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RESEARCH ARTICLE

# $\mathfrak{N}_{\overline{S}}\ \mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ and $\mathfrak{N}_{\overline{\mathbf{QS}}}\mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$- Spaces in $\mathfrak{D}_{\mathfrak{L}}$-Nano Topological Space

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## Abstract

**Objectives:** The purpose of this study is to propose a new kind of Nano Ideal topological space called $\mathfrak{D}_{\mathfrak{L}}$-Nano topological space, explores several novel forms of open and closed sets in the context of $\mathfrak{D}_{\mathfrak{L}}$-Nano topological space, and generate $\mathfrak{N}_{\overline{S}}\ \mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ and $\mathfrak{N}_{\overline{\mathbf{QS}}}\mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$-spaces in $\mathfrak{D}_{\mathfrak{L}}$-Nano topological space. **Methods:** The $\mathfrak{D}_{\mathfrak{L}}$-lower, $\mathfrak{D}_{\mathfrak{L}}$-upper and $\mathfrak{D}_{\mathfrak{L}}$-boundary regions have been established for any subset of the specified universal set and any binary relation defined on it. The associated $\mathfrak{D}_{\mathfrak{L}}$-Nano topological space can be determined by these regions in addition with universal set and the empty set. **Findings:** The relationship between $\mathfrak{D}_{\mathfrak{L}}$-Nano open sets and other sets in $\mathfrak{D}_{\mathfrak{L}}$-Nano topological spaces are found. Additionally, we figure out the forms of $\mathfrak{N}_{\overline{S}}\ \mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ and $\mathfrak{N}_{\overline{\mathbf{QS}}}\mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$-spaces. **Novelty:** Nano-open sets were used in the formation of normal spaces in Nano topological spaces, whereas in the current research work $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}$ and $\mathfrak{N}_{\bar{S}}\mathfrak{D}_{\mathfrak{L}}$-open sets have been used to generate $\mathfrak{N}_{\overline{S}}\ \mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ and $\mathfrak{N}_{\overline{\mathbf{QS}}}\mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ -spaces.

**Keywords:** $\mathfrak{D}_{\mathfrak{L}}$-Nano topological space; $\mathfrak{D}_{\mathfrak{L}}$-lower, $\mathfrak{D}_{\mathfrak{L}}$-upper; $\mathfrak{D}_{\mathfrak{L}}$-boundary region, $\mathfrak{N}_{\overline{S}}\ \mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ **-**space, $\mathfrak{N}_{\overline{\mathbf{QS}}}\mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$-space

## 1 Introduction

Nano Ideal topological structure has emerged as an essential feature of Nano topological spaces. The Various types of generalized closed sets and open sets in Nano Ideal topological spaces were analyzed[1–5]. The characterizations of different normal spaces in Nano Ideal topological spaces have been studied[6–8]. The key contribution of the current work is that we provide a novel method to generate Nano Ideal topological space referred to as $\mathfrak{D}_{\mathfrak{L}}$ - Nano topological space. In section 2, we present a new method to construct $\mathfrak{D}_{\mathfrak{L}}$ - Nano topological space. In section 3, we introduce new types of open sets $\mathfrak{N}_{\alpha}\mathfrak{D}_{\mathfrak{L}}$, $\mathfrak{N}_{s}\mathfrak{D}_{\mathfrak{L}}$, $\mathfrak{N}_{p}\mathfrak{D}_{\mathfrak{L}}$, $\mathfrak{N}_{\gamma}\mathfrak{D}_{\mathfrak{L}}$ and $\mathfrak{N}_{\beta}\mathfrak{D}_{\mathfrak{L}}$-open sets, and investigate their properties. Further, we describe and characterize new classes of normal spaces namely $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ and $\mathfrak{N}_{\overline{S}}\ \mathfrak{D}_{\mathfrak{L}}\mathfrak{N}\ell$ -spaces.

# 2 Methodology

## 2.1 $\mathfrak{D}_{\mathfrak{L}}$ Nano topology

**Definition: 2.1.** Let $\mathfrak{K}$ be an ideal in the universal set $\mathbb{U}$ and $\mathcal{M} \subseteq \mathbb{U}$. Then the Nano left dynamic lower, upper and boundary regions are defined as,

$\underline{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}) = \{\eta \in \mathbb{U} : \mathfrak{R}_{\mathrm{i}}'(\eta) \cap \mathcal{M}^c \in \mathfrak{K}$ and $\mathfrak{R}_{\mathrm{i}}'(\eta) \subseteq \mathcal{M}\}$, ,

$\overline{\overline{\mathcal{D}_{\mathfrak{L}}}}(\mathcal{M}) = \{\eta \in \mathbb{U} : \mathfrak{R}_{\mathrm{i}}'(\eta) \cap \mathcal{M} \notin \mathfrak{K}$ and $\mathfrak{R}_{\mathrm{i}}'(\eta) \cap \mathcal{M} \neq \emptyset\}$ where $\mathfrak{R}_i{}'(\eta) = \{\beta \in U : \mathfrak{R}_\ell(\eta) \subseteq \mathfrak{R}_\ell(\beta)\}$ and $\mathfrak{B}_{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}) = \overline{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}) - \mathfrak{D}_{\mathfrak{L}}(\mathcal{M}) R_i{}'(\eta) = (\beta \epsilon U : R_l(\eta) \subseteq R_l(\beta))$ . Then, the Nano left dynamic topology (called$\mathfrak{D}_{\mathfrak{L}}$ - Nano topology) on $\mathbb{U}$ with respect $\mathcal{M} \subseteq \mathbb{U}$ is defined as, $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\mathcal{M}) = \left\{\mathbb{U}, \emptyset, \overline{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}), \underline{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}), \mathfrak{B}_{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M})\right\}$. The collection $\mathfrak{B}_{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}) = \left\{\mathbb{U}, \mathfrak{D}_{\mathfrak{L}}(\mathcal{M}), \mathfrak{B}_{\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M})\right\}$ forms a basis for $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\mathcal{M})$ and the $\mathfrak{D}_{\mathfrak{L}}$ – Nano topological space is denoted by $(\mathbb{U}, \mathfrak{N}D_{\mathfrak{L}}(\mathcal{M}), \mathfrak{K})$.The elements of $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\mathcal{M})$ are called $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}$- open sets and its complements are called $\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}$- closed sets. Then, $\mathrm{int}_{\mathfrak{N}_{\mathfrak{L}}}(\mathcal{M}) = \cup_{\mathcal{P} \in \mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\mathcal{M})}\{\mathcal{P}/\mathcal{P} \subseteq \mathcal{M}\}$ and $cl_{\mathfrak{N}\mathfrak{D}_{\mathfrak{L}}}(\mathcal{M}) = \bigcap_{\mathcal{P} \in \mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\mathcal{M})}\{\mathcal{P}/\mathcal{P} \supseteq \mathcal{M}\}$

**Example 2.2 .** Let $= (\eta, \gamma, \varrho, \omega, \tau\}$ ,$M = \{\eta, \gamma\}$, $K = \{\emptyset, (\eta\}, (\xi\}, (\eta, \omega\}\}$ and the relation be $Y = \{(\eta, \eta), (\eta, \gamma), (\eta, \varrho), (\gamma, \gamma), (\varrho, \eta), (\upsilon, \omega), (\omega, \tau), (\tau, \gamma)\}$.Then, the corresponding $\mathfrak{D}_{\mathfrak{L}}$ - Nano topology is $ND_L(M) = \{U, \emptyset, (\eta, \gamma\}, (\eta, \gamma, \varrho\}, (\upsilon\}\}$.

**Definition 2.3.** Let $(\mathbb{U}, \mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\mathcal{F}), \mathfrak{K})$ be a $\mathfrak{D}_{\mathfrak{L}}$ - Nano topological space. An operator $\mathfrak{h}^{\mathfrak{D}_{\mathfrak{L}}} : \wp(\mathbb{U}) \to \wp(\mathbb{U})$ is known as the dynamic function of $\mathfrak{K}$ on $\mathbb{U}$ and defined as

$\mathfrak{S}^{D_{\mathfrak{L}}} = \{\eta \in \mathbb{U} : \mathcal{G} \cap \mathfrak{H} \notin \mathfrak{K}$ for all $\mathcal{G} \in \mathfrak{N}\mathfrak{D}_{\mathfrak{L}}(\eta)\}$. The corresponding closure operator of $\mathfrak{h}^{\mathfrak{D}_{\mathfrak{L}}}$ is defined as $\mathfrak{D}_{\mathfrak{L}}{}^*(\mathfrak{H}) = \mathfrak{H} \cup \mathfrak{H}^{D_{\mathfrak{L}}}$.

**Theorem 2.4.** Let $K_\alpha$ and $K_\beta$ be the two ideals on the universal set $U$ and $T_\alpha, T_\beta, H, D \subseteq U$ .Then,

1. $\emptyset^{D_L} = \emptyset$
2. If $T_\alpha \subset T_\beta$ then $T_\alpha{}^{D_L} \subset T_\beta{}^{D_L}$.
3. For any two ideals $K_\alpha \supseteq K_\beta$ on $U$, $H^{D_L}(K_\alpha) \supseteq H^{D_L}(K_\beta)$.
4. $H^{D_L} \subset D_L{}^*(H)$.
5. $(H^{D_L})^{D_L} \subset H^{D_L}$.
6. $H^{D_L} \cup D^{D_L} = (H \cup D)^{D_L}$.
7. $(H \cap D)^{D_L} = H^{D_L} \cap D^{D_L}$.
8. If $H \subset H^{D_L}$ then $H^{D_L} = D_L{}^*(H)$.

**Proof.** (1)Proof is trivial.(2) Let $T_\alpha \subset T_\beta$ and $\eta \in T_\alpha{}^{D_L}$ .Suppose $\eta \notin T_\beta{}^{D_L}$.Here $G \cap T_\alpha \subseteq G \cap T_\beta$ and $G \cap T_\beta \in K$ for all $G \in ND_L(\eta)$, we have $G \cap T_\alpha \in K$.Therefore $\eta \notin T_\alpha{}^{D_L}$ which is a contradiction. Hence, $T_\alpha{}^{D_L} \subset T_\beta{}^{D_L}$.(3)Let $K_\beta \subseteq K_\alpha$ and $\eta \in H^{D_L}(K_\alpha)$.Then $S \cap H \notin K_\alpha$ for all $S \in ND_L(\eta)$.Since $S \cap H \notin K_\beta$, $\eta \in H^{D_L}(K_\beta)$.Hence, $H^{D_L}(K_\alpha) \supseteq H^{D_L}(K_\beta)$.

(4)Let $\eta \in H^{D_L}$.Then for all $Z \in ND_L(\eta)$, $Z \cap H \notin K$ which results $Z \cap H = \emptyset$.Hence, $\eta \in D_L{}^*(H)$ and $H^{D_L} \subset D_L{}^*(H)$ .(5)Proof is trivial from the definition of $H^{D_L}$.

(6) Here,$H \subset H \cup D$ and $D \subset H \cup D$.Then $H^{D_L} \cup D^{D_L} \subseteq (H \cup D)^{D_L}$.

Let $\eta \in (H \cup D)^{D_L}$.Then for all $S \in ND_L(\eta)$, $S \cap (H \cup F) \notin K$ which implies $(S \cap H) \cup (S \cup D) \notin K$.Hence, $S \cap H$ *or* $S \cap D \notin K$ results that $\eta \in H^{D_L}$ or $\eta \in D^{D_L}$.

Therefore, $\eta \in H^{D_L} \cup D^{D_L}$.

(7) Here, $H \cap D \subset Q$ and $H \cap D \subset D$. Then, $(H \cap D)^{D_L} \subseteq H^{D_L} \cap D^{D_L}$.Conversely, let $\eta \in Q^{D_L} \cap F^{D_L}$.Then for all $Z \in ND_L(\eta)$, $S \cap Q$, $S \cap D \notin K$ results that $S \cap (H \cap D) \notin K$.Therefore, $\eta \in (H \cap D)^{D_L}$.

(8) For all $\mathfrak{G} \subseteq \mathbb{U}, \mathfrak{H} \subseteq \mathfrak{D}_\Omega{}^*(\mathfrak{H})$. Also, $\mathfrak{G} \subseteq \mathfrak{G}^{\mathfrak{D}_{\mathfrak{L}}}$ results that $\mathfrak{D}_{\mathfrak{L}}{}^*(\mathfrak{H}) \subseteq \mathfrak{D}_{\mathfrak{L}}{}^*(\mathfrak{G}^{D_{\mathfrak{L}}})$. Hence, $\tilde{\mathfrak{H}}^{\mathcal{D}_{\mathfrak{L}}} = \tilde{\mathfrak{D}}_{\mathfrak{L}}{}^*(\mathfrak{H})$.

**Theorem 2.5.** Let $(U, ND_L(F), K)$ be a $D_L$ - Nano topological space with $H, D \subseteq U$ .Then,

1. $H \subseteq D_L{}^*(H)$.
2. $D_L{}^*(\emptyset) = \emptyset$,$D_L{}^*(U) = U$.
3. If $H \subseteq D$ then $D_L{}^*(H) \subseteq D_L{}^*(D)$.
4. $D_L{}^*(H) \cup D_L{}^*(D) = D_L{}^*(H \cup D)$.
5. $D_L{}^*(D_L{}^*(H)) = D_L{}^*(H)$.
6. $D_L{}^*(H \cap D) \subseteq D_L{}^*(H) \cap D_L{}^*(D)$.
7. If $S$ is $ND_L$- open then $S \cap D_L{}^*(H) \subseteq D_L{}^*(S \cap H)$.

**Proof.** (1) Here $H \subseteq H \cup Q^{D_L} = D_L{}^*(H)$.(2) $D_L{}^*(\emptyset) = \emptyset \cup \emptyset^{D_L}$ and $D_L{}^*(U) = U \cup U^{D_L}$.

(3) If $H \subseteq F$ then $H^{D_L} \subseteq F^{D_L}$.Also $H \cup H^{D_L} \subseteq D \cup D^{D_L}$.Hence,${D_L}^*(H) \subseteq {D_L}^*(D)$.

(4) Here,${D_L}^*(H \cup D) = (H \cup D) \cup (H \cup D)^{D_L} = \left(H \cup H^{D_L}\right) \cup \left(D \cup D^{D_L}\right) = {D_L}^*(H) \cup {D_L}^*(D)$.(5) Consider ${D_L}^*({D_L}^*(H)) = {D_L}^*\left(H \cup H^{D_L}\right) = {D_L}^*(H)$.(6) Let $\eta \in {D_L}^*(H \cap D)$.Since $H \cap D \subseteq H$ and $H \cap D \subseteq D$, $(H \cap F)^{D_L} \subseteq D^{D_L}$ and $(H \cap D)^{D_L} \subseteq D^{D_L}$.

Also, ${D_L}^*(H \cap D) = (H \cap D) \cup (H \cap D)^{D_L} \subseteq H \cup H^{D_L} = {D_L}^*(H)$.

Similarly, ${D_L}^*(H \cap D) \subseteq {D_L}^*(D)$.Since $\in {D_L}^*(H \cap D)$ ,$\eta \in {D_L}^*(H)$

and $\eta \in {D_L}^*(D)$.Therefore $\eta \in {D_L}^*(H) \cap {D_L}^*(D)$.(7) Let $S$ be a $ND_L$-open.Then $S \cup \left(H \cup H^{D_L}\right) = (S \cap H) \cup (S \cap H^{D_L}) \cup (S \cap H)^{D_L}$. Hence, $S \cap {D_L}^*(H) \subseteq {D_L}^*(S \cap H)$.

**Remark 2.6.** Let $= (\eta, \gamma, \varrho, \omega, \tau\}$ ,$H = \{\eta, \gamma\}$,the relation $D = \{(\eta,\eta),(\eta,\gamma),(\eta,\varrho),(\gamma,\gamma),(\varrho,\eta),(\varrho,\omega),(\omega,\tau),(\tau,\gamma)\}$ and $K_\alpha = \{\emptyset,(\eta\},(\varrho\},(\omega\},(\eta,\omega\},(\eta,\varrho\},(\eta,\varrho,\omega\},(\varrho,\omega\}\}$ , $K_\beta = \{\emptyset,(\eta\},(\omega\},(\eta,\omega\}\}$.Then, the following facts have been observed.

i)If $T_\alpha = \{\gamma\}$ and $T_\beta = (\eta, \gamma, \varrho, \tau\}$ then ${T_\beta}^{D_L} \supseteq {T_\alpha}^{D_L}$

ii) $(\varrho\}^{D_L}(K_\beta) \supseteq (\varrho\}^{D_L}(K_\alpha)$.

iii) $\{\eta\} \subseteq \{\eta, \tau\}$ but ${D_L}^*(\eta, \tau\} \supseteq {D_L}^*(\eta\}$.

iv) ${D_L}^*((\eta,\gamma\}) \cap {D_L}^*(\eta\} \supseteq {D_L}^*((\eta,\gamma\} \cap (\eta\})$.

v)If $S = (\eta, \gamma, \varrho\}$ and $H = (\eta, \gamma\}$ then ${D_L}^*(S \cap H) \supseteq S \cap {D_L}^*(H)$.

## 3 Results and Discussion

**Definition 3.1.** A subset $Q$ of $(U, ND_L(G), K)$ is

a) $N_r D_L$-open if $Q = int_{ND_L}(cl_{ND_L}(Q))$.

b) $N_\alpha D_L$- open if $Q \subseteq int_{ND_L}\left(cl_{ND_L}(int_{ND_L}(Q))\right)$.

c) $N_s D_L$- open if $Q \subseteq cl_{ND_L}(int_{ND_L}(Q))$.

d) $N_p D_L$- open if $Q \subseteq int_{ND_L}(cl_{ND_L}(Q))$.

e) $N_\gamma D_L$ - open if $Q \subseteq cl_{ND_L}(int_{ND_L}(Q)) \cup int_{ND_L}(cl_{ND_L}(Q))$.

f) $N_\beta D_L$ - open if $Q \subseteq cl_{ND_L}(int_{ND_L}(cl_{ND_L}(Q)))$.

**Theorem 3.2.**

a) Every $ND_L$- open set is $N_\alpha D_L$ -open.

b) Every $N_\alpha D_L$- open set is $N_p D_L$ -open.

c) Every $N_p D_L$ - open set is $N_\gamma D_L$ -open.

d) Every $N_\beta D_L$ - open set is $N_\gamma D_L$ -open.

e) Every $N_s D_L$ - open set is $N_\gamma D_L$ -open.

**Proof.** (a)Let $Z$ be a $ND_L$- open set. Then, $Z = int_{ND_L}(Z)$ and

$cl_{ND_L}(Z) = cl_{ND_L}(int_{ND_L}(Z))$.Also, $Z \subseteq cl_{ND_L}(Z) = cl_{ND_L}(int_{ND_L}(Z))$ and

$int_{ND_L}(Z) \subseteq int_{ND_L}\left(cl_{ND_L}(int_{ND_L}(Z))\right)$.Hence, $Z$ is $N_\alpha D_L$ -open. (b) Let $F$ be $N_\alpha D_L$- open set. Then $F \subseteq int_{ND_L}\left(cl_{ND_L}(int_{ND_L}(F))\right)$.

Since $int_{ND_L}(F) \subseteq F$, $cl_{ND_L}(int_{ND_L}(F)) \subseteq cl_{ND_L}(F)$.

Also, $int_{ND_L}\left(cl_{ND_L}(int_{ND_L}(F))\right) int_{ND_L}(cl_{ND_L}(F))$.

This implies that $F \subseteq \left(int_{ND_L}(cl_{ND_L}(F)\right)$.Hence, $F$ is $N_p D_L$-open. (c) Let $L$ be a $N_p D_L$- open set. So, $L \subseteq \left(int_{ND_L}(cl_{ND_L}(F)\right)$.Also, $L \subseteq \left(int_{ND_L}(cl_{ND_L}(F)\right) \cup cl_{ND_L}(int_{ND_L}(F))$.

Hence, $L$ is $N_\gamma D_L$-open. (d) Let $O$ be $N_\beta D_L$ - open set.

Then,$O \subseteq cl_{ND_L}(int_{ND_L}(cl_{ND_L}(Q)))$. Also, $O \subseteq (int_{ND_L}(cl_{ND_L}(Q))$.

Therefore, $O \subseteq int_{ND_L}\left(cl_{ND_L}(int_{ND_L}(F))\right)$. Hence, $O$ is $N_\gamma D_L$-open. (e) Let $M$ be a $N_s D_L$- open set. Then $M \subseteq cl_{ND_L}(int_{ND_L}(M))$.

Also, $M \subseteq cl_{ND_L}(int_{ND_L}(M)) \cup int_{ND_L}(cl_{ND_L}(M))$.Hence, $M$ is $N_\gamma D_L$-open.

**Remark 3.3 .** From the example 2.2, we have the following facts.

i) $\{\eta\}$ is $N_\alpha D_L$ - open but not $ND_L$- open.

ii) $\{\gamma, \varrho\}$ is $N_p D_L$ - open but not $N_\alpha D_L$ – open.

iii) $(\eta,\gamma,\omega\}$ is $N_\gamma D_L$-open but not $N_p D_L$- open.
iv) $\{\eta,\varrho,\tau\}$ is $N_\gamma D_L$-open but not $N_\beta D_L$ - open.
v) $(\gamma,\varrho,\omega\}$ is $N_\gamma D_L$-open but not $N_s D_L$- open.

**Definition 3.4 .** Let $Q$ be a subset of $(U, ND_L(G), K)$.Then,
1. $Q$ is $D_L{}^*$-closed if $D_L{}^*(Q) = Q$ whenever $Q \subseteq S$, $S$ is $ND_L$- open.
2. $Q$ is $N_{\bar{G}} D_L$ - closed if $D_L{}^*(Q) \subseteq S$ whenever $Q \subseteq S$, $S$ is $ND_L$- open. The collection of all $N_{\bar{S}} D_L$-closed sets are denoted by $D_L{}^* C(G)$.
3. $Q$ is $N_{\bar{S}} D_L$-closed if $D_L{}^*(Q) \subseteq S$ whenever $Q \subseteq S$, $S$ is $N_s D_L$- open.The collection of all $N_{\bar{S}} D_L$-closed sets are denoted by $N_{\bar{S}} D_L C(G)$.

**Definition 3.5 .** Let $\Lambda : (U_a, N^a, K_a) \to (U_b, N^b, K_b)$ be a map. Then,
1. $\Lambda$ is $N_{\bar{S}} D_L$-continuous function if $\Lambda^{-1}(Q)$ is $N_{\bar{S}} D_L$- open subset of $(U_a, N^b, K_b)$ whenever $Q$ is $ND_L$-open subset of $(U_b, N^a, K_a)$.
2. $\Lambda$ is $N_{\bar{S}} D_L$ - irresolute if $\Lambda^{-1}(Q)$ is $N_{\bar{S}} D_L$ - open whenever $Q$ is $N_{\bar{S}} D_L$-open.

**Example 3.6 .** Let $U = (\eta,\gamma,\varrho,\omega,\tau\}$,$S = \{\eta,\gamma\}$, $K = \{\emptyset,(\eta\},(\omega\},(\eta,\omega\}\}$ and the relation be $Y = \{(\eta,\eta),(\eta,\gamma),(\eta,\varrho),(\gamma,\gamma),(\varrho,\eta),(\varrho,\xi),(\xi,\tau),(\tau,\gamma)\}$.Then $ND_L(G) = \{U,\emptyset,(\eta,\gamma\},(\eta,\gamma,\varrho\},(\varrho\}\}$.
$D_L{}^* C(S) = \{U,\emptyset,(\eta\},(\omega\},(\eta,\omega\},(\xi,\tau\},(\eta,\omega,\tau\},(\varrho,\omega,\tau\},(\eta,\gamma,\omega,\tau\},\{\eta,\varrho,\omega,\tau\}$ and,
$N_{\bar{S}} D_L C(S) = \{U,\emptyset,(\eta\},(\gamma\},(\omega\},(\eta,\omega\},(\eta,\tau\},(\gamma,\omega\},(\gamma,\tau\},(\omega,\tau\},(\eta,\omega,\tau\},(\varrho,\xi,\tau\},(\eta,\gamma,\omega,\tau\}(\eta,\upsilon,\omega,\tau\}\{\gamma,\varrho,\omega,\tau\}\}$.Also, the mapping $\Lambda_\alpha : U \to U$ as defined by $\Lambda_\alpha(\eta) = (\eta,\gamma\}$,
$\Lambda_\alpha(\gamma) = (\eta,\gamma,\varrho\}, \Lambda_\alpha(\varrho) = (\varrho\}, \Lambda_\alpha(\omega) = (\omega\}, \Lambda_\alpha(\tau) = (\tau\}$, is both $N_{\bar{S}} D_L$-continuous and $N_{\bar{S}} D_L$ - irresolute.

**3.1. $N_{\bar{S}} D_L Nl$-spaces**

**Definition 3.7 .** Let $(U, ND_L(G), K)$ be a dynamic Nano topological space. Then,
1. $U$ is $ND_L$-Normal space ( known as $ND_L Nl$-space )if for any pair of non-empty disjoint $ND_L$- closed sets $W_\eta$ and $W_\psi$ of $(U, ND_L(G), K)$, there exist $ND_L$-open sets $Y_\eta$ and $Y_\psi$ of $(U, ND_L(G), K)$ with $W_\eta \subset Y_\eta$ and $W_\psi \subset Y_\psi$.
2. $U$ is $N_{\bar{S}} D_L$-Normal space( known as $N_{\bar{S}} D_L Nl$-space ) if for any pair of non-empty disjoint $ND_L$- closed sets $W_\eta$ and $W_\psi$ of $(U, ND_L(G), K)$, there exist $N_{\bar{S}} D_L$-open sets $Y_\eta$ and $Y_\psi$ of $(U, ND_L(G), K)$ with $W_\eta \subset Y_\eta$ and $W_\psi \subset Y_\psi$.

**Theorem 3 .8.**Every $ND_L Nl$-space is $N_{\bar{S}} D_L Nl$-space.

**Proof .** Let $(U, ND_L(G), K)$ be a $ND_L Nl$-space and $W_\eta$ and $W_\psi$ be two disjoint non-empty $ND_L$- closed sets. Then there exists a disjoint non–empty pair of $ND_L$-open sets $Y_\eta$ and $Y_\psi$ of $(U, ND_L(G), K)$ with $W_\eta \subset Y_\eta$ and $W_\psi \subset Y_\psi$. Also, every $ND_L$- open sets are $N_{\bar{S}} D_L$-open sets. Then, $Y_\eta$ and $Y_\psi$ are $N_{\bar{S}} D_L$-open sets. So, $(U, ND_L(G), K)$ is $N_{\bar{S}} D_L Nl$-space.

**Remark 3.9 .** From the example 2.2,$\{\rho,\upsilon,\tau\}$ is $N_{\bar{S}} D_L$-open but not $ND_L$-open. Therefore every $N_{\bar{S}} D_L Nl$-space need not be a $ND_L Nl$-space.

**Theorem 3.10.** If $(U, ND_L(G), K)$ is $N_{\bar{S}} D_L Nl$-space then for every pair of $ND_L$-open sets $W_\eta$ and $W_\psi$ with $W_\eta \cap W_\psi = U$, there exist $N_{\bar{S}} D_L$-closed sets $Y_\eta$ and $Y_\psi$ with $Y_\eta \subset W_\eta$and $Y_\psi \subset W_\psi$ and $Y_\eta \cap Y_\psi = U$.

**Proof.** Let $W_\eta$ and $W_\psi$ be $ND_L$-open sets in $(U, ND_L(G), K)$ with $W_\eta \cap W_\psi = U$.Then $W_\eta{}^c$ and $W_\psi{}^c$ are disjoint $ND_L$ – closed sets. Since $(U, ND_L(G), K)$ is $N_{\bar{S}} D_L Nl$-space, there exists a pair of $N_{\bar{S}} D_L$-open sets $L_\eta$ and $L_\psi$ such that $W_\eta{}^c \subset L_\eta$ and $W_\psi{}^c \subset L_\psi$. This implies that$Y_\eta = W_\eta{}^c$ and$Y_\psi = W_\psi{}^c$ are $N_{\bar{S}} D_L$-closed sets such that $Y_\eta \subset W_\eta$ and $Y_\psi \subset W_\psi$ with $Y_\eta \cap Y_\psi = U$.

**Theorem 3.11.** Let $\Lambda : (U_a, N^a, K_a) \to (U_b, N^b, K_b)$ be a $ND_L$- continuous bijective function and $N_{\bar{S}} D_L$-open function. If $U_a$ is a $ND_L Nl$-space, then $U_b$is a $N_{\bar{S}} D_L Nl$ -space.

**Proof.** Let $W_\eta$ and $W_\psi$ be two disjoint $ND_L$- closed sets in $(U_b, N^b, K_b)$.
Since $\Lambda$ is $ND_L$-continuous bijective,$\Lambda^{-1}(W_\eta)$ and $\Lambda^{-1}(W_\psi)$ are disjoint $ND_L$- closed sets

in $(U_a, N^a, K_a)$. Also since $U_a$ is $ND_L Nl$-space, there exists a disjoint $ND_L$-open sets $Y_\eta$ and $Y_\psi$ with $\Lambda^{-1}(W_\eta) \subset Y_\eta$ and $\Lambda^{-1}(W_\psi) \subset Y_\psi$ .Then, $W_\eta \subset \Lambda(Y_\eta)$ and $W_\psi \subset \Lambda(Y_\psi)$.
Also, $\Lambda(Y_\eta)$ and $\Lambda(Y_\psi)$ are $ND_L$-open sets in $(U_b, N^b, K_b)$.
Since $\Lambda$ is bijective, $\Lambda(Y_\eta) \cap \Lambda(Y_\psi) = \emptyset$. Hence, $(U_b, N^b, K_b)$ is $N_{\bar{S}}D_L\, Nl$-space.

**Theorem 3.12.**Let $\Lambda : (U_a, N^a, K_a) \to (U_b, N^b, K_b)$ be $N_{\bar{S}}D_L$-continuous and $ND_L$-closed bijection and $U_b$ be a $ND_L Nl$-space. Then $U_a$ is a $N_{\bar{S}}D_L\, Nl$-space.

**Proof:**
Let $W_\eta$ and $W_\psi$ be two disjoint $ND_L$- closed sets in $(U_b, N^b, K_b)$.Then $\Lambda(W_\eta)$and $\Lambda(W_\psi)$ are disjoint closed sets in $(U_b, N^b, K_b)$.Since $U_b$ is $ND_L Nl$-space,there exist disjoint $ND_L$-open sets $Y_\eta$ and $Y_\psi$with $\Lambda(W_\eta) \subset Y_\eta$ and $\Lambda(W_\psi) \subset Y_\psi$.Therefore $W_\eta \subset \Lambda^{-1}(Y_\eta)$ and $W_\psi \subset \Lambda^{-1}(Y_\psi)$.Since f is $N_{\bar{S}}D_L$-continuous, $\Lambda^{-1}(Y_\eta)$ and $\Lambda^{-1}(Y_\psi)$ are $N_{\bar{S}}D_L$-open sets in $(U_a, N^a, K_a)$. Also, $\Lambda^{-1}(Y_\eta) \cap \Lambda^{-1}(Y_\psi) = \emptyset$.
Hence $(U_a, N^a, K_a)$ is $N_{\bar{S}}D_L\, Nl$-space.

**Theorem 3.13.**If $\Lambda : (U_a, N^a, K_a) \to (U_b, N^b, K_b)$ is $N_{\bar{S}}D_L$-irresolute, $ND_L$-closed bijection and $U_b$ is$N_{\bar{S}}D_L\, Nl$-space, then $U_a$is a $N_{\bar{S}}D_L\, Nl$-space.

**Proof.**Let $W_\eta$ and $W_\psi$ be two disjoint $ND_L$- closed sets in $(U_b, N^b, K_b)$.
Here $\Lambda$ is $ND_L$-closed bijection, $\Lambda(W_\eta)$ and $\Lambda(W_\psi)$ are disjoint $ND_L$- closed sets in $(U_b, N^b, K_b)$.Since $(U_b, N^b, K_b)$ is $N_{\bar{S}}D_L\, Nl$-space, there exist disjoint $N_{\bar{S}}D_L$-open sets in $Y_\eta$and $Y_\psi$such that $\Lambda(W_\eta) \subset Y_\eta$ and $\Lambda(W_\psi) \subset Y_\psi$.Since $\Lambda$ is $N_{\bar{S}}D_L$-irresolute, $\Lambda^{-1}(Y_\eta)$ and $\Lambda^{-1}(Y_\psi)$ are $N_{\bar{S}}D_L$-open sets in $(U_a, N^a, K_a)$.Also $\Lambda^{-1}(Y_\eta) \cap \Lambda^{-1}(Y_\psi) = \emptyset$.
Hence, $(U_a, N^a, K_a)$ is $N_{\bar{S}}D_L\, Nl$-space.

**3.2.$N_{\overline{QS}}D_L$ $Nl$-spaces**

**Definition 3.14.** A subset $Q_\zeta$ of $(U, ND_L(G), K)$ is a $N_{\bar{\pi}}D_L$-open set if it is the finite union of $N_r D_L$-opensets.

**Definition 3.15.** A $D_L$-nano topological space $(U, ND_L(G), K)$ is $N_{\overline{QS}}D_L\, Nl$-space if for any pair of disjoint $N_{\bar{\pi}}D_L$-closed sub sets $W_\eta$, $W_\psi$ of $U$,there exist disjoint $N_{\bar{S}}D_L$-open sets $Y_\eta$, $Y_\psi$such that$W_\eta \subseteq Y_\eta$and $W_\psi \subseteq Y_\psi$.

**Theorem 3.16.**If $(U, ND_L(G), K)$ is a $D_L$-nano topological space,then the following are equivalent.

a) $U$ is $N_{\overline{QS}}D_L\, Nl$-space.

b) For any pair of $N_{\bar{\pi}}D_L$-open sets $W_\eta$ and $W_\psi$of $U$ with $W_\eta \cup W_\psi = U$,there are $N_{\bar{S}}D_L$-closed subsets $Y_\eta$, $Y_\psi$of $U$ such that $Y_\eta \subseteq W_\eta$ and $Y_\psi \subseteq W_\psi$with $Y_\eta \cup Y_\psi = U$.

c) For every $N_{\bar{\pi}}D_L$-closed set $W_\eta$ and every $N_{\bar{\pi}}D_L$-open set $Y_\eta$with $W_\eta \subseteq Y_\eta$,there is $C_\eta$ with $W_\eta \subseteq C_\eta \subseteq N_{\bar{S}}D_L C(C_\eta) \subseteq Y_\eta$ .

d) For any pair of disjoint $N_{\bar{\pi}}D_L$-closed subsets $W_\eta$and $W_\psi$ of $U$, there exist $N_{\bar{S}}D_L$-open subsets $C_\eta$ and $C_\psi$ of $U$ with $W_\eta \subseteq C_\eta$, $W_\psi \subseteq C_\psi$ and $N_{\bar{S}}D_L C(C_\eta) \cap N_{\bar{S}}D_L C(C_\psi) = \emptyset$.

**Proof .** Assume (a).
Let $W_\eta$ and $W_\psi$ be $N_{\bar{\pi}}D_L$-open subsets of $N_{\overline{QS}}D_L\, Nl$-space $U$ with $W_\eta \cup W_\psi = U$.Then ${W_\eta}^c$ and ${W_\psi}^c$are the $N_{\bar{\pi}}D_L$-closed subsets of $U$.Also,there exist disjoint $N_{\bar{S}}D_L$-open subsets $L_\eta$, $L_\psi$ of $U$ with ${W_\eta}^c \subseteq L_\eta$ and ${W_\psi}^c \subseteq L_\psi$ .Let $Y_\eta = {L_\eta}^c$ and $Y_\psi = {L_\psi}^c$.Then $L_\eta$and $L_\psi$ are $N_{\bar{S}}D_L$-closed subsets of $U$ with $L_\eta \subseteq W_\eta$and $L_\psi \subseteq W_\psi$ .Hence (b) is proved.

Assume (b).Let $W_\eta$be any $N_{\bar{\pi}}D_L$-closed set and $C_\eta$be any $N_{\bar{\pi}}D_L$-open subset of $U$ with $W_\eta \subseteq Y_\eta$.Then $W_\eta{}^c$ and $C_\eta$ are $N_{\bar{\pi}}D_L$-open subsets of $U$ with $W_\eta{}^c \cup Y_\eta = U$.Then there exist $N_{\bar{S}}D_L$-closed subsets of $U L_\eta$, $L_\psi$of $U$ with $L_\eta \subseteq W_\eta{}^c$ and $L_\psi \subseteq Y_\eta$ and $L_\eta \cup L_\psi = U$. Also, $W_\eta \subseteq L_\eta{}^c$ , $Y_\eta{}^c \subseteq L_\psi{}^c$ and $L_\eta{}^c \cap L_\psi{}^c = \emptyset$.Let $C_\eta = L_\eta{}^c$ and $C_\psi = L_\psi{}^c$.Then $C_\eta$and $C_\psi$ are disjoint $N_{\bar{S}}D_L$-open sets with $W_\eta \subseteq L_\eta$ and $Y_\eta{}^c \subseteq C_\psi$ which gives $C_\psi{}^c \subseteq Y_\eta$.Since $C_\psi{}^c$is $N_{\bar{S}}D_L$-closed, $N_{\bar{S}}D_L C(C_\eta) \subseteq Y_\eta$.

Hence (c) is proved.

Assume (c).Let $W_\eta$ and $W_\psi$be any pair of disjoint $N_{\bar{\pi}}D_L$-closed sets of $U$.

Then $W_\eta \subseteq W_\psi{}^c$and $W_\psi{}^c$is $N_{\bar{\pi}}D_L$-open. After that, there is a $N_{\bar{S}}D_L$-open sets $Y_\eta$ of $U$ with $W_\eta \subseteq Y_\eta \subseteq N_{\bar{S}}D_L C(Y_\eta) \subseteq W_\psi{}^c$.Also $W_\eta \subseteq Y_\eta$, and $W_\psi \subseteq (N_{\bar{S}}D_L C(Y_\eta))^c$.

Let $Y_\psi = (N_{\bar{S}}D_L C(Y_\eta))^c$.Then $Y_\psi$ is $N_{\bar{S}}D_L$-open subset of $U$.

Therefore, $W_\eta \subseteq Y_\eta$and $W_\psi \subseteq Y_\psi$ with $N_{\bar{S}}D_L C\left(Y_\eta\right) \cap N_{\bar{S}}D_L C\left(Y_\psi\right) = \emptyset$.

Hence (d) is proved. From the definition of $N_{\bar{QS}}D_L Nl$-space, (d)⇒(a) can be easily verified.

**Theorem 3.17.** If $\Lambda : (U_a, N^a, K_a) \to (U_b, N^b, K_b)$ is $ND_L$-open and $ND_L$-continuous bijective function and $(U_a, N^a, K_a)$ is $N_{\bar{QS}}D_L Nl$-space,then $\Lambda(U_a)$ is $N_{\bar{QS}}D_L Nl$-space.

**Proof.**Let $L_\eta$and $L_\psi$be two disjoint $N_{\bar{\pi}}D_L$-closed subsets of $\Lambda(U_a)$.Then,$\Lambda^{-1}(L_\eta)$ and $\Lambda^{-1}(L_\psi)$ are disjoint $N_{\bar{\pi}}D_L$-closed subsets of $U_a$.Since $U_a$is a $N_{\bar{QS}}D_L Nl$-space,there exist $N_{\bar{S}}D_L$-open subsets $Y_\eta$and $Y_\psi$of $U_a$with $\Lambda^{-1}(L_\eta) \subseteq Y_\eta$, $\Lambda^{-1}(L_\psi) \subseteq Y_\psi$ and $\Lambda\left(L_\eta\right) \cap \Lambda\left(L_\psi\right) = \emptyset$.So, $\Lambda\left(L_\eta\right)$ and $\Lambda\left(L_\psi\right)$ are disjoint $N_{\bar{S}}D_L$-open sub sets of $\Lambda(U_a)$.Hence, $\Lambda(U_a)$ is $N_{\bar{QS}}D_L Nl$-space.

## 4 Conclusion

The present work explores the concept of $D_L$-Nano topological space and analyses two distinct normal spaces $ND_L Nl$ and $N_{\bar{QS}}D_L Nl$-spaces in the $D_L$-Nano topological space. Our research results contribute to the stream of investigations concerning several kinds of descriptions of $D_L$-Nano topological spaces. Future studies can characterize the structure of regular spaces and investigate the real - life applications of $D_L$-Nano topological spaces.

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