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RESEARCH ARTICLE

A New Parametrically Extended Exponential Information Measure on Fuzzy Set Framework

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Abstract

Objective: To generalize an existing fuzzy information measure and verify its fundamental characteristics. **Methods:** Both objective and descriptive methodologies, as well as problem-posing and problem-solving techniques, have been employed. New fuzzy information measures are established using a set of attributes proposed by De Luca and Termini as a standard. **Findings:** The fuzzy information measure is widely used for characterizing and quantifying fuzzy unpredictability and vagueness. A novel parametrically extended exponential information measure on a fuzzy set framework has been presented to address this. Once the validity of the proposed information measure is established, several features are also provided. **Novelty:** A new parametrically extended exponential information measure on fuzzy set framework has been introduced. The unique quality of the suggested fuzzy information measure is found in its parameters. The proposed measure of information is substantial and justifiable; it reduces to the present information measure if the parameters are substituted with a suitable value. Furthermore, certain notable and interesting properties of this information measure have been studied. Consequently, from an application standpoint, the stated information measure is more adaptable. **Mathematics Subject Classification 2020:** 94 D05

Keywords: Fuzzy set; Information measure; Fuzzy entropy; Parametric exponential measure; Uncertainty

1 Introduction

Mathematicians encountered enormous difficulties because of fuzzy uncertainty and mathematical ambiguity starting in the early 19th century. Shannon[1] developed the idea of entropy to express the degree of uncertainty in a probability distribution as follows:

$\mathrm{H(P)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{p_i.logfop_i}$ provides the information found in every observation including the probability distribution $P = (P_1, P_2, P_3, \ldots\ldots, P_n)$ of discrete random variables X. The mathematical expression for the fuzziness of a fuzzy set is called the fuzzy entropy. Measures of information and fuzziness refer to the degree of vagueness and ambiguity in an experiment's probabilistic uncertainty and its removal, respectively. Many fields, including finance, statistical mechanics, pattern recognition, image processing, decision-making, clustering, and so on, frequently use the degree of uncertainty. Zadeh[2] makes use of entropy, the fundamental concept of information theory, initially. The level of uncertainty in a fuzzy set serves as a logical guideline for analyzing systems that feature vagueness and ambiguity. De Luca and Termini[3] presented a framework of fuzzy information measures based on Shannon's entropy. Numerous researchers have created novel entropy measures for fuzzy sets. Exponential entropy is one of the important fuzzy information measure that Bhandari and Pal[4] introduced. Some novel fuzzy information measures that correspond to probabilistic measures were defined by Kapur[5], Verma and Sharma[6], Bhagwan Dass, Vijay Prakash Tomar, Krishan Kumar[7], Hooda[8], and Fan and Ma[9]. Fuzzy set- based generalized parametric entropy measures have defined by Krishan Kumar, Ashok Kumar, Komal Kumar Deswal, Bhagwan Das[10]. A new generalized intuitionistic fuzzy information measure was given by Priya Arora and V. P. Tomar[11]. Ting-Ting Xu, Hui Zhang, Bo-Quan Li[12] presented an axiomatic framework of fuzzy information and hesitancy information in a fuzzy domain. Yang, Lan Zhen, Xiao Ying Gong, Xian Jie Wang, and Sheng Qiao An.[13] defined generalized exponential measure of intuitionistic fuzzy information. Anjali Patel, Subhankar Jana, Juthika Mahanta[14] used intuitionistic fuzzy information measures to assess the medical waste treatment techniques. A unique multi-criteria decision-making technique based on intuitionistic imprecise information has been presented by Rajesh Joshi[15] and is being used to identify machine defects. Verma, R.[16] proposed fuzzy MABAC method based on new exponential fuzzy information measures. Liu, M., Zhang, X. & Mo, Z.[17] defined probabilistic hesitant fuzzy MEREC-TODIM decision-making based on improved distance measures. Gleb Beliakov, Jian-Zhang Wu, Weiping Ding[18] explained the representation, optimization and generation of fuzzy measures. Dinesh, Satish Kumar[19] applied exponential measures in multi-criteria decision-making problems. Tripathi, D. K., Nigam, S. K., Rani, P., & Shah, A. R.[20] used intuitionistic fuzzy parametric divergence measures in decision-making problems. In this article, a novel parametrically extended exponential information measure on fuzzy set framework has been introduced by using the idea of above-defined entropies. Mishra, Arunodaya Raj, Pratibha Rani, Abbas Mardani, Reetu Kumari, Edmundas Kazimieras Zavadskas, and Dilip Kumar Sharma[21] applied exponential fuzzy measures in multi-criteria service quality in vehicle insurance firms. The suggested entropy measure is significant and legitimate; if the parameters are changed to an appropriate value, it reduces to the current entropy measure. Furthermore, certain notable and interesting properties of the proposed entropy measures have been studied. Consequently, from an application standpoint, the stated entropy measure is more adaptable. In Section 2, we define some terms related to fuzzy set theory and information theory. In Section 3, we introduce a new extended parametric exponential information measure with degrees α and β on fuzzy set framework and verify its axiomatic validity. Some important properties of the proposed measure have been discussed, and finally, we conclude our research in Section 4.

2 Methodology

Some fundamental ideas about fuzzy set theory and fuzzy information theory are covered in this section.

Definition 1.

Let $\mathrm{X} = (\mathrm{x_1}, \mathrm{x_2} \ldots\ldots\ldots, \mathrm{x_n})$ be a discrete domain. A fuzzy set A is given as

$$A = \{< x_i, u_A(x_i) > x_i \in X\}$$

where, $\mu_A(x_i): X \to [0,1]$ is the membership function of A. The level of belongingness of $x_i \in X$. is denoted by membership value $\mu_{\mathrm{A}}(\mathrm{x_i})$.

Definition 2.

Let M is a real-valued function defined on the collection of fuzzy sets (FS(X)) such that $M : FS(X) \to R^+$ and if M fulfills the following axioms then it is referred to as fuzzy information measure.

$(\mathrm{B_1})\ M(A) = 0$ if A is a crisp set

$(\mathrm{B_2})\ M(A)$ assumes a unique maximum if and only if $\mu_{\mathrm{A}}(\mathrm{x_i}) = 0.5$

$(\mathrm{B_3})\ M(A^*) \leq M(A)$, where A^* is the sharpened version of A

$(B_4)\ M(A) = M(A^C)$

De Luca and Termini[3] suggested fuzzy information measure utilizing the concept of the same degree of uncertainty of $\mu_{\mathrm{A}}(\mathrm{x_i})$ and $1 - \mu_A(x_i)$ as

$$H(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i).\ln \mu_A(x_i) + (1 - \mu_A(x_i), \ln(1 - \mu_A(x_i))\right] \tag{1}$$

Later, two measures of fuzzy information were proposed by Bhandari and Pal[4] as

$$H_{\alpha}(A)=\frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\ln\ \mu^{\alpha}{}_{A}(x_i)+(1-\mu_A(x_i))^{\alpha}\right] \tag{2}$$

$$H_{e}(A)=\frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[\mu_A(x_i).e^{1-\mu_A(x_i)}+(1-\mu_A(x_i).e^{\mu_A(x_i)-1}\right] \tag{3}$$

Following this, Kapur[5] established the fuzzy set's information measures as

$$H_{k\alpha}(A)=\frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(X_i)+(1-\mu_A(X_i))^{\alpha}-1\right]$$

$$H_{\alpha,\beta}(A)=\frac{1}{\alpha+\beta-2}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(X_i)+(1-\mu_A(X_i))^{\alpha}+\mu_A^{\beta}(X_i)+(1-\mu_A(X_i))^{\beta}-2\right]$$

$$H_{\alpha}^{\beta}(A)=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(X_i)+(1-\mu_A(X_i))^{\alpha}-\mu_A^{\beta}(X_i)-(1-\mu_A(X_i))^{\beta}\right]$$

Verma and Sharma[6] proposed exponential information measures for the fuzzy set as

$$H_{e}A=\frac{1}{n(e^{1-0.5^{\alpha}}-1)}\sum_{i=1}^{n}\left[\mu_A(X_i).e^{1-\mu_A^{\alpha}(X_i)}+(1-\ \mu_A(X_i)).e^{1-(1-\mu_A(X_i))^{\alpha}}-1\right] \tag{4}$$

Later Bhagwan Dass, Vijay Prakash Tomar, Krishan Kumar[7] also suggested exponential information measures for the fuzzy set as

$$\mathrm{B_e}(\mathrm{A})=\frac{1}{\mathrm{n}\left(\mathrm{e}^{1\text{-}0.5^{\text{-}\alpha}}\text{-}\mathrm{e}^{1\text{-}0.5^{\text{-}\beta}}\right)}\sum_{\mathrm{i=1}}^{\mathrm{n}}\begin{bmatrix}\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{1\text{-}\mu_{\mathrm{A}}^{\alpha}(\mathrm{x_i})}+\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{1\text{-}\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\alpha}}\\ \text{-}\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{1\text{-}\mu_{\mathrm{A}}^{\beta}(\mathrm{x_i})}\text{-}\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{1\text{-}\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\beta}}\end{bmatrix}\ \alpha>0,\beta>0 \tag{5}$$

3 Result and Discussion

3.1 New Parametrically Extended Exponential Information Measure

After analyzing the above-described concepts, a new parametrically extended exponential information measure on fuzzy set framework is defined as

$$\mathrm{M_e}(\mathrm{A})=\frac{1}{\mathrm{n}\left(\mathrm{e}^{\text{-}\alpha}\text{-}\mathrm{e}^{\text{-}\beta}\right)}\sum_{\mathrm{i=1}}^{\mathrm{n}}\begin{bmatrix}\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{\mu_{\mathrm{A}}^{\alpha}(\mathrm{x_i})}+\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\alpha}}\\ \text{-}\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{\mu_{\mathrm{A}}^{\beta}(\mathrm{x_i})}\text{-}\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\beta}}\end{bmatrix}\ \alpha>0,\beta>0 \tag{6}$$

Theorem1: This measure establishes that $\mathrm{M_e}(\mathrm{A})$ on fuzzy set A is a valid information measure.

Proof: For measure $\mathrm{M_e}\,(\mathrm{A}) to\ be\ valid,\ we\ need\ to\ satisfy\ four\ properties.$

$\mathrm{M_1}$: To prove $\mathrm{M_e}(\mathrm{A})=0$ if and only if $\mu_{\mathrm{A}}(\mathrm{x_i})=0$ or $\mu_{\mathrm{A}}(\mathrm{x_i})=1$

First, we suppose that $\mathrm{M_e}(\mathrm{A})=0$

$$\Rightarrow\frac{1}{\mathrm{n}(\mathrm{e}^{\text{-}\alpha}\text{-}\mathrm{e}^{\text{-}\beta})}\sum_{\mathrm{i=1}}^{\mathrm{n}}\begin{bmatrix}\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{\mu_{\mathrm{A}}^{\alpha}(\mathrm{x_i})}+\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\alpha}}\\ \text{-}\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{\mu_{\mathrm{A}}^{\beta}(\mathrm{x_i})}\text{-}\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\beta}}\end{bmatrix}=0$$

$$\Rightarrow\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{\mu_{\mathrm{A}}^{\alpha}(\mathrm{x_i})}+\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\alpha}}=\mu_{\mathrm{A}}(\mathrm{x_i}).\mathrm{e}^{\mu_{\mathrm{A}}^{\beta}(\mathrm{x_i})}+\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right).\mathrm{e}^{\left(1\text{-}\mu_{\mathrm{A}}(\mathrm{x_i})\right)^{\beta}}$$

This is true if $\mu_A(x_i)=0$ or $\mu_A(x_i)=1$.

Conversely: Let $\mu_A(x_i)=0$ or $\mu_A(x_i)=1$

$$\Rightarrow \mu_A(x_i).e^{\mu_A^{\alpha}(x_i)}+\left(1-\mu_A(x_i)\right).e^{\left(1-\mu_A(x_i)\right)^{\alpha}}=0$$
$$\Rightarrow \mu_A(x_i).e^{\mu_A^{\beta}(x_i)}+\left(1-\mu_A(x_i)\right).e^{\left(1-\mu_A(x_i)\right)^{\beta}}=0$$

$\Rightarrow M_e(A)=0$.

Hence $M_e(A)=0$ iff $\mu_A(x_i)=0$ or $\mu_A(x_i)=1$.

M_2: To prove $M_e(A)$ is maximum at $\mu_A(x_i)=0.5$

Now

$$\frac{\partial M_e(A)}{\partial \mu_A(x_i)}=\frac{1}{n\left(e^{-\alpha}-e^{-\beta}\right)}\sum_{i=1}^{n}\begin{bmatrix} e^{\mu_A^{\alpha}(x_i)}+\alpha\mu_A^{\alpha}(x_i).e^{\mu_A^{\alpha}(x_i)}+\alpha\left(1-\mu_A(x_i)\right)^{\alpha}.e^{\left(1-\mu_A(x_i)\right)^{\alpha}} \\ -e^{\left(1-\mu_A(x_i)\right)^{\alpha}}-e^{\mu_A^{\beta}(x_i)}+\beta\mu_A^{\beta}(x_i).e^{\mu_A^{\beta}(x_i)} \\ -\beta\left(1-\mu_A(x_i)\right)^{\beta}.e^{\left(1-\mu_A(x_i)\right)^{\beta}}+e^{1-\left(1-\mu_A(x_i)\right)^{\beta}} \end{bmatrix}$$

Case - 1: Let $0\leq\mu_A(x_i)<0.5$ then, we have $\frac{\partial M_e(A)}{\partial \mu_A(x_i)}>0$ for all $0<\alpha$ and $0<\beta$

Therefore $M_e(A)$ is an increasing function.

Case - 2: Let $0.5<\mu_A(x_i)\leq 1$ then, we have $\frac{\partial M_e(A)}{\partial \mu_A(x_i)}<0$ for all $0<\alpha$ and $0<\beta$

Therefore $M_e(A)$ is a decreasing function.

Case - 3: Let $\mu_A(x_i)=0.5$ then, we have $\frac{\partial M_e(A)}{\partial \mu_A(x_i)}=0$ for all $0<\alpha$ and $0<\beta$

It is clear that $M_e(A)$ is a concave function.

M_3 : We know that $M_e(A)$ is an increasing function for $0\leq\mu_A(x_i)<0.5$ and a decreasing function for $0.5<\mu_A(x_i)\leq 1$. So using the above concept, we have

$$\mu_{A^*}(x_i)\leq\mu_A(x_i)\Rightarrow M_e\left(A^*\right)\leq M_e(A) \text{ and}$$
$$\mu_{A^*}(x_i)\geq\mu_A(x_i)\Rightarrow M_e\left(A^*\right)\leq M_e(A)$$
$$\Rightarrow M_e\left(A^*\right)\leq M_e(A) \text{ in both cases.}$$

M_4: Assuming that A^C is the complement of A, we can use the membership function to get $\mu_A(x_i)=1-\mu_A(x_i)$. Hence $M_e\left(A^C\right)=M_e(A)$.

This proves that $M_e(A)$ is a valid fuzzy information measure.

3.2 Various Characteristics of New Measure of Information

Definition3. Let FS(X) be the collection of all fuzzy sets in the domain X and $R,S,T\in FS(X)$ is given by

$$R=\{(x,\mu_R(x)):x\in X\}$$
$$S=\{(x,\mu_S(x)):x\in X\}$$
$$T=\{(x,\mu_T(x)):x\in X\}$$

then the definitions of the subsequent set operations are given as:

a. $R\cup S=\{(x,max(\mu_R(x),\mu_S(x))):x\in X\}$
b. $R\cap S=\{(x,min(\mu_R(x),\mu_S(x))):x\in X\}$
c. $(R\cup S)\cup T=\left\{x,\max\left(\max\left(\mu_R(x),\mu_S(x)\right),\mu_T(x)\right):x\in X\right\}$
d. $(R\cap S)\cap T=\left\{x,\min\left(\min\left(\mu_R(x),\mu_S(x)\right),\mu_T(x)\right):x\in X\right\}$
e. $R^C=\left\{x,\mu_{R^C}(x)=1-\mu_R(x):x\in X\right\}$

Some important characteristic of the proposed exponential fuzzy information measure are given as follows:

Theorem2. For fuzzy set $R,S,T\in FS(X)$ prove that

$$\text{(I)}\quad M_e(R\cup S)+M_e(R\cap S)=M_e(R)+M_e(S)$$

$$\text{(II)}\ M_e((R\cup S)\cup T)=M_e(R\cup(S\cup T))$$

$$\text{(III)}\ M_e((R\cap S)\cap T)=M_e(R\cap(S\cap T))$$

$$\text{(IV)}\ M_e(R)=M_e\left(R\cap R^C\right)=M_e\left(R\cup R^C\right)=M_e\left(R^C\right)$$

$$\text{(V)}\ M_e(R\cap S)=M_e\left(R^C\cup S^C\right)^C$$

$$\text{(VI)}\ M_e(R\cup S)=M_e\left(R^C\cap S^C\right)^C$$

Proof: We suppose that

$$X_\phi=\{x_i : x_i\in X, \mu_R(x_i)\geq\mu_S(x_i)\geq\mu_T(x_i)\}$$
$$X_\psi=\{x_i : x_i\in X, \mu_R(x_i)\geq\mu_S(x_i)\geq\mu_T(x_i)\}$$

where $\mu_R(x_i)$, $\mu_S(x_i)$ and $\mu_T(x_i)$ are the membership functions.

(I) We know that

$$M_e(R\cup S)=\frac{1}{n(e^{-\alpha}e^{-\beta})}\sum_{i=1}^{n}\begin{bmatrix}\mu_{R\cup S}(x_i).e^{\mu^{\alpha}_{R\cup S}(x_i)}\\ +(1-\mu_{R\cup S}(x_i)).e^{(1-\mu^{\alpha}_{R\cup S}(x_i)}\\ -\mu_{R\cup S}(x_i).e^{\mu^{\beta}_{R\cup S}(x_i)}\\ -(1-\mu_{R\cup S}(x_i)).e^{(1-\mu^{\beta}_{R\cup S}(x_i)}\end{bmatrix}$$

$$\Rightarrow M_e(R\cup S)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\begin{bmatrix}\sum_{x_i=X_\phi}\begin{bmatrix}\mu_R(x_i).e^{\mu_R^{\alpha}(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R\left(1-(x_i)\right)^{\alpha}}\\ -\mu_R(x_i).e^{\mu_R^{\beta}(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^{\beta}}\end{bmatrix}\\ +\sum_{x_i=X_\psi}\begin{bmatrix}\mu_S(x_i).e^{\mu_S^{\alpha}(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^{\alpha}}\\ -\mu_S(x_i).e^{\mu_S^{\beta}(x_i)}-\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^{\beta}}\end{bmatrix}\end{bmatrix} \tag{7}$$

$$\text{and } M_e(R\cap S)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\begin{bmatrix}\sum_{x_i=X_\phi}\begin{bmatrix}\mu_S(x_i).e^{\mu_S^{\alpha}(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^{\alpha}}\\ -\mu_S(x_i).e^{\mu_S^{\beta}(x_i)}-\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^{\beta}}\end{bmatrix}\\ +\sum_{x_i=X_\psi}\begin{bmatrix}\mu_R(x_i).e^{\mu_R^{\alpha}(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R\left(1-(x_i)\right)^{\alpha}}\\ -\mu_R(x_i).e^{\mu_R^{\beta}(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^{\beta}}\end{bmatrix}\end{bmatrix} \tag{8}$$

Adding **Equations (7)** and **(8)** we obtain

$$M_e(R\cup S)+M_e(R\cap S)$$

$$=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_S(x_i).e^{\mu_S^\alpha(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\alpha}\\-\mu_S(x_i).e^{\mu_S^\beta(x_i)}-\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_S(x_i).e^{\mu_S^\alpha(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\alpha}\\-\mu_S(x_i).e^{\mu_S^\beta(x_i)}-\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_R(x_i).e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\end{array}\right]$$

$$\Rightarrow M_e(R\cup S)+M_e(R\cap S)=M_e(R)+M_e(S)$$

$$M_e(R\cup S)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_S(x_i).e^{\mu_S^\alpha(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\alpha}\\-\mu_S(x_i).e^{\mu_S^\beta(x_i)}-\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\beta}\end{array}\right]\end{array}\right]$$

$$M_e((R\cup S)\cup T)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_T(x_i).e^{\mu_T^\alpha(x_i)}+\left(1-\mu_T(x_i)\right).e^{\left(1-\mu_T(x_i)\right)^\alpha}\\-\mu_T(x_i).e^{\mu_T^\beta(x_i)}-\left(1-\mu_T(x_i)\right).e^{\left(1-\mu_T(x_i)\right)^\beta}\end{array}\right]\end{array}\right] \tag{9}$$

Now, $M_e(S\cup T)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_S(x_i).e^{\mu_S^\alpha(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\alpha}\\-\mu_S(x_i).e^{\mu_S^\beta(x_i)}-\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_T(x_i).e^{\mu_T^\alpha(x_i)}+\left(1-\mu_T(x_i)\right).e^{\left(1-\mu_T(x_i)\right)^\alpha}\\-\mu_T(x_i).e^{\mu_T^\beta(x_i)}-\left(1-\mu_T(x_i)\right).e^{\left(1-\mu_T(x_i)\right)^\beta}\end{array}\right]\end{array}\right]$

$$M_e(R\cup(S\cup T))=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_T(x_i).e^{\mu_T^\alpha(x_i)}+\left(1-\mu_T(x_i)\right).e^{\left(1-\mu_T(x_i)\right)^\alpha}\\-\mu_T(x_i).e^{\mu_T^\beta(x_i)}-\left(1-\mu_T(x_i)\right).e^{\left(1-\mu_T(x_i)\right)^\beta}\end{array}\right]\end{array}\right] \tag{10}$$

By **Equation (9)** and **(10)** we can easily prove that

$$M_e((R\cup S)\cup T)=M_e(R\cup(S\cup T))$$

From above outcomes it is clear that

$$M_e((R\cap S)\cap T)=M_e(R\cap(S\cap T)).$$

We know that

$$M_e\left(R\cup R^C\right)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).\,e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).\,e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).\,e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_{R^C}(x_i).\,e^{\mu_{R^C}^\alpha(x_i)}+\left(1-\mu_{R^C}(x_i)\right).\,e^{\left(1-\mu_{R^C}(x_i)\right)^\alpha}\\-\mu_{R^C}(x_i).\,e^{\mu_{R^C}^\beta(x_i)}-\left(1-\mu_{R^C}(x_i)\right).e^{\left(1-\mu_{R^C}(x_i)\right)^\beta}\end{array}\right]\end{array}\right] \tag{11}$$

Or

$$M_e\left(R\cup R^C\right)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_{R^C}(x_i).\,e^{\mu_{R^C}^\alpha(x_i)}+\left(1-\mu_{R^C}(x_i)\right).\,e^{\left(1-\mu_{R^C}(x_i)\right)^\alpha}\\-\mu_{R^C}(x_i).e^{\mu_{R^C}^\beta(x_i)}-\left(1-\mu_{R^C}(x_i)\right).\,e^{\left(1-\mu_{R^C}(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).\,e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).\,e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).\,e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\end{array}\right] \tag{12}$$

After solving both **Equations (11)** & **(12)** we get

$$M_e(R)=M_e\left(R\cup R^C\right)$$

Using above result we can prove that

$$M_e(R)=M_e\left(R\cup R^C\right)=M_e\left(R\cap R^C\right)=M_e\left(R^C\right)$$

We know that

$$M_e\left(R^C\cup S^C\right)=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_{S^C}(x_i).\,e^{\mu_{S^C}^\alpha(x_i)}+\left(1-\mu_{S^C}(x_i)\right).\,e^{\left(1-\mu_{S^C}(x_i)\right)^\alpha}\\-\mu_{S^C}(x_i).e^{\mu_{S^C}^\beta(x_i)}-\left(1-\mu_{S^C}(x_i)\right).e^{\left(1-\mu_{S^C}(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_{R^C}(x_i).\,e^{\mu_{R^C}^\alpha(x_i)}+\left(1-\mu_{R^C}(x_i)\right).\,e^{\left(1-\mu_{R^C}(x_i)\right)^\alpha}\\-\mu_{R^C}(x_i).e^{\mu_{R^C}^\beta(x_i)}-\left(1-\mu_{R^C}(x_i)\right).e^{\left(1-\mu_{R^C}(x_i)\right)^\beta}\end{array}\right]\end{array}\right]$$

$$M_e\left(R^C\cup S^C\right)^C=\frac{1}{n(e^{-\alpha}-e^{-\beta})}\left[\begin{array}{l}\sum_{x_i=X_\phi}\left[\begin{array}{l}\mu_R(x_i).e^{\mu_R^\alpha(x_i)}+\left(1-\mu_R(x_i)\right).e^{\mu_R(1-(x_i))^\alpha}\\-\mu_R(x_i).\,e^{\mu_R^\beta(x_i)}-\left(1-\mu_R(x_i)\right).e^{\left(1-\mu_R(x_i)\right)^\beta}\end{array}\right]\\+\sum_{x_i=X_\psi}\left[\begin{array}{l}\mu_S(x_i).e^{\mu_S^\alpha(x_i)}+\left(1-\mu_S(x_i)\right).e^{\left(1-\mu_S(x_i)\right)^\alpha}\\-\mu_S(x_i).e^{\mu_S^\beta(x_i)}-\left(1-\mu_S(x_i)\right).\,e^{\left(1-\mu_S(x_i)\right)^\beta}\end{array}\right]\end{array}\right]$$

Hence the result.
(VI) By the above result clearly $M_e(RUS)=M_e(R^C\cup S^C)$

4 Conclusion

A new parametrically extended exponential information measure on a fuzzy set framework has been presented. The suggested information measure, which reduces to the current information measure after entering an appropriate value for the parameters, is significant and legitimate. Additionally, a few noteworthy and intriguing characteristics of this information measure have been investigated. As a result, from the perspective of application, the defined entropy measure is more adaptable. In many domains, parametric exponential fuzzy measures are an invaluable tool, particularly when handling imprecision and uncertainty. Parametric exponential fuzzy measures are useful in multi-criteria decision-making (MCDM) because they capture the relative relevance of multiple criteria, which facilitates the aggregation of preferences and leads to more nuanced decision outcomes. These measures can be applied in the insurance and financial industries to enhance risk assessment and management strategies by accounting for the fuzzy nature of data and forecasting future outcomes uncertainty. They can be applied to sectors like image processing and sensor networks to aggregate information while controlling the inherent uncertainties. These measures can enhance the controller robustness in fuzzy control systems by integrating uncertain information. Also the proposed information measure can be generalized as intuitionistic fuzzy information measures by introducing the degree of hesitation.

References

1) Shannon CE. A Mathematical Theory of Communication. *Bell System Technical Journal*. 1948;27(3):379–423. Available from: https://doi.org/10.1002/j.1538-7305.1948.tb01338.x.
2) Zadeh LA. Fuzzy sets. *Information and Control*. 1965;8(3):338–353. Available from: https://doi.org/10.1016/S0019-9958(65)90241-X.
3) De Luca A, Termini S. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. *Information and Control*. 1972;20(4):90199–90203. Available from: https://doi.org/10.1016/S0019-9958(72)90199-4.
4) Bhandari D, R N. Some new information measures for fuzzy sets. *Information Sciences*. 1993;67(3):209–228. Available from: https://doi.org/10.1016/0020-0255(93)90073-U.
5) Kapur JN. Measures of Fuzzy Information. *Mathematical Sciences Trust Society*. 1997. Available from: https://books.google.co.in/books?id=BjUHNAAACAAJ.
6) Verma R, Sharma BD. On Generalized Exponential Fuzzy Entropy. *World Academy of Science, Engineering and Technology*. 2011;60:1402–1405. Available from: https://www.researchgate.net/publication/234062704_On_Generalized_Exponential_Fuzzy_Entropy.
7) Dass B, Tomar VP, Kumar K. Generalized Parametric Exponential Entropy Measure of degree on Fuzzy Set. *TURCOMAT*. 2021;12(11):459–468. Available from: https://doi.org/10.17762/turcomat.v12i11.5908.
8) Hooda DS. On generalized measures of fuzzy entropy. *Mathematica Slovaca*. 2004;54(3):315–325. Available from: https://www.researchgate.net/publication/268165998_On_generalized_measures_of_fuzzy_entropy.
9) Fan JL, Ma YL. Some new fuzzy entropy formulas. *Fuzzy Sets and Systems*. 2002;128(2):277–284. Available from: https://doi.org/10.1016/S0165-0114(01)00127-0.
10) Kumar K, Kumar A, Deswal KK, Dass B. Fuzzy set based generalized parametric exponential entropy measure. In: INTERNATIONAL CONFERENCE ON ADVANCES IN MULTI-DISCIPLINARY SCIENCES AND ENGINEERING RESEARCH: ICAMSER-2021. 2022;p. 20027–20027. Available from: https://doi.org/10.1063/5.0095492.
11) Arora P, Tomar VP. Novel Generalized Divergence Measure on Intuitionistic Fuzzy Sets and Its Application. *Advances in Fuzzy Systems*. 2021;2021:1–10. Available from: https://doi.org/10.1155/2021/5544993.
12) Xu TT, Zhang H, Li BQ. Axiomatic framework of fuzzy entropy and hesitancy entropy in fuzzy environment. *Soft Computing*. 2021;25(2):1219–1238. Available from: https://doi.org/10.1007/s00500-020-05216-9.
13) Yang LZ, Gong XY, Wang XJ, An SQ. Generalized Exponential Entropy on Intuitionistic Fuzzy Sets. *AMM*. 2014;556:4097–4102. Available from: https://doi.org/10.4028/www.scientific.net/AMM.556-562.4097.
14) Patel A, Jana S, Mahanta J. Intuitionistic fuzzy EM-SWARA-TOPSIS approach based on new distance measure to assess the medical waste treatment techniques. *Applied Soft Computing*. 2023;144:110521–110521. Available from: https://doi.org/10.1016/j.asoc.2023.110521.
15) Joshi R. A new multi-criteria decision-making method based on intuitionistic fuzzy information and its application to fault detection in a machine. *Journal of Ambient Intelligence and Humanized Computing*. 2020;11(2):739–753. Available from: https://doi.org/10.1007/s12652-019-01322-1.
16) Verma R. Fuzzy MABAC method based on new exponential fuzzy information measures. *Soft Comput*. 2021;25(14):9575–9589. Available from: https://doi.org/10.1007/s00500-021-05739-9.
17) Liu M, Zhang X, Mo Z. Probabilistic Hesitant Fuzzy MEREC-TODIM Decision-Making Based on Improved Distance Measures. *Int J Fuzzy Syst*. 2024;26(7):2370–2393. Available from: https://doi.org/10.1007/s40815-024-01741-z.
18) Beliakov G, Wu JZ, Ding W. Representation, optimization and generation of fuzzy measures. *Information Fusion*. 2024;106:102295–102295. Available from: https://doi.org/10.1016/j.inffus.2024.102295.
19) Dinesh S, Kumar. A new exponential knowledge and similarity measure with application in multi-criteria decision-making. *Decision Analytics Journal*. 2024;10:100407–100407. Available from: https://doi.org/10.1016/j.dajour.2024.100407.
20) Tripathi DK, Nigam SK, Rani P, Shah. New intuitionistic fuzzy parametric divergence measures and score function-based CoCoSo method for decision-making problems. *Decis Mak Appl Manag Eng*. 2023;6(1):535–563. Available from: https://doi.org/10.31181/dmame0318102022t.
21) Mishra AR, Rani P, Mardani A, Kumari R, Zavadskas EK, Sharma DK. An Extended Shapley TODIM Approach Using Novel Exponential Fuzzy Divergence Measures for Multi-Criteria Service Quality in Vehicle Insurance Firms. *Symmetry*. 2020;12(9):1452–1452. Available from: https://dx.doi.org/10.3390/sym12091452.