2}} e^{(-\frac{i}{2}x^2 cot\theta)} S_{1,\alpha,\beta,a,b,\theta}\varphi(x)$$

$$= (a-b+k+1)x^q D_x^k x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x) + x^{q+1} D_x^{k+1} x^{-\frac{\alpha+a-\beta-b+1}{2}} e^{(-\frac{i}{2}x^2 cot\theta)}\varphi(x).$$

from which we can have,

$$\rho_{q,k,\theta}^{1,\alpha,\beta,a,b}(S1,\alpha,\beta,a,b,\theta\varphi) \leq (a-b+k+1)\,\rho_{q,k,\theta}^{1,\alpha+1,\beta,a,b}(\varphi) + \rho_{q+1,k+1,\theta}^{1,\alpha+1,\beta,a,b}(\varphi), \forall q,k. \tag{29}$$

Therefore, from Equation (29), we conclude the theorem.

Similarly we can conclude that, The differential operator $R_{2,\alpha,\beta,a,b,\theta}$ is a continuous and linear map from $H_{2,\alpha+1,\beta,a,b}^{\theta}(I)$ into $H_{2,\alpha,\beta,a,b}^{\theta}(I)$.

Proof can be easily verified using the above technique.

Corollary 2. i. The differential operator $S_{1,\alpha,\beta,a,b,\theta}$ is a continuous and linear map from $H_{1,\alpha,\beta-1,a,b}^{\theta}(I)$ or $H_{1,\alpha,\beta,a+1,b}^{\theta}(I)$ or $H_{1,\alpha,\beta,a,b-1}^{\theta}(I)$ into $H_{1,\alpha,\beta,a,b}^{\theta}(I)$.

ii. The differential operator $R_{2,\alpha,\beta,a,b,\theta}$ is a continuous and linear map from $H_{1,\alpha,\beta-1,a,b}^{\theta}(I)$ or $H_{1,\alpha,\beta,a+1,b}^{\theta}(I)$ or $H_{1,\alpha,\beta,a,b-1}^{\theta}(I)$ into $H_{2,\alpha,\beta,a,b}^{\theta}(I)$.

Next, we define the adjoint of the differential operator defined in section 2.2.

Definition. 1. Let $R^*_{2,\alpha,\beta,a,b,\theta}$ be an adjoint of $R_{1,\alpha,\beta,a,b,\theta}$ defined on $\left(H^{\theta}_{1,\alpha+1,\beta,a,b}\right)^{\prime}(I)$ as

$$\left\langle R^*_{2}{}_{,\alpha,\beta,a,b,\theta}\, f,\varphi\right\rangle = \left\langle f,\ R_{1,\alpha,\beta,a,b,\theta}\varphi\right\rangle,\ f\in\ (H^{\theta}{}_{1,\alpha,\beta,a,b})^{\backprime}(I),\ \varphi\in H^{\theta}{}_{1,\alpha+1,\beta,a,b}(I)$$

2. Let $R_{1,\alpha,\beta,a,b,\theta}$be an adjoint of $R^*_{2}{}_{,\alpha,\beta,a,b,\theta}$ defined on $(H^{\theta}_{2},\alpha,\beta,a,b)^{\backprime}(I)$ as

$$\left\langle R_{1,\alpha,\beta,a,b,\theta}\, f,\varphi\right\rangle = \left\langle f,\ R^*_{2,\alpha,\beta,a,b,\theta}\varphi\right\rangle,\ f\in\ (H^{\theta}{}_{2,\alpha+1,\beta,a,b})^{\backprime}(I),\ \varphi\in H^{\theta}{}_{2,\alpha+1,\beta,a,b}(I)$$

3. Let $S^*_{2}{}_{,\alpha,\beta,a,b,\theta}$be an adjoint of $S_{2,\alpha,\beta,a,b,\theta}$ defined on $(H^{\theta}{}_{1,\alpha,\beta,a,b})^{\backprime}(I)$ as

$$\left\langle S^*_{2}{}_{,\alpha,\beta,a,b,\theta}\, f,\varphi\right\rangle = \left\langle f,\ S_{1,\alpha,\beta,a,b,\theta}\varphi\right\rangle,\ f\in\ (H^{\theta}{}_{1,\alpha,\beta,a,b})^{\backprime}(I),\ \varphi\in H^{\theta}{}_{1,\alpha+1,\beta,a,b}(I)$$

4. Let $S_{1,\alpha,\beta,a,b,\theta}$be an adjoint of $S_{2,\alpha,\beta,a,b,\theta}$ defined on $(H^{\theta}{}_{2,\alpha+1,\beta,a,b})^{\backprime}(I)$ as

$$\left\langle S_{1,\alpha,\beta,a,b,\theta}\, f,\varphi\right\rangle = \left\langle f,\ S^*_{2,\alpha,\beta,a,b,\theta}\varphi\right\rangle,\ f\in\ (H^{\theta}{}_{2,\alpha+1,\beta,a,b})^{\backprime}(I),\ \varphi\in H^{\theta}{}_{2,\alpha+1,\beta,a,b}(I)$$

4 Conclusion

This study presents and characterizes a pair of fractional powers of generalized Hankel-Clifford transformations and their associated differential operators. We investigated their mathematical features and proved their suitability for Zemanian type spaces. Our discoveries build on the current foundation of Hankel-Clifford transformations, giving new insights and approaches for covering complex problems in mathematical analysis and associated domains. Future research might investigate the possible uses of these transformations across other fields of science and technology.

Remark. The study is motivated by Prasad, A. and Kumar, S. [1] and we strongly believe that the result obtained in our study is stronger than it and can be reduced to[1] by putting the value $\alpha = \frac{1}{4} + \frac{\mu}{2}, \beta = \frac{1}{4} - \frac{\mu}{2}$ and $a = \frac{1}{4} + \frac{v}{2}, b = \frac{1}{4} - \frac{v}{2}$.

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