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Degree, Neighbourhood, and Connectedness in Fuzzy Digraph

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Abstract

Objective: In the context of fuzzy digraphs, degree neighborhood, and connectedness play important roles in understanding the structure and relationships within the graph. Here are the objectives of degree, neighborhood, and connectedness in fuzzy digraphs: 1. Degree Neighborhood: The degree neighborhood of a vertex in a fuzzy digraph refers to the set of vertices that are directly connected to it. In fuzzy digraphs, the concept of degree neighborhood is extended to include the strength or degree of connection between vertices, which is represented by fuzzy values. 2. Connectedness: Connectedness in fuzzy digraphs refers to the existence of paths between any pair of vertices, considering the fuzzy nature of relationships. **Methods:** By using fuzzy vertex and fuzzy edges in fuzzy digraph to find degree (in-degree and out-degree), neighborhood (in-neighborhood and out-neighborhood), and to find connectedness in fuzzy digraph. **Findings:** In this study, some types of fuzzy digraphs are introduced, neighborhood (in-neighborhood and out-neighborhood) and degree (in-degree and out-degree) in fuzzy digraphs, and strong, weak, unilateral connectedness in fuzzy digraphs are introduced. **Novelty:** To show that let $G_D : (\sigma, \mu)$ be a fuzzy digraph with fuzzy vertex set σ and fuzzy edge set μ. In this paper, some types of fuzzy digraphs are introduced, neighborhood (in-neighborhood and out-neighborhood) and degree (in-degree and out-degree) in fuzzy digraph and strong, weak, unilateral connectedness in fuzzy digraph are introduced. Furthermore, this paper establishes various results and theorems, and some properties of different fuzzy digraphs are discussed.

Keywords: Fuzzy digraph; Degree in a fuzzy digraph; Neighborhood in a fuzzy digraph; Fuzzy diwalk; Fuzzy dicycle; Fuzzy diacyclic digraph

1 Introduction

The concept of the theory of graphs and its applications was established by Berge in 1962. The notion of graph theory was introduced by Hararyin 1992. A rough fuzzy digraph was initiated by Muhammad Akram, and Fariha Zafar [1]. J. N. Mordeson and P. Nairfirst introduced the concept of directed graphs (digraphs) in a fuzzy environment in 1996. Yang B, Hu BQ. [2] introduced the idea of fuzzy neighborhood operators and derived fuzzy coverings in 2019. Mary JD, Shanmugapriya R.[3] introduced the concept of the connectedness of fuzzy graphs and the resolving number of fuzzy digraphs in 2021. Nithyanandham and Augustin[4] were introduced in "Bipolar intuitionistic fuzzy graph based decision-making model to identify flood vulnerable region" in 2023. Improved digraph and matrix assessment model using bipolar fuzzy numbers was also initiated by Zafar, F[5] in 2024. In 2023, Deva Nithyanandham, and Felix Augustin[6], also introduced "A bipolar fuzzy p-competition graph-based ARAS technique for prioritizing COVID-19 vaccines". In 2024, J. Ai, S. Gerke, G. Gutin, S. Wang, A. Yeo, and Y. Zhou[7], were initiated the concept of "On Seymour's and Sullivan's second neighborhood conjectures". Maryam A, Rahim MT, and Hussain F[8] initiated the concept of Properties of Cubic Fuzzy Competition Graphs with Applications in 2024. Islam SR, Pal M[9]. Stated neighborhood and competition graphs under the fuzzy incidence graph and its application in 2024. Muthupandiyan S, Ismayil AM.[10] introduced the idea of upper g-eccentric domination in fuzzy graphs. X. Zhang, B. Chen and D. Miao,[11] initiated the idea of "Systematic Feature Selection Based on Three-Level Improvements of Fuzzy Dominance Three-Way Neighborhood Rough Sets," in 2024. Z. Yuan, P. Hu, H. Chen, Y. Chen and Q. Li,[12] stated "DFNO: Detecting Fuzzy Neighborhood Outliers," in 2024. Yuzhang, B., Jusheng, M[13] found the idea Adaptive intuitionistic fuzzy neighborhood classifier in the year 2024.

2 Methodology

In this study, the concepts of degree (in-degree and out-degree), neighborhood (in-neighborhood and out-neighborhood), and strong, weak, and unilateral connected fuzzy digraphs are introduced. This work is to provide general results of fuzzy diwalk, fuzzy dicycle, and fuzzy acyclic-directed graph to facilitate further developments on these concepts.

Preliminaries

This section comes up with the definition of fuzzy digraph, and its order and size.

Definition 2.1:

Let X be any set of the universe of discourse U. A fuzzy subset σ of X is a function from X into the closed interval $[0,1]$. Let σ be a fuzzy subset of a set S and μ be a fuzzy relation on $S\times S$. Then μ is known as fuzzy relation on σ if $\mu\,(x,y) \le \sigma\,(x) \wedge \sigma(y)$ for all $x,\ y \in S$.

1. μ is called reflexive if $\mu\,(x,x) = \sigma\,(x)$ for all $x \in S$.
2. μ is called symmetric if $\mu\,(x,y) = \mu\,(y,x)$ for all $x,\ y \in S$.
3. μ is called transitive if $\mu^2 \subseteq \mu.\ \ i.e.\ \ \mu\,(x,\ z) \ge sup_{y\in S}\,[\mu\,(x,\ y) \wedge \mu\,(y,\ z)]\ \ \forall x,\ z \in S$
4. The strong fuzzy relation on σ is the fuzzy relation μ defined by $\mu\,(x,y) = \sigma\,(x) \wedge \sigma(y)$ for all $x,\ y \in S$.

Definition 2.2 :

Let V be a non-empty set. A fuzzy graph is a pair of function $G : (\sigma,\ \mu)$ where $\sigma : V \to [0,\ 1]$ and $\mu : V\times V \to [0,\ 1]$ such that $\mu\,(u,\ v) \le \sigma(u) \wedge \sigma(v)$ for all $u,\ v \in V$. The underlying crisp graph of the fuzzy graph $G : (\sigma,\ \mu)$ is indicated as $G^* : (\sigma^*,\ \mu^*)$ where $\sigma^* = \{u|\sigma(u) > 0\}$ is referred to as the nonempty set V of nodes and $\mu^* = \{(u,\ v) > 0\}= E\ \subseteq V\times V$. The crisp graph $G^*\,(V,\ E)$ is a unique case of the fuzzy graph G with every vertices and edges of $(V,\ E)$ with a degree of membership 1.

Definition 2.3:

A fuzzy digraph or directed fuzzy graph (DFG) $G_D\,(\sigma,\ \mu)$ is defined on $(V,\ E)$ consist of a non-empty finite set $\sigma : V \to [0,\ 1]$ and $\mu :\ E \subseteq V\times V \to [0,\ 1]$ such that for all $x,\ \ y \in V$, $\mu\,(x,y) \le \sigma\,(x) \wedge \sigma(y)$ and $V\times V$ is an ordered pair of distinct vertices. The set of all vertices in V are known as vertex sets and E are known as edge sets. We write frequently $G_D(\sigma,\ \mu)$ this implies σ and μ are vertex set and edge set of G_D respectively.

Note: In a fuzzy digraph, μ is not necessarily symmetric. i.e., If $(x,y) \in E$ then x directed to y but (y,x) need not be in E.

Definition 2.4:

An edge (u,v) the initial vertex u is called tail and the following vertex v are called heads. The edge (u,v) leaves u and enters v. The head and tail of an edge are called end vertices and the end vertices are adjacent, i.e., u is adjacent to v and v is adjacent to u. If $(u.v)$ is an edge then u dominates v (v is dominated by u) and indicated it with $u \to v$. A vertex u is incident to an edge v if u is the head or tail of v. An edge (x,y) is denoted by xy.

Definition 2.5 :

The order of the fuzzy digraph G_D is defined and denoted by $\sum_{x\in V}\sigma(x)=p$ and the size of G_D is defined and denoted by $\sum_{x,\ y\in V}\mu(x,\ y)=q$.

Example: 2.1 Consider the fuzzy digraph G_D given in Figure 1.

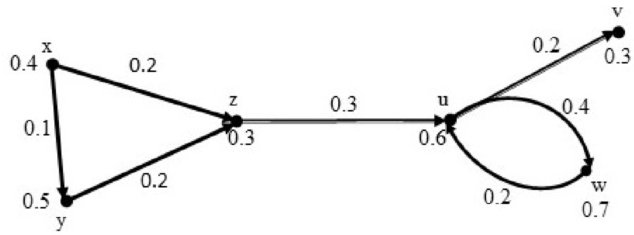

Fig 1. Directed fuzzy digraph $G_D(\sigma,\mu)$

The order of $G_D=|D|=\sum_{x\in V(G_D)}\sigma(x)=2.8=p$

The size of $G_D=|D|=\sum_{(x,\ y)\in E(G_D)}\mu(x,\ y)=1.6=q$

Definition 2.6 :

An out-degree of a vertex u in a directed fuzzy graph is defined by $d^{+}_{G_D}(u)=\sum_{(u,v)\in\mu_D}\mu_D(u,v)$. The in-degree of a vertex u in a directed fuzzy graph is defined by $d^{-}_{G_D}(u)=\sum_{(v,u)\in\mu_D}\mu_D(v,u)$. The sets $d^{+}_{G_D}(v)$ and $d^{-}_{G_D}(v)$ and $d_{G_D}(v)=d^{+}_{G_D}(v)+d^{-}_{G_D}(v)$ are called the out-degree, in-degree, and degree of v respectively.

Example: 2.2

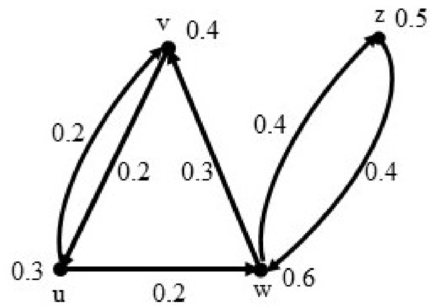

Fig 2. Out-neighborhood and in-neighborhood

In Figure 2, $d^{+}_{G_D}(u)=\sum_{(u,v)\in\mu_D}\mu_D(u,v)=0.4$,$d^{-}_{G_D}(u)=\sum_{(v,u)\in\mu_D}\mu_D(v,u)=0.2$,$d^{+}_{G_D}(v)=\sum_{(v,u)\in\mu_D}\mu_D(v,u)=0.2$,$d^{-}_{G_D}(v)=\sum_{(u,v)\in\mu_D}\mu_D(u,v)=0.5$,$d^{+}_{G_D}(w)=\sum_{(w,v)\in\mu_D}\mu_D(w,v)=0.7$,$d^{-}_{G_D}(w)=\sum_{(u,w)\in\mu_D}\mu_D(u,w)=0.6$,$d^{+}_{G_D}(z)=\sum_{(z,w)\in\mu_D}\mu_D(z,w)=0.4$,$d^{-}_{G_D}(z)=\sum_{(w,z)\in\mu_D}\mu_D(w,z)=0.4$. $d_{G_D}(u)=d^{+}_{G_D}(u)+d^{-}_{G_D}(u)$= 0.6, $d_{G_D}(v)=d^{+}_{G_D}(v)+d^{-}_{G_D}(v)=0.7$, $d_{G_D}(w)=d^{+}_{G_D}(w)+d^{-}_{G_D}(w)=1.3$, $d_{G_D}(z)=d^{+}_{G_D}(z)+d^{-}_{G_D}(z)=0.8$,

Note: $\sum_{u\in V}d^{+}_{G_D}(u)=\sum_{u\in V}d^{-}_{G_D}(u)=q$.

3 Neighborhood and Various Degrees of a Fuzzy Digraph

In this section, neighborhood and various degrees of a fuzzy digraph are discussed.

Definition 3.1:

For any vertex $v \in G_D$, $N^+_{G_D}(u) = \{v \in V - \{u\}\ (u,v) \in E\}$ and $N^-_{G_D}(u) = \{w \in V - \{v\}\ (w,u) \in E\}$. The sets $N^+_{G_D}(v)$, $N^-_{G_D}(v)$ and $N_{G_D}(v) = N^+_{G_D}(v) \cup N^-_{G_D}(v)$ is known as out-neighborhood, in-neighborhood and neighborhood of u respectively.

Example: 3.1

In Figure 2 , $N^+_{G_D}(u) = \{v,w\} \Rightarrow |N^+_{G_D}(u)| = 1.0$, $N^-_{G_D}(u) = \{v\} \Rightarrow |N^-_{G_D}(u)| = 0.4$,$N^+_{G_D}(w) = \{v, z\} \Rightarrow |N^+_{G_D}(w)| = 0.9$, $N^-_{G_D}(w) = \{u, z\} \Rightarrow |N^-_{G_D}(w)| = 0.8$, $N^+_{G_D}(z) = \{w\} \Rightarrow |N^+_{G_D}(z)| = 0.6$,$N^-_{G_D}(z) = \{w\} \Rightarrow |N^-_{G_D}(z)| = 0.6$,$N^+_{G_D}(v) = \{u\} \Rightarrow |N^+_{G_D}(v)| = 0.3$,$N^-_{G_D}(v) = \{u,w\} \Rightarrow |N^-_{G_D}(v)| = 0.9$. $N_{G_D}(u) = N^+_{G_D}(u) \cup N^-_{G_D}(u)$= 1.4, $N_{G_D}(w) = N^+_{G_D}(w) \cup N^-_{G_D}(w) = 1.7$, $N_{G_D}(z) = N^+_{G_D}(z) \cup N^-_{G_D}(z) = 1.2$, $N_{G_D}(v) = N^+_{G_D}(v) \cup N^-_{G_D}(v) = 1.2$.

Definition 3.2:

The minimum (min) out-degree and minimum in-degree of G_D is defined by

$\delta^+_d(G_D) = \wedge\{d^+_{G_D}(x) : x \in V(G_D)\}$ and $\delta^-_d(G_D) = \wedge\left\{d^-_{G_D}(x) : x \in V(G_D)\right\}$ respectively.

The min-semi-degree of G_D is $\delta^0(G_D) = \wedge\{\delta^+_d(G_D), \delta^-_d(G_D)\}$.

Finally, the min-degree of G_D is $\delta(G_D) = \wedge\left\{d^+_{G_D}(v) + d^-_{G_D}(v) : v \in V(G_D)\right\}$.

Similarly, we can define the maximum (max) out-degree and maximum in-degree of G_D is

$\triangle^+_d(G_D) = \vee\{d^+_{G_D}(x) : x \in V(G_D)\}$ and $\triangle^-_d(G_D) = \vee\left\{d^-_{G_D}(x) : x \in V(G_D)\right\}$.

The max-semi-degree of G_D is $\triangle^0(G_D) = \vee\{\triangle^+_d(G_D), \triangle^-_d(G_D)\}$.

Finally, the maximum degree of G_D is $\triangle(G_D) = \vee\left\{d^+_{G_D}(v) + d^-_{G_D}(v) : v \in V(G_D)\right\}$.

Example: 3.2 Consider the Figure 2, $\delta^+_d(G_D) = 0.2$ and $\delta^-_d(G_D) = 0.2$ is the min-out-degree and min-in-degree. $\Delta^+_d(G_D) = 0.4$ and $\Delta^-_d(G_D) = 0.4$ is the max-out-degree and max-in-degree.

Definition 3.3:

A G_D is a regular if $\delta^0(G_D) = \Delta^0(G_D)$ .G_D is also called $\delta^0(G_D)$-regular.

Definition 3.4 :

A vertex u is called isolated if $d^+_{G_D}(u) = d^-_{G_D}(u) = 0$. A vertex y is called semi-isolated if either $d^+_{G_D}(y) = 0$ or $d^-_{G_D}(y) = 0$.

Definition 3.5:

An H_D is a fuzzy subdigraph of a fuzzy digraph G_D is defined by $V(H_D) \subseteq V(G_D)$, $E(H_D) \subseteq E(G_D)$ and each edge in $E(H_D)$ has both end-vertices in $V(H_D)$. If $V(H_D) = V(G_D)$, then H_D is a fuzzy spanning subdigraph of G_D. Every edge of $E(G_D)$ with both end-vertices in $V(H_D)$ is in $E(H_D)$ and H_D is fuzzy induced by $X = V(H_D)$ (we write $H_D = G_D(X)$) also H_D is an fuzzy induced subdigraph of G_D.

4 Types of Fuzzy Digraphs

In this section, different types of fuzzy digraphs are discussed and come up with new concepts of strong, weak, and unilateral connections in some fuzzy digraphs. Some working terminologies and some insightful thoughts in the form of theorems, and remarks with examples are given.

Definition 4.1 :

A fuzzy digraph is called pseudo if it contains self-loop and parallel edges. A pseudo-fuzzy digraph without a self-loop is called a fuzzy multidigraph and pseudo fuzzy digraph without a self-loop and parallel edges is called a simple fuzzy digraph. A fuzzy digraph is called an oriented fuzzy digraph if it has no cycle of length two and no pair of edges of the form (x, y) and (y, x).

Example: 4.1 Example for (a) Pseudo fuzzy digraph and (b) Oriented fuzzy digraph

From the Figure 3**(a)**

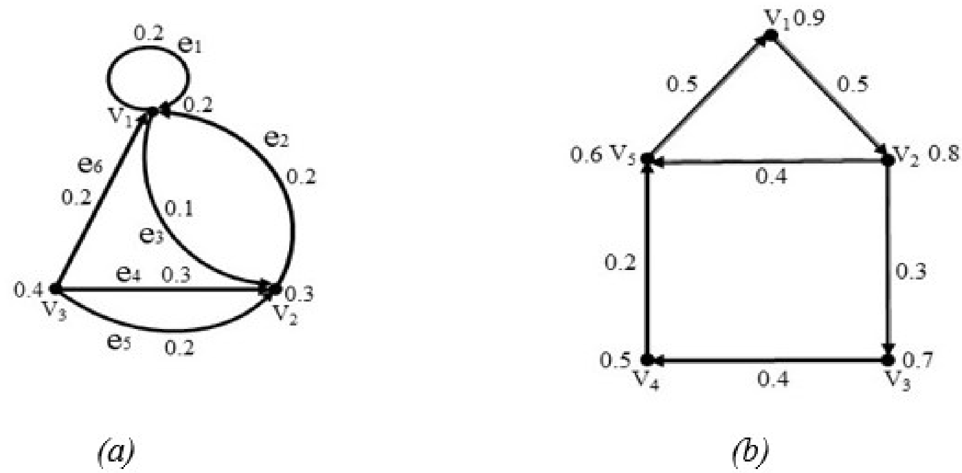

Fig 3.

1. e_1 is a self-loop, e_4, e_5 is a parallel edge but not e_2 and e_3. Hence Fig.(a) is a pseudo-fuzzy digraph.
2. If the self-loop (edge e_1) has been removed, then it is an example of a fuzzy multidigraph.
3. If the self-loop (edge e_1) and the parallel edge e_5 has been removed, it is an example of the simple fuzzy digraph.
4. The vertices V_1, $V_2$are symmetric but not parallel edges.

Figure 3**(b)** is an example of an oriented fuzzy digraph since there is no cycle of length two and no pair of edges of the form $(x,\, y)$ and $(y,\, x)$.

Definition 4.2 :

A tournament fuzzy digraph is known as an oriented fuzzy digraph if each pair of distinct vertices is adjacent.

Example: 4.2

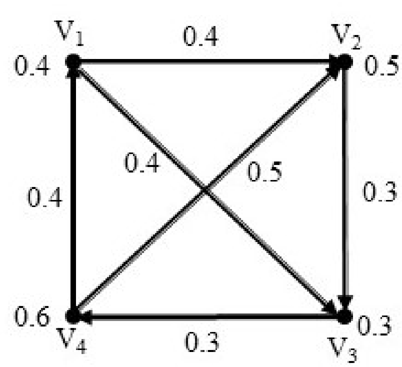

Fig 4. Tournament fuzzy digraph

Definition 4.3:

A fuzzy digraph $G_D : (\sigma,\, \mu)$ is a complete fuzzy digraph if $\mu_D\,(x,\, y) = \sigma\,(x) \wedge \sigma(y) \forall x,\, y \in \sigma^*$ and μ_D is symmetric.

Example: 4.3

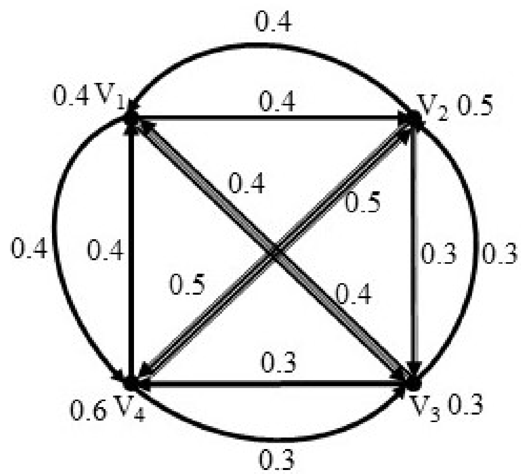

Fig 5. Complete fuzzy digraph

Definition 4.4 :

The set of vertices $\{x_i : i = 1, 2, 3, \ldots, n\}$ is denoted by $V(W)$, the set of edges $\{a_j : k = 1, 2, 3 \ldots, n-1\}$ is denoted by $E(W)$. We say that W is a fuzzy diwalk from x_1 to x_n or an (x_1, x_n)-fuzzy diwalk. If fuzzy diwalk W is open, then the vertex x_1 is the starting vertex of W, the vertex x_n is the end vertex of W. The strength of the path is defined by $min\mu(x_{i-1}, x_i), i = 1, 2, 3 \ldots, n$ and it is denoted by $\mu_s(x_1, x_n)$.

Remark 4.1: There is no path between x_i and x_j then $\mu_s(x_i, x_j) = 0$.

Definition 4.5:

A fuzzy trail is a fuzzy diwalk in which all edges are distinct.

Definition 4.6 :

A k-cycle is a cycle of length k. The smallest integer k for which G_D has a k-cycle is the girth of G_D.

Definition 4.7 :

If the vertices of the fuzzy diwalk W are distinct, then W is called a fuzzy directed path or fuzzy dipath.

Definition 4.8 :

A diwalk in a fuzzy digraph is an alternating sequence of vertices and edges, $v_0, x_1, v_1, \ldots, x_n, v_n$ in which each edge x_i $(v_{i-1}v_i)$ is in E. A fuzzy closed diwalk has the same vertices first and last and fuzzy spanning diwalk contains all the vertices. A fuzzy dipath is fuzzy diwalk in which all the vertices are distinct.

Definition 4.9 :

A semi fuzzy diwalk is again an alternating sequence $v_0, x_1, v_1, \ldots, x_n, v_n$ of vertices and edges, but each edge x_i may be either $(v_{i-1}v_i) \in E$ or $(v_i v_{i-1}) \in E$.

Definition 4.10 :

A fuzzy spanning closed diwalk of a fuzzy digraph is a fuzzy diwalk that visits all the vertices of the fuzzy digraph and returns back to the starting vertex.

Definition 4.11 :

A fuzzy digraph is strong or strongly connected, if every two vertices are mutually reachable. It is unilateral or unilaterally connected, if for any two vertices, at least one is reachable from the other. It is weak or weakly connected if every two vertices are joined by a semi fuzzy dipath.

Example: 4.4

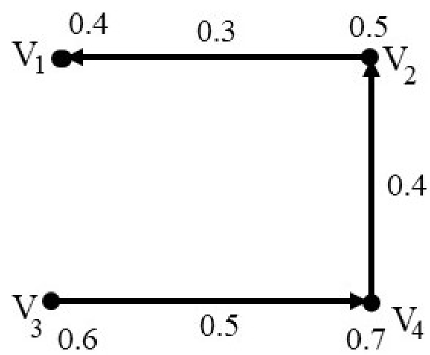

Fig 6. Strongly connected fuzzy digraph

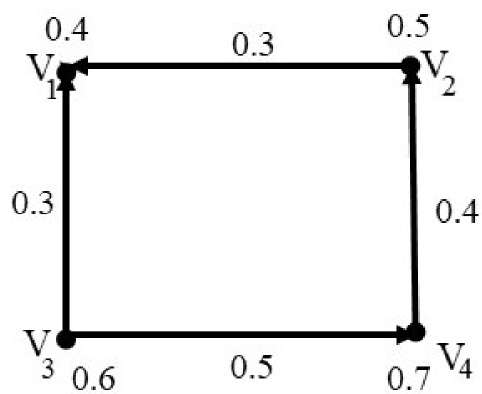

Fig 7. Unilateral connected fuzzy digraph

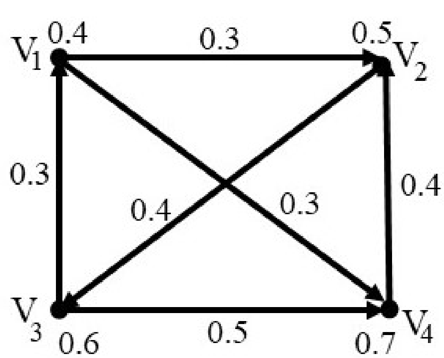

Fig 8. Weakly connected fuzzy digraph

Note: Every strong fuzzy digraph is unilateral and every unilateral is weak, but the converse is not true.

Definition 4.12 :

If the vertices $(x_{1,}x_{2,}x_{3,} \ldots, x_n)$ are distinct, $n \geq 3$ and $(x_1 = x_n)$, $C\left[x_i, x_j\right] = x_i x_{i+1} \ldots x_j$. C is a fuzzy directed cycle of length $j-i(=k)$ or k-fuzzy dicycle.

Definition 4.13 :

A G_D is called vertex fuzzy directed pancyclic as if for each $x \in V\left(G_D\right)$ and each integer $k \in \{3, 4, \ldots n\}$,∃ a $k-$fuzzy dicycle through x in G_D.

Definition 4.14 :

A vertex basis of fuzzy digraph G_D is a minimal collection of vertices from which all vertices are reachable. Thus, a set S of vertices of a fuzzy digraph G_D is a vertex basis if every vertices of G_D is reachable from the vertices of S and no vertices of S is reachable from any other.

Example: 4.5

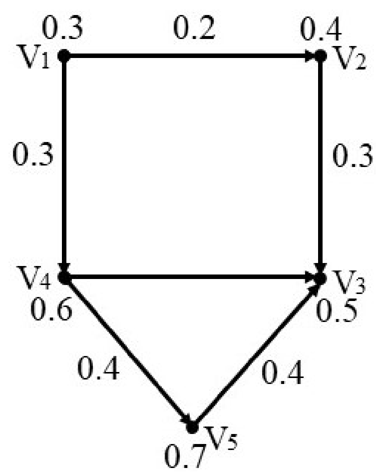

Fig 9. Fuzzy vertex basis

In Figure 9, v_2, v_3, v_4, v_5 are reachable from v_1. So v_1 is the vertex basis.

Definition 4.15 :

A fuzzy digraph is said to be di-acyclic if it has no fuzzy dicycle.

5 Results and Discussion

Theorem 5.1:

For every fuzzy digraph G_D we have

$\sum_{u\in V} N^+_{G_D}(u) + \sum_{u\in V} N^-_{G_D}(u) = \sum_{u\in V} N_{G_D}(u)$.

Proof: Let G_D be a fuzzy digraph and then the out-neighbourhood of vertex u is $N^+_{G_D}(u) = \sum_{u\in V} N^+_{G_D}(u)$, the in-neighbourhood of vertex of u is $N^-_{G_D}(u) = \sum_{u\in V} N^-_{G_D}(u)$. Therefore the union of $N^+_{G_D}(u)$ and $N^-_{G_D}(u) = N_{G_D}(u)$. Hence $\sum_{u\in V} N^+_{G_D}(u) + \sum_{u\in V} N^-_{G_D}(u) = \sum_{u\in V} N_{G_D}(u)$.

Proposition 5.1: For every fuzzy multidigraph G_D we have

$\sum_{x\in V(D)} d^+_{G_D}(x) = \sum_{x\in V(D)} d^-_{G_D}(x) = \left|\mu_{G_D}\right|$.

Proof. Let D be a fuzzy multidigraph and then the out-degree of a vertex x is $d^{+}_{G_D}(x) = \sum_{(x,y)\in\mu_D} \mu_D(x,y)$, the in-degree of a vertex x is $d^{-}_{G_D}(x) = \sum_{(y,x)\in\mu_D} \mu_D(y,x)$. Therefore the sum of $d^{+}_{G_D}(x)$ and the sum of $d^{-}_{G_D}(x) = \mu_{G_D}$.

Hence, $\sum_{x\in V(D)} d^{+}_{G_D}(x) = \sum_{x\in V(D)} d^{-}_{G_D}(x) = \left|\mu_{G_D}\right|$.

Proposition 5.2: Let G_D be a fuzzy digraph and let x, y be a pair of distinct vertices in G_D. If G_D has an (x,y)-fuzzy diwalk W, then G_D contains an (x,y)-fuzzy dipath P such that $E(P) \subseteq E(W)$. If G_D has a closed (x,x)-fuzzy diwalk W, then G_D contains a fuzzy dicycle C through x such that $E(C) \subseteq E(W)$.

Proof: Consider a fuzzy diwalk W from x to y of minimum length of all (x,y)-fuzzy diwalks whose edges belong to $E(W)$. We show that P is a fuzzy dipath. Let $P = x_1x_2x_3...x_n$, where $x = x_1$ and $y = x_n$. If $x_i = x_j$ for some $1 \le i < j \le n$, then the fuzzy diwalk $P[x_1, x_i]\ P[x_{j+1}, x_n]$ is shorter than P. It contradicts itself. Hence, every vertices of P are distinct, so P is a fuzzy dipath with $E(P) \subseteq E(W)$.

Let $W = z_1z_2z_3...z_n$ be a fuzzy diwalk from $x = z_1$ to itself $(x = z_n)$. Since G_D has no loop, $z_{n-1} \neq z_n$. Let $y_1y_2y_3...y_m$ be a shorter fuzzy diwalk from $y_1 = z_1$ to $y_t = z_{n-1}$. We have proved above that $y_1y_2y_3...y_m$ is a fuzzy dipath. Thus, $y_1y_2y_3...y_ty_1$ is a cycle through $y_1 = x$.

Theorem 5.2:

Every fuzzy tournament fuzzy digraph is traceable.

Proof: Let G_T be a fuzzy tournament digraph with a vertex set $\{v_i : i \in [n]\}$. Imagine that the vertices of G_T are labeled in a manner that the number of reverse edges, i.e., edges of the form v_jv_i, $j > i$, is minimum. Then, $v_1v_2v_3...v_n$ is a Hamilton fuzzy dipath in G_T. If this is no the case, there exist a subscript $i < n$ such that $v_i\ v_{i+1} \notin E(G_T)$. Thus $v_{i+1}v_i \in E(G_T)$. So, in this case, it contradicts that we can change the vertices v_i and v_{i+1} in the labeling and the reduction of the number of reverse edges.

Theorem 5.3 :

A fuzzy digraph is strong iff it has a fuzzy spanning closed diwalk.

Proof: Let G_D be a fuzzy digraph with vertex set $V_D = \{v_0, v_1, v_3, \ldots, v_n\}$ edge set $E_D = \{x_1, x_2, x_3, x_4, x_5, \ldots, x_m\}$.

Let G_D is strong, then for every $v_iv_{i-1}\ \forall\, i$, is mutually reachable, we prove that G_D has a fuzzy spanning closed diwalk. By the method of contradiction, there is no fuzzy spanning diwalk between any two vertices say v_i, v_{i-1}. Then v_i is not reachable to v_{i-1}. Which is a contradiction. Therefore G_D has a fuzzy spanning closed diwalk.

Conversely, Let G_D has a fuzzy spanning closed diwalk. Then we prove that G_D is strong, suppose G_D is not strong then at least two of the vertices have no fuzzy closed walk. Hence G_D is strong.

Theorem 5.4:

A fuzzy digraph is unilateral (or weak) iff it has a fuzzy spanning (or spanning semi) diwalk.

Proof: Let G_D be a fuzzy digraph with a vertex set V_D and edge set E_D. Let G_D is unilateral (or weak), then for any two $v_iv_{i-1}\forall i$, is reachable. We prove that G_D has a fuzzy spanning (or spanning semi) diwalk. By the method of contradiction, if there is no fuzzy spanning (or spanning semi) diwalk between any two vertices v_iv_{i-1}. Then v_i is not reachable to v_{i-1}, which is a contradiction.

$\therefore G_D$ has a fuzzy spanning (or spanning semi) diwalk. Converse is obvious.

Theorem 5.5:

A fuzzy digraph G_D is fuzzy dicycle iff $d^{+}_{D}(v) = d^{-}_{D}(v)$ and all the vertices are strongly connected.

Proof: Let G_D be a fuzzy dicycle. G_D is a strong, then for every $v_iv_{i-1}\forall i$ is reachable. We prove that $d^{+}_{D}(v) = d^{-}_{D}(v)$ and all the vertices are strongly connected. Suppose, G_D is a fuzzy acyclic digraph $d^{+}_{D}(v) \neq d^{-}_{D}(v)$ and it is also not strongly connected which is a contradiction in to other assumptions.

Conversely, if there exists a fuzzy dicycle, then we have $d^{+}_{D}(v) = d^{-}_{D}(v)$ for any vertex $v \in V$.

On the other hand if $d^{+}_{D}(v) = d^{-}_{D}(v)$ and all the vertices are strongly connected.

Theorem 5.6: (Moon's theorem)

Each strong fuzzy tournament fuzzy digraph G_T is vertex fuzzy directed pancyclic.

Proof: Let $x \in V(G_T)$ and $|V(G_T^*)| \geq 3$. By using mathematical induction. First, we prove that G_T has a $3-$fuzzy dicycle through x. Since G_T is strong, both $O = N_{G_T}^{+}(x)$ and $I = N_{G_T}^{-}(x)$ are non-empty. Therefore $(O \cup I)$ is non-empty; let $y, z \in (O \cup I)$. Then $xyzx$ is a 3-fuzzy dicycle through x. Let $C_T^* = x_0.x_1.. .x_m$ be a fuzzy dicycle in G_T with $x_i = x_0 = x_m$ and $m \in \{3, 4, ... n-1\}$.

We prove that G_T has an $(m+1)$-fuzzy dicycle through x. If one exists vertex $y \in V(G_T^*) - V(C_T^*)$ dominates a vertex in C_T and y is dominated by a vertex in C_T, then $\exists$ an index i such that $x_i \rightarrow y$ and $y \rightarrow x_{i+1}$. Therefore $C[x_0, x_i]yC[x_{i+1}, x_m]$ is an $m+1$-fuzzy dicycle through x. Thus we assume that every vertex is in $V(G_T^*) - V(C_T^*)$ dominates a vertex in C_T or dominated by each vertex in C_T. The vertices from outside C_T that dominates all vertices from $V(C_T^*)$ form a set R; the set of the vertices in $V(G_T^*) - V(C_T^*)$ form a set S. Since G_T is strong, both S and R are not equal to $\emptyset$ and the set (S, R) is not equal to $\emptyset$. Thus taking $s, r \in (S, R)$ we see that $x_0 srC[x_2, x_0]$ is a $(m+1)$-fuzzy dicycle through $x = x_0$.

Corollary 5.1: (Camion's theorem) Each strong fuzzy tournament is Hamiltonian.

Corollary 5.2: Every strong semi-complete fuzzy digraph is vertex-pancyclic.

Theorem 5.7:

An di-acyclic fuzzy digraph has at least one vertex v of G_D such that $d_D^{+}(v) = 0$, and one vertex v of G_D such that $d_D^{-}(v) = 0$.

Proof: Let G_D be an acyclic fuzzy digraph with a vertex set V_D and edge set E_D with maximal fuzzy dipath.

By method of contradiction, if there exists a fuzzy dicycle, then each vertex contains at least $d_D^{+}(v) = 1$ and at least $d_D^{-}(v) = 1$ which is a contradiction. Converse is obvious.

Theorem 5.8:

Every acyclic fuzzy digraph has a unique vertex basis containing all vertices of $d_D^{-}(v) = 0$.

Proof: Let G_D be an acyclic fuzzy digraph with a vertex set V_D and edge set E_D with a unique vertex basis. By method of contradiction, if there exists a fuzzy dicycle then each vertex contains at least $d_D^{+}(v) = 1$ and at least $d_D^{-}(v) = 1$ which is a contradiction. Therefore, a unique vertex basis no longer exists.

Therefore G_D is an acyclic fuzzy digraph. Converse is obvious.

6 Conclusion

From previous studies, this study is centered on fuzzy digraphs and introduced some types of fuzzy digraphs, neighborhood (in-neighborhood and out-neighborhood) and degree (in-degree and out-degree) in fuzzy digraphs. Also, it introduced strong, weak, unilateral connectedness in fuzzy digraphs and we discussed using some standard fuzzy digraphs like fuzzy diwalk. Fuzzy dicycle, acyclic fuzzy digraph, and gives some insightful thoughts in the form of theorems and remarks with examples. We can expand on this idea in the future by utilizing domination in digraphs.

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