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RESEARCH ARTICLE

Numerical Analysis for a System of Partially Singularly Perturbed Convection-Diffusion Delay Differential Equations

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Abstract

Objectives: To find efficient numerical method to solve a system of n- partially singularly perturbed delay differential equations of convection-diffusion type with large spatial delay. **Methods**: The integration of a classical finite difference scheme with a piecewise uniform Shishkin mesh enhances the resolution of sharp gradients and boundary layers, providing improved accuracy. **Findings**: The approach is shown to achieve first-order convergence in the maximum norm, with uniform convergence independent of the perturbation parameters, ensuring robustness across various perturbation values. **Novelty**: The method addresses both boundary and interior layers effectively, which are complex due to the presence of singular perturbation parameters and large spatial delay. The theoretical results are supported by numerical example, validating the practical effectiveness of the proposed method.

Keywords: Singularly perturbed delay differential equation; Boundary layers; Interior layers; Finite difference scheme; Maximum norm; Shishkin mesh; Parameter uniform convergence

1 Introduction

A class of differential equations, which contains a small positive parameter ε $(0< \varepsilon \ll 1)$ in leading term with shifting parameter is known as a system of singularly perturbed delay differential equations. Numerical analysis for this type of system involves developing and applying specialized techniques to address the challenges raised by singular perturbations and delays. Delays impact various fields like dengue virus transmission[(1)], neuronal variability model[(2)] and some more. Several research works are dealing with singularly perturbed delay type equations to adopt suitable computational methods. Specifically, Joy D[(3)] proposed Gauss Elimination method employed with non-polynomial spline and with uniform mesh in solving problems involving large delay. Hassen[(4)] used fitted operator method on uniform mesh to solve parabolic convection—diffusion problem. Woldaregay and Mesfin Mekuria[(5)], Tesfaye[(6)] solved singularly perturbed time delay problems using Crank-Nicolson method for temporal discretization with uniform mesh. Hailu E A, Duressa[7] proposed weighted average method with piecewise uniform mesh in treating large spatial delay. Woldaregay M M[8] discussed convection-diffusion type problem in which delay occurs at both convection and reaction terms, with finite difference scheme on uniform and piecewise uniform Shishkin meshes. Saravana Sankar R[9] presented a parameter uniform numerical scheme utilizing fitted mesh method to solve singularly perturbed convection-diffusion type problems.

In the previous works, many authors have used uniform meshes for spatial discretization in finding approximate solution of singularly perturbed delay problems. Hence using adaptive mesh methods, finite difference schemes, and by carefully handling delays and stability issues, accurate and efficient solutions can be achieved. With these literature and motivated by the work in[9], in this research work we consider a system of partially singularly perturbed delay differential equations of convection –diffusion type on the interval [0, 2]. Here some of the equations are combined with distinct singular perturbation parameters, some of them are combined with the same singular perturbation parameter and remaining equations are non-singularly perturbed. To develop a parameter uniform numerical scheme for this system, we use piecewise uniform Shishkin mesh, which finer the layer regions, so it enables a numerical method employed with this type of mesh will give numerical approximations which inherit the stability properties of the exact solution by preserving the monotonicity of the original problem. Thus we suggest a classical finite difference scheme with a piecewise uniform Shishkin mesh to solve this type of system and this approach ensures that the complex interactions between convection, diffusion and delays are effectively captured and analyzed. The method is proved to be first order convergent in the maximum norm in the perturbation parameters. A numerical illustrations is given to establish the validity of the proposed method and further we use double mesh algorithm from[10] to obtain the order of convergence and error constant for different values of the perturbation parameter and mesh points.

The problem under consideration is

$$\vec{L}\vec{y}(s) = \varepsilon\vec{y}^{''}(s) + A(s)\vec{y}^{'}(s) - B(s)\vec{y}(s) - D(s)\vec{y}(s-1) = \vec{f}(s)\, on\ (0,2) \tag{1}$$

$$with\ \vec{y} = \vec{\varrho}\ on\ i\ and\ \vec{y}(2) = \vec{\rho} \tag{2}$$

For all $s\epsilon[0,\ 2]$, $\vec{y} = (y_1, y_2, \ldots, y_n)^T$, $\vec{f} = (f_1, f_2, \ldots, f_n)^T.\varepsilon,\ A,\ B\, and\, D$ are $n \times n$ matrices,
$\varepsilon = diag(\vec{\varepsilon}),\ \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n), A = diag(\vec{a}),\ \vec{a} = (a_1, a_2, \ldots, a_n),$
$B(s) = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & & \ldots \\ b_{n1} & \ldots. & \ldots & b_{nn} \end{pmatrix}$ and $D(s) = diag\left(\vec{d}\right),\ \vec{d} = (d_1, d_2, \ldots, d_n)$. The function$\vec{\varrho}$ is sufficiently differentiable on $[-1,0]$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ are small positive parameters with $\varepsilon_i < \varepsilon_j$, for $1 \le i < j \le k$ and $\varepsilon_i = \varepsilon_k$, for $i = k+1,\ k+2, \ldots,\ l$ and $\varepsilon_i = 1$ for $i = l+1,\ l+2, \ldots,\ n$. Further it is assumed that, the functions a_i, b_{ij}, d_i and f_i for all i, j are sufficiently smooth on $[0,2]$ and also they satisfy

$$a_i(s) \ge \alpha > 0, b_{ii}(s) + b_{ij}(s) \ge \beta > 0 \tag{3}$$

$$b_{ij}(s),\ d_i(s) \le 0,\ for\ \ 1 \le i \ne j \le n\ and\ b_{ii}(s) > \textstyle\sum_{i \ne j} \left|b_{ij}(s) + d_i(s)\right| \tag{4}$$

$$and\ 0 < \gamma < \min_{s \in [0,2]} \left(\textstyle\sum_{j=1}^{n} b_{ij}(s) + d_i(s)\right)\ for\ i = 1,\ 2, \ldots, n. \tag{5}$$

From these assumptions that the problem Equations (1) and (2) has a unique solution $\vec{y}$ and $\vec{y} \in C^2$,
where $C = C^0[0,2] \cap C^1(0,2) \cap C^2((0,1) \cup (1,2))$.

The problem Equations (1) and (2) can be written in equivalent form as

$$\vec{L}_1\vec{y}(s) = \varepsilon\vec{y}^{''}(s) + A(s)\vec{y}^{'}(s) - B(s)\vec{y}(s) = \vec{f}(s) + D(s)\vec{\varrho}(s-1) = \vec{g}(s)\ on\ (0,1) \tag{6}$$

$$\vec{L}_2\vec{y}(s) = \varepsilon\vec{y}^{''}(s) + A(s)\vec{y}^{'}(s) - B(s)\vec{y}(s) - D(s)\vec{y}(s-1) = \vec{f}(s)\ on\ (1,2) \tag{7}$$

$$\vec{y}(0)=\vec{\varrho}(0),\vec{y}(2)=\vec{\rho},\ \vec{y}(1-)=\vec{y}(1+),\ \vec{y}^{'}(1-)=\vec{y}^{'}(1+). \tag{8}$$

And when perturbation parameters are equal to zero, the corresponding reduced problem is given by

$$\varepsilon_0\vec{y}_0^{''}(s)+A(s)\vec{y}_0^{'}(s)-B(s)\vec{y}_0(s)=\vec{g}(s)\,on\,(0,1)$$

$$\varepsilon_0\vec{y}_0^{''}(s)+A(s)\vec{y}_0^{'}(s)-B(s)\vec{y}_0(s)-D(s)\vec{y}_0(s-1)=\vec{f}(s)\,on\,(1,2)$$

$\vec{y}_0(2)=\vec{\rho}, y_{0i}(0)=y_i(0)$, for $i=l+1,\ l+2,\ldots,\ n, \vec{y}_0=(y_{01},y_{02},..,y_{0n})^T$ and $\varepsilon_0=diag(\vec{\varepsilon}_0)$ with $\vec{\varepsilon}_0=(0,0,..,0,\varepsilon_{l+1},\ldots,\varepsilon_n)$. If $y_i(0)\neq y_{0i}(0)$ for any i such that $1\leq i\leq l$, then a boundary layer of width O(ε_i) is expected near $s=0$ in each of the solution component y_k, $1\leq k\leq i$, also at $s=1$ each of the solution component y_k, $1\leq k\leq i$, exhibits an interior layer of width O(ε_i) for any i such that $1\leq i\leq l$.

2 Methodology

The article contains two parts namely Analytical results and Numerical results. In Analytical results Maximum Principle, Stability result, Shishkin decomposition of the solution of Equation (1) and estimates of the derivatives of the components of the solution are discussed. Under Numerical results, the method that comprises of the appropriate Shishkin mesh and the Classical finite difference scheme, discrete Maximum Principle, discrete Stability result, the parameter uniform convergence of the method are presented.

Analytical Results

Maximum Principle: Let the conditions Equations (3), (4) and (5) hold. Let $\vec{\delta}=(\delta_1,\delta_2,\ldots,\delta n)^T$ be any function in the domain of $\vec{L}$ such that $\vec{\delta}(0)\geq\vec{0}$, $\vec{\delta}(2)\geq\vec{0}$, $\vec{L}_1\vec{\delta}\leq\vec{0}$ on $(0,1)$, $\vec{L}_2\vec{\delta}\leq\vec{0}$ on $(1,2)$, $\left[\vec{\delta}\right](1)=\vec{0}$ and $\left[\vec{\delta}'\right](1)\leq\vec{0}$ then $\vec{\delta}\geq\vec{0}$ on [0,2].

Proof . Let i^*, s^* be such that $\delta_{i^*}(s^*) = \min\limits_{\substack{s\in[0,2]\\ i=1,2,\ldots,n}} \delta_i(s)$.

If $\delta_{i^*}(s^*)\geq 0$, then there is nothing to prove. Suppose $\delta_{i^*}(s^*)<0$ then $s^*\notin\{0,2\}$ and $\delta_{i^*}^{''}(s^*)\geq 0$, $\delta_{i^*}^{'}(s^*)=0$. If $s^*\epsilon(0,1)$, then

$(\vec{L}_1\vec{\delta})_{i^*}(s^*)=\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)+a_{i^*}(s^*)\delta_{i^*}^{'}(s^*)-\sum_{j=1}^{n}[b_{i^*j}(s^*)\delta_j(s^*)]$

$=\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)+a_{i^*}(s^*)\delta_{i^*}^{'}(s^*)-b_{i^*i^*}(s^*)\delta_{i^*}(s^*)-\sum_{j=1,j\neq i^*}^{n}[b_{i^*j}(s^*)\delta j(s^*)]$

$\geq\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)+a_{i^*}(s^*)\delta_{i^*}^{'}(s^*)-b_{i^*i^*}(s^*)\delta_{i^*}(s^*)-\sum_{j=1,j\neq i^*}^{n}[b_{i^*j}(s^*)\delta_{i^*}(s^*)]$

$\geq\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)$–$\sum_{j=1}^{n}\left[b_{i^*j}(s^*)\delta_{i^*}(s^*)\right]>0$, this is a contradiction.

If $s^*\epsilon(1,2)$, then

$(\vec{L}_2\vec{\delta})_{i^*}(s^*)=\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)+a_{i^*}(s^*)\delta_{i^*}^{'}(s^*)-\sum_{j=1}^{n}[b_{i^*j}(s^*)\delta_j(s^*)]-d_{i^*}(s^*)\delta_{i^*}(s^*-1)$

$=\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)+a_{i^*}(s^*)\delta_{i^*}^{'}(s^*)-b_{i^*i^*}(s^*)\delta_{i^*}(s^*)-\sum_{j=1,j\neq i^*}^{n}[b_{i^*j}(s^*)\delta_j(s^*)]$
$-d_{i^*}(s^*)\delta_{i^*}(s^*-1)$

$\geq\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)+a_{i^*}(s^*)\delta_{i^*}^{'}(s^*)-b_{i^*i^*}(s^*)\delta_{i^*}(s^*)-\sum_{j=1,j\neq i^*}^{n}[b_{i^*j}(s^*)\delta_{i^*}(s^*)]$
$-d_{i^*}(s^*)\delta_{i^*}(s^*)$ as $\delta_{i^*}(s^*)\leq\delta_j(s^*)$ and $\delta_{i^*}(s^*)\leq\delta_{i^*}(s^*-1)$

$\geq\varepsilon_{i^*}\delta_{i^*}^{''}(s^*)$–$(\sum_{j=1}^{n}b_{i^*j}(s^*)+d_{i^*}(s^*))\delta_{i^*}(s^*)>0$, this is a contradiction.

If $s^*=1$, in this case we discuss about the differentiability of δ_{i^*} at $s=1$. If $\delta_{i^*}^{'}(1)$ does not exist then $[\delta_{i^*}^{'}](1)=\delta_{i^*}^{'}(1+)-\delta_{i^*}^{'}(1-)>0$, which is a contradiction to $[\delta_{i^*}^{'}](1)\leq 0$. If δ_{i^*} is differentiable at $s=1$, then $a_{i^*}(1)\delta_{i^*}^{'}(1)-\sum_{j=1}^{n}b_{i^*j}(1)\delta_j(1)>0$ and all the entries of $A(s)$, $B(s)$ and $\delta_j(s)$ are in $C([0,2])$, there exist an interval $[1-h,1)$ on which $a_{i^*}(s)\delta_{i^*}^{'}(s)-\sum_{j=1}^{n}b_{i^*j}(s)\delta_j(s)>0$. If $\delta_{i^*}^{''}(\hat{x})\geq 0$ at any point $\hat{x}\in[1-h,1)$ then $(\vec{L}_1\vec{\delta})_{i^*}(\hat{x})\geq 0$, which is a contradiction. Thus we can assume that $\delta_{i^*}^{''}(s)<0$ on $(1-h,1)$ which implies that $\delta_{i^*}^{''}(s)$ is strictly decreasing on $[1-h,1)$

and since $\delta_{i^*}^{'}(1)=0,\ \delta_{i^*}^{'}\in C\left((0,2)\right),$ thus $\delta_{i^*}^{'}(s)>0$ on $[1-h,1]$, consequently the continuous function $\delta_{i^*}(s)$ cannot have minimum at $s=1$, which contradicts the assumption that $s^*=1$. Hence $\delta_{i^*}(s)$ on $[0,2]$.

Stability result: Let the conditions Equations (3), (4) and (5) hold. Let $\vec{\delta}$ be any function in C^2, such that $\left[\vec{\delta}\right](1)=\vec{0}$ and $\left[\vec{\delta}'\right](1)=\vec{0}$, then for each $i=1,2,\ldots,n$ and $s\in(0,2]$,

$$|\varrho_i(s)|$$
$$\le \max\left\{\|\vec{\wp}(0)\|,\|\vec{\varrho}(2)\|,\frac{1}{\beta}\|\vec{\mathcal{L}_1}\vec{\Theta}\Big)\Big\|,\frac{1}{\gamma}\Big\|\vec{\mathcal{L}_2}\vec{\Theta}\|\right\}.$$

Proof: Let $\mathrm{M}=\max\left\{\|\vec{\varrho}(0)\|,\|\vec{\varrho}(2)\|,\frac{1}{\beta}\|\vec{\mathcal{L}_1}\vec{\varrho}\Big)\Big\|,\frac{1}{\gamma}\Big\|\vec{\mathcal{L}_2}\vec{\varrho}\|\right\}$

Define two functions $\vec{\theta}^{\pm}(s)=M\vec{e}\pm\vec{\delta}(s),\vec{e}=(1,1,\ldots,1)^T$. Then $\vec{\theta}^{\pm}(0)\ge\vec{0},\vec{\theta}^{\pm}(2)\ge\vec{0}$.

For $s\epsilon(0,1)$, $\quad\vec{L}_1\vec{\theta}^{\pm}(s)=-B(s)M\vec{e}\pm\vec{L}_1\vec{\delta}(s)\le\vec{0}$.

For $s\epsilon(1,\ 2)$, $\quad\vec{L}_2\vec{\theta}^{\pm}(s)=-B(s)M\vec{e}-D(s)M\vec{e}\pm\vec{L}_2\vec{\delta}(s)\le\vec{0}$.

Also, $\left[\vec{\theta}^{\pm}\right](1)=\pm\left[\vec{\delta}\right](1)=\vec{0}$ and $\left[\vec{\theta}^{\pm\prime}\right](1)=\pm\left[\vec{\delta}'\right](1)=\vec{0}$.

From maximum principle it follows that $\vec{\theta}^{\pm}(s)\ge\vec{0}$ on $[0,\ 2]$ and proves the result.

Estimates of the solution of Equations (1) and (2) and its derivatives: Let the conditions Equations (3), (4) and (5) hold. Let $\vec{y}$ be the solution of Equations (1) and (2) . Then for all $s\in[0,\ 2]$ and $i=1,2,\ldots,n$,

$|y_i(s)|\le Cmax(\mathrm{k}\vec{Y}(0)\mathrm{k},\mathrm{k}\vec{Y}(2)\mathrm{k},\mathrm{k}\vec{f}\,\mathrm{k})$,

$|y_i^{(k)}(s)|\le C\varepsilon_i^{-k}(\mathrm{k}\vec{y}(0)\mathrm{k}+\mathrm{k}\vec{y}(2)\mathrm{k}+\varepsilon\_i\,\mathrm{k}\vec{f}\,\mathrm{k})$,for $k=1,\ 2$,

$|y_i^{(3)}(s)|\le C\varepsilon_i^{-2}\varepsilon_1^{-1}(\mathrm{k}y(0)\mathrm{k}+\mathrm{k}\vec{y}(2)\mathrm{k}+\varepsilon_i\,\mathrm{k}\vec{f}\,\mathrm{k})+\varepsilon_i^{-1}\,\mathrm{k}f_i'\,\mathrm{k}$

Proof. The results are proved by using the procedure adopted in[9].

Shishkin decomposition of the solution

The solution $\vec{y}$ of the problem Equations (1) and (2) can be decomposed into $\vec{y}=\vec{v}+\vec{\omega}$,

where $\vec{v}=(v_1,\ v_2,\ldots,\ v_n)^T$, the smooth component is the solution of

$$\vec{L}_1\vec{v}=\vec{g}on(0,\ 1),\ \vec{v}(0)=\vec{\chi}\ ,\vec{v}(1-)=\vec{y}_0\ (1-),\tag{9}$$

where $\vec{\chi}\ =(\chi_1,\chi_2,\ldots,\chi_l,\ y_{l+1}(0),\ldots,\ y_n(0))^T$, $\chi_1,\ \chi_2,\ldots,\chi_l$ are to be chosen.

$$\vec{L}_2\vec{v}=\vec{f}on(1,\ 2),\vec{v}(1+)=\vec{y}_0\ (1+),\ \vec{v}(2)=\vec{\rho}\tag{10}$$

and $\vec{\omega}=(\omega_1,\ \omega_2,\ldots,\ \omega_n)^T$, $the\ singular\ component$ is the solution of

$$\vec{L}_1\vec{\omega}=\vec{0}\ on\ (0,\ 1),\ \ \vec{L}_2\vec{\omega}=\vec{0}\ on\ (1,\ 2)\tag{11}$$

with $\vec{\omega}(0)=\vec{y}(0)-\vec{v}(0),\vec{\omega}(2)=\vec{0}$, $(\vec{\omega}](1)=-(\vec{v}](1)$, $(\vec{\omega}']\,(1)=-(\vec{v}'](1)$.

Estimates of the smooth component and its derivatives

For a proper choice of $\vec{\chi}$, the solution of the problem Equations (9) and (10) satisfies

$\left|v_i^{(k)}(s)\right|\le C\left(1+\varepsilon_i^{2-k}\right)$, if $1\le i\le l$, $\quad and$

$\left|v_i^{(k)}(s)\right|\le C$, if $l+1\le i\le n$, for $0\le k\le 3$ and for $s\in[0,\ 2]$.

Proof. First, the estimates of the bounds on the smooth component and their derivatives are proved for interval [0,1]. Using these bounds, the estimates of the bounds on[1,2] are proved. According to the layer pattern of the solution, first the decomposition is done for all the components of $\vec{v}$ with ε_k. Next the decomposition is done with $\varepsilon_{k-1}$for the first $(k-1)$ components of $\vec{v}$, then the decomposition is done with ε_{k-2} for the first $(k-2)$ components of $\vec{v}$ and so on. We proceed the decomposition as follows.

First, the smooth component $\vec{v}$ is decomposed into $\vec{v}=\vec{p}_n+\varepsilon_k\vec{q}_n+\varepsilon_k^2\vec{z}_n$,

where $\vec{p}_n=(p_{n1},\ p_{n2},\ldots,\ p_{nn})^T$ is the solution of

$$\varepsilon_0\vec{p}_n^{''}(s)+A(s)\vec{p}_n^{'}(s)-B(s)\vec{p}_n(s)=\vec{g}(s)\,on\,(0,1)\tag{12}$$

$$\varepsilon_0\vec{p}_n^{''}(s)+A(s)\vec{p}_n^{'}(s)-B(s)\vec{p}_n(s)-D(s)\vec{p}_n(s-1)=\vec{f}(s)\,on\,(1,2),\tag{13}$$

$p_{ni}(0) = y_i(0)$, for $i = l+1,\ l+2, \ldots,\ n, \vec{p}_n(2) = \vec{\rho},$ where $\varepsilon_0 = diag(0,\ 0,\ \ldots,\ \varepsilon_{l+1}, \ldots,\ \varepsilon_n)$
and $\vec{q}_n = \left(q_{n1},\ q_{n2},\ \ldots,\ q_{nn}\right)^T$ is the solution of

$$\varepsilon_0 \vec{q}_n^{''}(s) + A(s)\vec{q}_n^{'}(s) - B(s)\vec{q}_n(s) = -\varepsilon_k^{-1}\varepsilon_0^{'}\vec{p}_n^{''}(s)\ on\ \ (0,\ 1) \tag{14}$$

$$\varepsilon_0 \vec{q}_n^{''}(s) + A(s)\vec{q}_n^{'}(s) - B(s)\vec{q}_n(s) - D(s)\vec{q}_n(s-1) = \ \varepsilon_k^{-1}\varepsilon_0^{'}\vec{p}_n^{''}(s)\ \ on\ \ (1,\ 2), \tag{15}$$

$q_{ni} = 0$, for $i = l+1,\ l+2, \ldots,\ n, \vec{q}_n(2) = \vec{0}$ and $\varepsilon_0^{'} = diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l,\ 0, \ldots, 0)$
$\vec{z}_n = (z_{n1},\ z_{n2},\ \ldots,\ z_{nn})^T$ is the solution of

$$\vec{L}_1 \vec{z}_n(s) = -\varepsilon_k^{-1}\varepsilon_0^{'}\vec{q}_n^{''}(s)\, on\, (0,1) \tag{16}$$

$$\vec{L}_2 \vec{z}_n(s) = -\varepsilon_k^{-1}\varepsilon_0^{'}\vec{q}_n^{''}(s)\, on\, (1,2),\ \vec{z}_n(2) = \vec{0}\ and\ \vec{z}_n(0)\ remains\ to\ be\ chosen. \tag{17}$$

Since $\varepsilon_k^{-1}\varepsilon_0^{'}$ is a matrix with bounded entries and hence from the equations Equations (12) and (14) we have for —Then proceeding in a similar way as in[9], by the proper choice of $\chi_l, \chi_{l-1}, \ldots, \chi_1$ the required bounds for $1 \le i \le n,\ \ 0 \le k \le 3$ are proved.

Estimates of the bounds on the singular component and its derivatives.

Let$\beta_1^i(s), \beta_2^i(s), 1 \le i \le n$be the layer functions defined on $[0,\ 1]$ and $[1,\ 2]$respectively as $\beta_1^i(s) = exp(-\frac{\alpha s}{\varepsilon_i})$ on $[0,1]$and $\beta_2^i(s) = exp\left(-\frac{\alpha(s-1)}{\varepsilon_i}\right)$ on $[1,2]$.

Let $\vec{\omega}$ be the solution of Equation (11) , then the following estimates hold.

On the interval $[0,\ 1]$, for $1 \le i \le k$,

$$(\omega_i(s)| \le C\left[\beta_1^k(s) + \varepsilon_k + \varepsilon_k^2\left(1 - \beta_1^k(s)\right)\right], \left|\omega_i^{'}(s)\right| \le C\ \left[\varepsilon_i^{-1}\beta_1^i(s) + \varepsilon_k^{-1}\beta_1^k(s)\right],$$

$\left|\omega_i^{(2)}(s)\right| \le C\ \sum_{r=i}^{k}\varepsilon_r^{-2}\beta_1^r(s), \left|\omega_i^{(3)}(s)\right| \le C\varepsilon_i^{-1}\left[\sum_{r=1}^{i-1}\varepsilon_r^{-1}\beta_1^r(s) + \sum_{r=i}^{k}\varepsilon_r^{-2}\beta_1^r(s)\right],$
for $k \le i \le l$,
$|\omega_i(s)| \le C\left[\beta_1^k(s) + \varepsilon_k + \varepsilon_k^2\left(1 - \beta_1^k(s)\right)\right],\ \left|\omega_i^{(p)}(s)\right| \le C\ \varepsilon_k^{-p}\beta_1^k(s)$, for $p = 1,\ 2,$
$\left|\omega_i^{(3)}(s)\right| \le C\varepsilon_k^{-1}\left[\sum_{r=1}^{k-1}\varepsilon_r^{-1}\beta_1^r(s) + \varepsilon_k^{-2}\beta_1^k(s)\right],$
and for $l+1 \le i \le n,$
$|\omega_i(s)| \le C\left[\varepsilon_k + \varepsilon_k^2\left(1 - \beta_1^k(s)\right)\right], \left|\omega_i^{'}(s)\right| \le C\varepsilon_k, \left|\omega_i^{(2)}(s)\right| \le C\ \left[\varepsilon_k + \beta_1^k(s)\right],$
$\left|\omega_i^{(3)}(s)\right| \le C\sum_{r=1}^{k}\varepsilon_r^{-1}\beta_1^r(s).$

On the interval $[1,\ 2]$, for $1 \le i \le k$,

$$|\omega_i(s)| \le C\left[\beta_2^k(s) + \varepsilon_k + \varepsilon_k^2\left(1 - \beta_2^k(s)\right)\right],\ \ \left|\omega_i^{'}(s)\right| \le C\ \left[\varepsilon_i^{-1}\beta_2^i(s) + \varepsilon_k^{-1}\beta_2^k(s)\right],$$

$\left|\omega_i^{(2)}(s)\right| \le C\ \sum_{r=i}^{k}\varepsilon_r^{-2}\beta_2^r(s), \left|\omega_i^{(3)}(s)\right| \le C\varepsilon_i^{-1}\left[\sum_{r=1}^{i-1}\varepsilon_r^{-1}\beta_2^r(s) + \sum_{r=i}^{k}\varepsilon_r^{-2}\beta_2^r(s)\right],$
for $k \le i \le l,$
$|\omega_i(s)| \le C\left[\beta_2^k(s) + \varepsilon_k + \varepsilon_k^2\left(1 - \beta_2^k(s)\right)\right], \left|\omega_i^{(p)}(s)\right| \le C\ \varepsilon_k^{-p}\beta_2^k(s)$ for $p = 1,2,$
$\left|\omega_i^{(3)}(s)\right| \le C\varepsilon_k^{-1}\left[\sum_{r=1}^{k-1}\varepsilon_r^{-1}\beta_2^r(s) + \varepsilon_k^{-2}\beta_2^k(s)\right],$
and for $l+1 \le i \le n,$
$|\omega_i(s)| \le C\left[\varepsilon_k + \varepsilon_k^2\left(1 - \beta_2^k(s)\right)\right], \left|\omega_i^{'}(s)\right| \le C\varepsilon_k, \left|\omega_i^{(2)}(s)\right| \le C\ \left[\varepsilon_k + \beta_2^k(s)\right],$
$\left|\omega_i^{(3)}(s)\right| \le C\sum_{r=1}^{k}\varepsilon_r^{-1}\beta_2^k(s).$

Proof. First the bounds of $\vec{\omega}$ and its derivatives are estimated in $[0,1]$. Next these results are used to get the estimates in $[1,2]$.Let $s\epsilon[0,1]$. Now consider the barrier functions

$\vec{3}^{\pm}(s) = \left(\varsigma_1(s), \varsigma_2(s), .., \varsigma_n(s)\right)^T \pm \vec{\omega}$ (s), where $\varsigma_i(s) = C_1\beta_1^k(s) + C_2\varepsilon_k(1-s) + C_3\varepsilon_k^2\left(1 - \beta_1^k(s)\right), 1 \le i \le l, \varsigma_i(s) = C_2\varepsilon_k(1-s) + C_3\varepsilon_k^2\left(1 - \beta_1^k(s)\right), l+1 \le i \le n$. For proper choice of the constants C_1, C_2 andC_3, $\vec{3}^{\pm}(0) \ge \vec{0}, \vec{3}^{\pm}(1) \ge \vec{0}$ and

$\vec{L}_1\vec{3}^{\pm} \leq \vec{0}$ on $(0,\ 1)$. Then the results on the interval $[0,\ 1]$ are proved by the methods adapted in[9]. Then using these results, proceeding in a similar way, it is not hard to derive the estimates for the interval $[1,\ 2]$.

Numerical Method

Fitted mesh method – using Shishkin mesh. We discretize the solution space by using a piecewise uniform Shishkin mesh with N mesh intervals and to tackle the delay term, one nodal point is considered at delay point, in particular at mid-point of the mesh. Let $\Omega^N = \Omega_1^N \cup \Omega_2^N$, where $\Omega_1^N = \{s_j\}_{j=1}^{\frac{N}{2}-1}$, $\Omega_2^N = \{s_j\}_{j=\frac{N}{2}+1}^{N-1}$ and $s_{\frac{N}{2}} = 1$. Then, $\overline{\Omega}_1^N = \{s_j\}_{j=0}^{\frac{N}{2}}$, $\overline{\Omega}_2^N = \{s_j\}_{j=\frac{N}{2}}^{N}$, $\overline{\Omega}_1^N \cup \overline{\Omega}_2^N = \overline{\Omega}^N = \{s_j\}_{j=0}^{N}$ and $\Gamma^N = (0, 2\}$ and N the number of mesh elements is taken as a multiple of$4k$.Since in the considered problem the layers occur at the right side of the boundary point 0 and interior point 1, so $\overline{\Omega}^N$ is divided into $2k+2$ subintervals as $[0,\ \rho_1]$, $[\rho_1, \rho_2]$,....,$[\rho_{k-1}, \rho_k]$, $[\rho_k,\ 1]$, $[1,\ 1+\rho_1]$, $[1+\rho_1,\ 1+\rho_2]$,.....,$[1+\rho_{k-1},\ 1+\rho_k]$, $[1+\rho_k,\ 2]$, in which the intervals $[\rho_k,\ 1]$, $[1+\rho_k,\ 2]$ are outer layer regions(each contains $\frac{N}{4}$ mesh points)and the remaining subintervals are called inner layer regions (each contains $\frac{N}{4k}$ mesh points). The transition parameters ρ_m, for $m = 1,\ 2,\ \ldots,\ k$ are defined by ρ_k = $min\left\{\frac{1}{2},\ \frac{2\varepsilon_k}{\gamma}\ lnN\right\}$,

ρ_m = $min\left\{\frac{m\rho_{m+1}}{m+1},\ \frac{2\varepsilon_m}{\gamma}\ lnN\right\}$, for $m = k-1,\ldots,2,1$.

The discrete problem

The discrete problem for upwind classical finite difference scheme is defined as

$$\vec{L}^N\vec{Y}(s_j) = \varepsilon\delta^2\vec{Y}(s_j) + A(s_j)D^+\vec{Y}(s_j) - B(s_j)\vec{Y}(s_j) - D(s_j)\vec{Y}(s_j-1) = \vec{f}(s_j)\ on\ \Omega^N \tag{18}$$

$\vec{Y}(s_j-1) = \vec{\varrho}(s_j-1)$ for $s_j \in \Omega_1^N$ and $\vec{Y} = \vec{y}$ on Γ^N.

The problem Equation (18) can have an equivalent form as,

$$\vec{L}_1^N\vec{Y}(s_j) = \varepsilon\delta^2\vec{Y}(s_j) + A(s_j)D^+\vec{Y}(s_j) - B(s_j)\vec{Y}(s_j) = \vec{g}(s_j)\ on\ \Omega_1^N \tag{19}$$

$$\vec{L}_2^N\vec{Y}(s_j) = \mathcal{E}\delta^2\vec{Y}(s_j) + A(s_j)D^+\vec{Y}(s_j) - B(s_j)\vec{Y}(s_j) - D(s_j)\vec{Y}(s_j-1) = \vec{f}(s_j)\ on\ \Omega_2^N \tag{20}$$

$\vec{Y} = \vec{y}$ on Γ^N, $D^-\vec{Y}\left(s_{\frac{N}{2}}\right) = D^+\vec{Y}\left(s_{\frac{N}{2}}\right)$, where $\vec{Y}(s_j) = \left(Y_1(s_j),\ Y_2(s_j),\ \ldots,\ Y_n(s_j)\right)^T$, and $0 \leq j \leq N-1$,

$D^+\vec{Y}(s_j) = \frac{\vec{Y}(s_{j+1})-\vec{Y}(s_j)}{h_{j+1}}$, $D^-\vec{Y}(s_j) = \frac{\vec{Y}(s_j)-\vec{Y}(s_{j-1})}{h_j}$, $\delta^2\vec{Y}(s_j) = \frac{1}{\bar{h}_j}\left(D^+\vec{Y}(s_j) - D^-\vec{Y}(s_j)\right]$,

where $h_j = s_j - s_{j-1}$, $\bar{h}_j = \frac{h_j+h_{j+1}}{2}$. This is used to compute numerical approximation to the solution of Equations (1) and (2) . The discrete maximum principle and the discrete stability results are analogous to those for continuous case with the given hypothesis. Hence, we just state the results without proof.

Discrete Maximum P rinciple

Let the conditions Equations (3), (4) and (5) hold. Then, for any mesh function $\vec{\psi} = (\psi_1, \psi_2, \ldots, \psi_n)^T$, the inequalities $\vec{\psi} \geq \vec{0}$ on Γ^N, $\vec{L}_1^N\vec{\psi} \leq \vec{0}$ on Ω_1^N, $\vec{L}_2^N\vec{\psi} \leq \vec{0}$ on Ω_2^N and $D^+\vec{\psi}\left(s_{\frac{N}{2}}\right) - D^-\vec{\psi}\left(s_{\frac{N}{2}}\right) \leq \vec{0}$ imply that $\vec{\psi} \geq \vec{0}$ on $\overline{\Omega}^N$.

Discrete Stability Result

Let the conditions Equations (3), (4) and (5) hold. Then, for any mesh function $\vec{\psi} = (\psi_1, \psi_2, \ldots, \psi_n)^T$, satisfying

$D^+\vec{\psi}\left(s_{\frac{N}{2}}\right) = D^-\vec{\psi}\left(s_{\frac{N}{2}}\right)$,

$|\psi_i(s_j)| \leq max\left\{ \text{k}\vec{\psi}(s_0)\text{k}, \text{k}\vec{\psi}(s_N)\text{k}, \frac{1}{\beta}\,\text{k}L_1^{\to N}\vec{\psi}\,\text{k}_{\Omega_1^N}, \frac{1}{Y}\,\text{k}L_2^{\to N}\vec{\psi}\,\text{k}_{\Omega_2^N} \right\}$, , for $i = 1,\ 2,\ldots,n$ and $0 \leq j \leq N$.

Shishkin decomposition in discrete case As in the continuous case the discrete solution $\vec{Y}$ can be decomposed into $\vec{V}$ and $\vec{W}$, which are difined to be the solutions of the following discrete problems.

$\vec{L}_1^N\vec{V}(s_j) = \vec{g}(s_j)$, $s_j \in \Omega_1^N$, $\vec{V}(0) = \vec{v}(0)$, $\vec{V}\left(s_{\frac{N}{2}-1}\right) = \vec{v}(1-)$

$\vec{L}_2^N\vec{V}(s_j) = \vec{f}(s_j)$, $s_j \in \Omega_2^N$, $\vec{V}\left(s_{\frac{N}{2}+1}\right) = \vec{v}(1+)$, $\vec{V}(2) = \vec{v}(2)$,

and $\vec{L}_1^N\vec{W}(s_j) = \vec{0}$, $s_j \in \Omega_1^N$, $\vec{W}(0) = \vec{\omega}(0)$, $\vec{L}_2^N\vec{W}(s_j) = \vec{0}$, $s_j \in \Omega_2^N$, $\vec{W}(2) = \vec{\omega}(2)$.

$D^-\vec{V}\left(s_{\frac{N}{2}}\right) + D^-\vec{W}\left(s_{\frac{N}{2}}\right) = D^+\vec{V}\left(s_{\frac{N}{2}}\right) + D^+\vec{W}\left(s_{\frac{N}{2}}\right)$.

The error at each point $s_j \in \overline{\Omega}^N$ is denoted by $\vec{e}(s_j) = \vec{Y}(s_j) - \vec{y}(s_j)$, then the local truncation error has the decomposition $\vec{L}^N\vec{e}(s_j) = \vec{L}^N\left(\vec{V}-\vec{v}\right)(s_j) + \vec{L}^N\left(\vec{W}-\vec{\omega}\right)(s_j)$, for $j \neq \frac{N}{2}$.

The local truncation error for $\vec{V}$ and $\vec{W}$ (for $j \neq \frac{N}{2}$) are found as in[9] and hence we state the following theorem

Theorem1. Let the conditions Equations (3), (4) and (5) hold. If $\vec{v},\ \vec{\omega}$ are the smooth and singular components of the solution of Equations (6) and (7) and if $\vec{V},\ \vec{W}$ are the smooth and singular components of the solution of Equations (19) and (20) then, for $i = 1,\ 2, \ldots, n,\ \ j \neq \frac{N}{2}$,

$$\left(\vec{L}_1^N\left(\vec{V}-\vec{v}\right)(s_j)\right| \leq CN^{-1},\ \left(\vec{L}_1^N\left(\vec{W}-\vec{\omega}\right)(s_j)\right| \leq CN^{-1}lnN,\ 0 \leq j \leq \frac{N}{2}-1$$

and

$$\left(\vec{L}_2^N\left(\vec{V}-\vec{v}\right)(s_j)\right| \leq CN^{-1},\ \left(\vec{L}_2^N\left(\vec{W}-\vec{\omega}\right)(s_j)\right| \leq CN^{-1}lnN,\ \ \frac{N}{2}+1 \leq j \leq N.$$

The essentially first order parameter-uniform error estimate is proved here, using discrete maximum principle by constructing a suitable barrier functions and the mesh function. The method is as follows:

Define a set of discrete barrier functions on $\overline{\Omega}^N$, for $i = 1,\ 2, .., n$ by

$$\omega_i(s_j) = \begin{cases} \dfrac{\pi^{\mathrm{j}}{}_{\mathrm{k=1}}\left(1+\dfrac{\mathrm{yh_k}}{\sqrt{2\varepsilon_\mathrm{i}}}\right)}{\pi^{\frac{\mathrm{N}}{2}}{}_{\mathrm{k=1}}\left(1+\dfrac{\mathrm{yh_k}}{\sqrt{2\varepsilon_\mathrm{i}}}\right)} & if\ 0 \leq j \leq \dfrac{N}{2} \\ \dfrac{\pi^{\mathrm{N-1}}{}_{\mathrm{k=}j}\left(1+\dfrac{\mathrm{yh_{k+1}}}{\sqrt{2\varepsilon_\mathrm{i}}}\right)}{\pi^{N-1}{}_{\mathrm{k=}\frac{N}{2}}\left(1+\dfrac{\mathrm{yh_{k+1}}}{\sqrt{2\varepsilon_\mathrm{i}}}\right)} & if\ \dfrac{N}{2} \leq j \leq N \end{cases} \tag{21}$$

From the definition of the barrier functions and by the assumption that $\varepsilon_i < \varepsilon_{i+!}$,

$$0\ \leq \omega_i(s_j) < \omega_{i+1}(s_j) \leq 1\ and\ 0\ \leq \omega_i(s_j) < \omega_{i+1}(s_j) \leq 1\ on\ \overline{\Omega}^N \tag{22}$$

For $i = 1,\ \ 2,\ \ldots, n$ and $s_j \in \overline{\Omega}_1^N$

$$D^+\omega_i(s_j) = \frac{\gamma\omega_i(s_j)}{\sqrt{2\varepsilon_i}},\ \ D^-\omega_i(s_j)\ = \frac{\gamma\,\omega_i(s_j)}{\sqrt{2\varepsilon_i}(1+\gamma h_j/\sqrt{2\varepsilon_i})},\ \ \ \delta^2\omega_i(s_j) \leq \frac{\gamma^2\omega_i(s_j)}{\varepsilon_i} \tag{23}$$

Also, for $i = 1,\ \ 2,\ \ \ldots,\ \ n$ and $s_j \in \overline{\Omega}_2^N$ we have

$$D^+\omega_i(s_j) = -\frac{\gamma\omega_i(s_j)}{\sqrt{2\varepsilon_i}(1+\gamma h_{j+1}/\sqrt{2\varepsilon_i})},\ \ D^-\omega_i(s_j)\ = -\frac{\gamma\,\omega_i(s_j)}{\sqrt{2\varepsilon_i}},\ \ \ \delta^2\omega_i(s_j) \leq \frac{\gamma^2\omega_i(s_j)}{\varepsilon_i} \tag{24}$$

From Equations (23) and (24)

$$\varepsilon_i\,\delta^2\omega_i(s_j) \leq\ \gamma^2\omega_i(s_j)\ \ for\ s_j \in \overline{\Omega}^N. \tag{25}$$

Therefore, using the result Equation (25) in Equations (19) and (20)

$$\left(\vec{L}_1^N\vec{\omega}\right)_i(s_j) = \varepsilon_i\delta^2\omega_i(s_j) + a_i(s_j)\,D^+\omega_i(s_j) - \sum_{r=1}^{n} b_{ir}(s_j)\,\omega_r(s_j)$$

$$\leq \gamma^2\omega_i(s_j) + a_i(s_j)\,\frac{\gamma\omega_i(s_j)}{\sqrt{2\varepsilon_i}} - \sum_{r=1}^{n} b_{ir}(s_j)\,\omega_r(s_j)$$

$$\leq \gamma^2 \omega_i\left(s_j\right) + a_i\left(s_j\right) \frac{\gamma \omega_i(s_j)}{\sqrt{2\varepsilon_i}} - \sum_{r=1}^{n} b_{ir}\left(s_j\right) \omega_r\left(s_j\right) \tag{26}$$

$$\left(\vec{L}_2^N \vec{\omega}\right)_i\left(s_j\right) = \varepsilon_i \delta^2 \omega_i\left(s_j\right) + a_i\left(s_j\right) D^+ \omega_i\left(s_j\right) - \sum_{r=1}^{n} b_{ir}\left(s_j\right) \omega_r\left(s_j\right) - d_i\left(s_j\right) \omega_i\left(s_j - 1\right)$$

$$\leq \gamma^2 \omega_i\left(s_j\right) + a_i\left(s_j\right) \frac{\gamma \omega_i\left(s_j\right)}{\sqrt{2\varepsilon_i}(1+\gamma h_{j+1}/\sqrt{2\varepsilon_i})} - \sum_{r=1}^{n} b_{ir}\left(s_j\right) \omega_r\left(s_j\right) - d_i\left(s_j\right) \omega_i\left(s_j - 1\right)$$

$$\leq \gamma^2 \omega_i\left(s_j\right) + a_i\left(s_j\right) \frac{\gamma \omega_i\left(s_j\right)}{\sqrt{2\varepsilon_i}(1+\gamma h_{j+1}/\sqrt{2\varepsilon_i})} - \sum_{r=1}^{n} b_{ir}\left(s_j\right) \omega_r\left(s_j\right) - d_i\left(s_j\right) \omega_i\left(s_j - 1\right). \tag{27}$$

Further at $j = \frac{N}{2}$, for $i = 1,\ 2,\ 3, \ldots,\ n$

$(D^+ - D^-) e_i\left(s_{\frac{N}{2}}\right) = (D^+ - D^-)(Y_i - y_i)\left(s_{\frac{N}{2}}\right) = -(D^+ - D^-)(y_i)\left(s_{\frac{N}{2}}\right).$

Let $h^* = max\{h^-_{\frac{N}{2}},\ h^+_{\frac{N}{2}}\}$.

Then

$$\left|(D^+ - D^-) e_i\left(s_{\frac{N}{2}}\right)\right| = \left|(D^+ - D^-)(y_i)\left(s_{\frac{N}{2}}\right)\right|$$

$$\leq \left|\left(D^+ - \frac{d}{ds}\right)(y_i)\left(s_{\frac{N}{2}}\right)\right| + \left|\left(D^- - \frac{d}{ds}\right)(y_i)\left(s_{\frac{N}{2}}\right)\right|$$

$$\leq \frac{1}{2} h^+_{\frac{N}{2}} \left|y_i^{''}(\eta_1)\right|_{\eta_1 \in (1,2)} + \frac{1}{2} h^-_{\frac{N}{2}} \left|y_i^{''}(\eta_2)\right|_{\eta_2 \in (0,1)}$$

$$\leq C h^* \max_{\eta \in (0,1) \cup (1,2)} \left|y_i^{''}(\eta)\right|.$$

Using the estimates of smooth and singular components of the solution,

$$\left|(D^+ - D^-) e_i\left(s_{\frac{N}{2}}\right)\right| \leq C \frac{h^*}{\varepsilon_i} \tag{28}$$

Also, in particular at $j = N/2$, from Equations (23) and (24)

$$(D^+ - D^-) \omega_i\left(s_j\right) = \left[-2 \frac{\gamma \omega_i\left(s_j\right)}{\sqrt{2\varepsilon_i}(1+\gamma h^*/\sqrt{2\varepsilon_i})}\right] \leq -\frac{C}{\sqrt{\varepsilon_i}} \tag{29}$$

Theorem 2. Let $\vec{y}(s_j)$ be the solution of the problem Equations (1) and (2) and $\vec{Y}(s_j)$ be the solution of the problem Equation (18) . Then, k$Y^{\rightarrow}(s_j) - y^{\rightarrow}(s_j)$k $\leq CN^{-1}\ \ln\ N,\ 0 \leq j \leq N$

Proof. Consider the mesh function $\vec{J}$ given by

$$J_i\left(s_j\right) = C_1 N^{-1} lnN + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \omega_i\left(s_j\right) \pm e_i\left(s_j\right), i = 1,\ 2, \ldots, n,\ 0 \leq j \leq N.$$

where C_1 and C_2 are constants such that C_2 is suitably chosen and $\kappa = \underset{i=1,2,..,k}{min} \varepsilon_i$.

Then, $J_i(0) = J_i(2) = C_1 N^{-1}$

From Equation (26) and Theorem 1

$$\left(\vec{L}_1^{N}\right)_i(s_j) = -\sum_{r=1}^{n} b_{ir}(s_j) C_1 N^{-1} lnN + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(\vec{L}_1^{N}\right)_i(s_j) \pm \left(\vec{L}_1^{N} \vec{e}\right)_i(s_j)$$

$$\leq -\sum_{r=1}^{n} b_{ir}(s_j) C_1 N^{-1} lnN + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(\left(\gamma^2 + \frac{a_i(s_j)\gamma}{\sqrt{2\varepsilon_i}}\right)_i (s_j) - \sum_{r=1}^{n} b_{ir}(s_j)_r (s_j)\right] \pm C N^{-1} lnN$$

$$\leq -\sum_{r=1}^{n} b_{ir}(s_j) C_1 N^{-1} lnN + C_2 Q_1 + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(\gamma^2 \omega_i(s_j) - \sum_{r=1}^{n} b_{ir}(s_j) \omega_r(s_j)\right] \pm C N^{-1} lnN$$

where Q_1 is a constant. Let $\mu_i(s_j) = \left[\gamma^2 \omega_i(s_j) - \sum_{r=1}^{n} b_{ir}(s_j) \omega_r(s_j)\right], i = 1,\ 2,\ ..,\ n.$

Then choosing $C_1 \geq C_2 (\mathrm{k}\mu^{\rightarrow} \mathrm{k} + Q_1) + C$, we have $\left(\vec{L}_1^{N}\right)_i(s_j) \leq 0$, on Ω_1^N, for $i = 1,\ 2, \ldots, n$.

And from Equation (27) , Theorem 1

$\left(\vec{L}_2^{N}\right)_i(s_j) = -\sum_{r=1}^{n} b_{ir}(s_j) C_1 N^{-1} lnN + d_i(s_j) C_1 N^{-1} lnN + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(\vec{L}_2^{N}\right)_i(s_j) \pm \left(\vec{L}_2^{N} \vec{e}\right)_i(s_j)$

$\leq -C_1 (\sum_{r=1}^{n} b_{ir}(s_j) + d_i(s_j)) N^{-1} lnN + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(\left(\gamma^2 - \frac{a_i(s_j)\gamma}{\sqrt{2\varepsilon_i}(1+\gamma h_{j+1}/\sqrt{2\varepsilon_i})}\right)_i (s_j) - \right.$ $\sum_{r=}^{n}$

$CN^{-1} lnN$

$\leq -C_1 \left(\sum_{r=1}^{n} b_{ir}(s_j) + d_i(s_j)\right) N^{-1} lnN + C_2 Q_2 + C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(\gamma^2 \omega_i(s_j)\right]$

$+ C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} \left(-\sum_{r=1}^{n} b_{ir}(s_j) \omega_r(s_j) - d_i(s_j) \omega_i(s_j - 1)\right] \pm C N^{-1} lnN,$

where Q_2 is a constant. Let $\eta_i(s_j) = \gamma^2 \omega_i(s_j) - \sum_{r=1}^{n} b_{ir}(s_j) \omega_r(s_j) - d_i(s_j) \omega_i(s_j - 1)$.

Then choosing – we have $\left(\vec{L}_2^{N}\right)_i(s_j) \leq 0$, on Ω_2^N, for $i = 1,\ 2,\ldots,$n.

Also, $D^+ J_i(1)$-$D^- J_i(1) = C_2 \frac{\kappa h^*}{\sqrt{\varepsilon_i}} (D^+ - D^-) w_i(1) \pm (D^+ - D^-) e_i\left(s_{\frac{N}{2}}\right)$

$\leq -\frac{\kappa h^*}{\sqrt{\varepsilon_i}} C_2 \frac{1}{\sqrt{\varepsilon_i}} C \pm C \frac{h^*}{\varepsilon_i} \leq 0.$ (using Equations (28) and (29))

Hence, by applying discrete maximum principle to $^{\rightarrow}$, it follows that $_i(s_j) \geq 0$, for $i = 1,\ 2, \ldots,\ n,$ $0 \leq j \leq N$. Also from Equation (22) , $\omega_i(s_j) \leq 1$, therefore for sufficiently large N, $\mathrm{k}\vec{Y}(s_j) - \vec{y}(s_j)\mathrm{k} \leq CN^{-1} \ \ln N$, which completes the proof.

3 Result and Discussion

Numerical illustration

The proposed numerical scheme is validated by the following example.

$$\varepsilon \vec{y}''(s) + A(s) \vec{y}'(s) - B(s) \vec{y}(s) = \vec{g}(s) \quad on\ (0,1)$$

$$\varepsilon \vec{y}''(s) + A(s) \vec{y}'(s) - B(s) \vec{y}(s) - D(s) \vec{y}(s-1) = \vec{f}(s) \quad on\ (1,2)$$

with $\vec{g}(s) = (s, 2,\ 1)^T,\ \vec{f}(s) = (2,\ 2,\ 2)^T, \varepsilon = diag(\varepsilon_1,\ \varepsilon_2,\ 1),\ A(s) = diag(1,\ 1,\ 1),$

$B(s) = \begin{pmatrix} 1.2 & -0.1 & -0.1 \\ -0.1 & 1.1 & -0.1 \\ -0.1 & -0.1 & 1.3 \end{pmatrix}$ and $D(s) = diag(-0.2,\ -0.2,\ -0.2), \vec{y}(0) = (1,\ 1,\ 1)^T, \vec{y}(2) = \vec{0}, \varepsilon_1 = \frac{\eta}{32}, \varepsilon_2 = \frac{\eta}{16}$ and $\eta = \frac{1}{4}$.

With the help of two mesh techniques[10], the maximum pointwise errors for various values of the singular perturbation parameters $\varepsilon_{1,}\ \varepsilon_2$ and mesh points N are calculated, the corresponding order of convergence and error constant are presented in Table 1.

Table 1. Values of T^N, ρ^N, H_ρ^N and ρ^*, $H_{\rho^*}^N$

η	Number of mesh points N			
	64	128	256	512
0.2500E+00	0.4039E-01	0.3159E-01	0.2358E-01	0.1750E-01
0.6250E-01	0.3348E-01	0.2483E-01	0.1716E-01	0.1124E-01
0.15625E-01	0.3138E-01	0.2278E-01	0.1536E-01	0.9455E-02
0.390625E-02	0.3081E-01	0.2222E-01	0.1487E-01	0.8992E-02
0.9765625E-03	0.3066E-01	0.2208E-01	0.1474E-01	0.8870E-02
N	0.4039E-01	0.3159E-01	0.2358E-01	0.1749E-01
ρ^N	0.3545E+00	0.4218E+00	0.4307E+00	
H_ρ^N	0.8097E+00	0.8097 E+00	0.7728E+00	0.7331 E+00
The order of $\vec{\varepsilon}$ - uniform convergence ρ^* =0.35451				
Computed $\vec{\varepsilon}$ - uniform error constant $H_{\rho^*}^N$ =0.80975				

Note: In Two-mesh algorithm,

$\vec{\varepsilon}$-uniform maximum pointwise two-mesh differences $\mathrm{T^N} = \max_{\vec{\varepsilon}} \left\| \overrightarrow{\mathrm{Y}}^{\mathrm{N}} - \overrightarrow{\mathrm{Y}}^{2\ \mathrm{N}} \right\|_{\Omega^{\mathrm{N}}}$

$\vec{\varepsilon}$-uniform order of convergence $\rho^* = \underset{N}{min} log_2 \frac{T^N}{T^{2N}}$

$\vec{\varepsilon}$-uniform error constant $H_{\rho^*} = \underset{N}{max} \frac{T^N N^{\rho^*}}{1-2^{-\rho^*}}$ where $\widehat{\overrightarrow{Y}}^{2N}$ is the piecewise linear interpolants of the solution $\overrightarrow{Y}^N$.

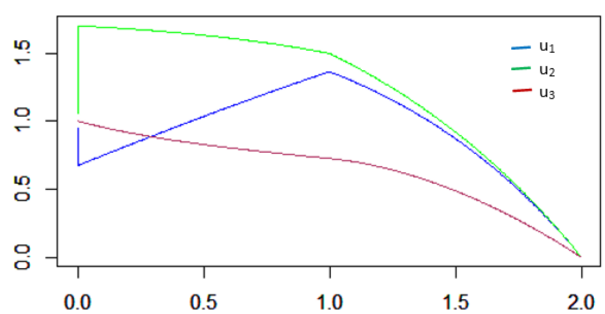

Fig 1. The solution components

Figure 1 shows that, for $N = 1024$, $\varepsilon_1 = \frac{\eta}{32}, \varepsilon_2 = \frac{\eta}{16}$ the solution components u_1, u_2 have boundary layers at the point 0 and interior layers at the point 1, also it shows that the solution component u_3 behaves smooth.

4 Conclusion

Numerical methods for solving singularly perturbed delay differential equations of convection-diffusion type are essential in capturing the complex behaviour induced by the small perturbation parameter, convection, diffusion and delays. Techniques like finite difference methods with piecewise uniform Shishkin meshes play a critical role in obtaining accurate and efficient solutions for these challenging problems.

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