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RESEARCH ARTICLE

Computations of Exponential Diophantine Rectangles over Gnomonic Numbers using Python

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Abstract

Objective: The main objective of this paper is to define and collect a new type of rectangle called the Exponential Diophantine Rectangle over Gnomonic numbers (figurate numbers that take the form $n^2-(n-1)^2, n\in N$). **Methods:** It is done by solving the two exponential Diophantine equations using Mihailescu's theorem, binomial expansion, and the basic theory of congruences. **Findings:** Here, it is proven that there are only four exponential Diophantine rectangles over Gnomonic numbers. Finally, it is validated using Python programming for a specific limit. **Novelty:** The concept of solving an exponential Diophantine equation, as well as creating rectangles under certain conditions using Diophantine equations, is already known in mathematics. But in this article, we connect the shapes and Diophantine equation which is known as Exponential Diophantine rectangles over Gnomonic numbers.
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1 Introduction

A particular concept of number theory is the Diophantine equation. It is a polynomial equation of several unknowns having the integral coefficients and needs to be solved over integers [(1)]. An exponential Diophantine equation is one among the Diophantine equations and it has the form $a^x+b^y=c^z$ with the unknowns in the exponents. A number of researchers are researching the exponential Diophantine equation in a variety of ways [(2–11)] for example in [(2)], Abdelkader Hamtat solved the exponential Diophantine equations on triangular numbers and in [(12)] the exponential Diophantine equation $M_p^x+\left(M_q+1\right)^y=z^2$ has been solved.

Geometrical shapes such as rectangles, triangles, and so on were collected by solving the Diophantine equations by various methods as in [(13)] and [(14)]. Also [(15)] is used for dealing congruences. In this article, we connect the rectangle with the exponential Diophantine equation, which was inspired from [(16)] and [(17)]. Here we define and collect a new type of rectangle called exponential Diophantine rectangle over G_n.

The solution to the exponential Diophantine equation entails Binomial expression, Mihailescu's Theorem, and some verified lemmas. Section 2 provides the fundamental definitions and lemmas necessary for the article. Section 3 is divided into two subsections 3.1 and 3.2, with each subsection containing the solution to the exponential Diophantine equations. Following the theorems 3.1.1, 3.1.2 and 3.1, we list the ED Rectangles over G_n using python programming.

2 Methodology

This section contains the fundamental definitions and a lemma required for this article.

Lemma 2.1 (Mihailescu's Theorem) [12]

$(a,b,x,y) = (3,2,2,3)$ is the unique solution for the exponential Diophantine equation $a^x + b^y = 1$, where $a,\ b,\ x,y \in Z$ such that $min\{a,b,x,y\} \geq 2$.

Definition 2.1 (Binomial Expansion)

For $x \in Z$ and $n \in N$ then the expansion is given as, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}$ and $(x-y)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} y^k x^{n-k}$

3 Results and Discussion

Exponential Diophantine Rectangles over G_n

This section defines Exponential Diophantine rectangles over G_n, provides some lemmas for solving exponential Diophantine equations, and is divided into two subsections that examine some theorems.

Definition 3.1 (Exponential Diophantine Rectangle over G_n)

An Exponential Diophantine rectangle over Gnomonic numbers (G_n) is defined as a rectangle with the length (l) and breadth (b) as $(l,b) = (3n + 2(x+y),\ 2n + (x+y))$ where $n \in N$ and x, y are non - negative integers such that

$$G_{n+1}^x + G_n^x = (n+1)^y \tag{1}$$

(or)

$$G_{n+1}^x - G_n^x = (n+1)^y \tag{2}$$

Notation. ED Rectangle over G_n - Exponential Diophantine Rectangle over Gnomonic numbers.

Lemma 3.1 For $n > 1$, the inequality $(1+n)^y > 2$ holds for all $y > 1$.

Proof: Let us prove this by induction hypothesis on n. For $n = 2$, the inequality $(1+n)^y > 2$ becomes $3^y > 2$. This is obviously true for $y > 1$. Now, assume that the inequality $(1+n)^y > 2$ holds for $n = k$. That is, $(k+1)^y > 2$ for $k > 1$. We have to show that the given inequality holds for $n = k+1$. We know that $(k+2) > (k+1)$ implies $(k+2)^y > (k+1)^y$. By the assumption we have $(k+2)^y > 2$, for $k, y > 1$.

Lemma 3.2 If $y > 2$, the inequality $(1+n)^y > 4n$ holds for all $n > 1$.

3.1 The equation $G_{n+1}^{x} + G_{n}^{x} = (n+1)^{y}$

Theorem 3.1.1 The exponential Diophantine equation $G_{n+1}^x + G_n^x = (n+1)^y$ has only two solutions $(x,y,n) = (0,1,1)$ and $(1,2,1)$ where $n \in N$ and x, y are non - negative integers.

Proof:

Case 1: For $x = 0,\ y = 1$. The Equation (1) becomes $n+1 = 2$ which implies $n = 1$.

Case 2: For $x = 1,\ y > 1$,Equation (1) becomes $4n = (n+1)^y$.

Subcase 1: If $n = 1$, it becomes $4 = 2^y$. From this we get, $y = 2$.

Subcase 2: If $n > 1$ in Equation (1), we get $4n = (1+n)^y$. By using definition 2.1, we have $\binom{y}{1} = 4$, therefore we get $y = 4$. Suppose $y = 4$, we have $4n = (1+n)^y$. By lemma 3.2, it is not possible.

Case 3: For $x = y = 0$, the Equation (1) leads to $2 = 1$. This is not possible.

Case 4: For $y = 0$ and $x = 1$, Equation (1) implies $4n = 1$. We get $n = \frac{1}{4} \notin N$.

Case 5: For $x = y = 1$and Equation (1) gives $3n = 1$, which is an impossible one.

Case 6: In this case we take $x=0$ and $y>1$. The Equation (1) is equivalent to $2=(n+1)^y$.

Subcase 1: If $n=1$, the equation $(n+1)^y=2$ transforms into $2^y=2 \implies y=1$. It is an impossible one.

Subcase 2: If $n>1$, the equation $(n+1)^y=2$ is not possible by lemma 3.1.

Case 7: For $y=0$ and $x>1$, the Equation (1) changes into $G_{n+1}^x+G_n^x=1$.

Subcase 1: If $n=1$, it becomes $3^x=0$. This is not a suitable one.

Subcase 2: If $n>1$implies $n+1>2$. From this we have, $G_n^x>G_n$ and $G_{n+1}^x>G_{\,n+1}$. The equation $G_{n+1}^x+G_n^x=1$ becomes1 $>G_n+G_{n+1}$. Finally, we get $1>4n$, which is impossible.

Case 8: For $y=1$ and $x>1$, Equation (1) is equivalent to $G_{n+1}^x+G_n^x=n+1$.

Subcase 1: Suppose if $n=1$, we get $3^x=1$. This satisfies only when $x=0$.

Subcase 2: If $n>1$. As in the case 7, we get $G_n^x>G_n$ and $G_{n+1}^x>G_{\,n+1}$. The equality $G_{n+1}^x+G_n^x=n+1$ becomes $n+1>G_{n+1}+G_n$, which will reduce into $1>3n$. Hence we get a contradiction.

Case 9: In this case, we have $x,y>1$

Subcase 1: For $n=1$, the Equation (1) reduces to $3^x+1=2^y$. By lemma 2.1, this makes an impossible one.

Subcase 2: This subcase deals with $n>1$. Now by using the binomial expansion in $(2n+1)^x+(2n-1)^x=(n+1)^y$, it becomes $\sum_{k=0}^{x}\binom{x}{k}2n^k(1)^{n-k}+\sum_{k=0}^{x}\binom{x}{k}(-1)^{x+k}2n^k(1)^{n-k}=\sum_{k=0}^{y}\binom{y}{k}n^k(1)^{n-k}$. Thus we obtain $1+(-1)^x=1$by comparing the constant term on both sides. It is an impossible one for all $x>1$.

3.2 The equation $G_{\mathbf{n+1}}^{\mathbf{x}}-G_{\mathbf{n}}^{\mathbf{x}}=(n+\mathbf{1})^{\mathbf{y}}$

Theorem 3.2.1 $(1,1,1)$ and $(2,3,1)$ are the only solutions for the exponential Diophantine equation $G_{n+1}^x-G_n^x=(n+1)^y$ where $x,\ y$ are non-negative integers and $n\in N$.

Proof:

Case 1: In this case, we take $x=y=1$. Equation (2) reduces to $G_{n+1}^x-G_n^x=n+1 \Longrightarrow n=1$.

Case 2: This case deals with $x,y>1$.

Subcase 1: If $n=1$, the Equation (2) is equivalent to $3^x-1=2^y$ and by lemma 2.1, we get $x=2$ and $y=3$.

Subcase 3: If $n>1$in Equation (2) and by applying definition 2.1, we get $\sum_{k=0}^{x}\binom{x}{k}2n^k(1)^{n-k}-\sum_{k=0}^{x}\binom{x}{k}(-1)^{x+k}2n^k(1)^{n-k}=\sum_{k=0}^{y}\binom{y}{k}n^k(1)^{n-k}$. On comparing the constant term, we have $1-(-1)^x=1$, an impossible one.

Case 3: For $x=y=0$, the Equation (2) becomes $0=1$.

Case 4: Putting $x=0$ and $y=1$ in the Equation (2), we get $n=-1\notin N$.

Case 5: For $y=0$ and $x=1$, the Equation (2) becomes $G_{(n+1)}-G_n=1$. This is an impossible one.

Case 6: For $x=0$ and $y>1$ in Equation (2), it leads to $(1+n)^y=0$. This not possible for all $n\geq 1$ and $y>1$.

Case 7: For $x>1$ and $y=0$, the Equation (2) reduces to $(2n+1)^x-(2n-1)^x=1$. The term $G_{n+1}^x-G_n^x\equiv 0\,(mod\,2)$. This is contradiction.

Case 8: Here in this case, take $x=1$ and $y>1$. The Equation (2) changes into $2=(n+1)^y$.

Subcase 1: For $n=1$, we get a contradiction.

Subcase 2: For $n>1$, this is not possible by lemma 3.1.

Case 9: For $x>1$ and $y=1$ in Equation (2), it becomes $G_{n+1}^x-G_n^x=n+1$.

Subcase 1: If $n=1$, Equation (2) reduces to $3^x=3$. This is a contradiction.

Subcase 2: If $n>1$ in $G_{n+1}^x-G_n^x=n+1$. Now by applying the binomial expansion and equating the coefficients of n on both sides we get $2x-(-1)^{x+1}2x=1$. This is not possible for all choices of x.

Theorem 3.1 There are exactly four Exponential Diophantine Rectangles over G_n having the sides as $(l,b)=(5,3),(9,5),(7,4),(13,7)$.

Proof: From the theorem 3.1.1 and theorem 3.2.1, there are only four solutions for Equations (1) and (2). Hence, we conclude that there are exactly four ED Rectangles over G_n.

3.2.1 Python Programming

This section displays the existence of Exponential Diophantine Rectangles over Gnomonic numbers. (Figures 1, 2, 3 and 4)

```
#ED rectangle over Gn
import math
def rectangle():
 print('x\ty\tn\tGm\tGn\t(l,b)')
 for x in range(0,m+1):
    for y in range (0,m+1):
        for n in range (0,m+1):
         Gm=2*n+1
         Gn=2*n-1
         l=3*n+2*(x+y)
         b=2*n+(x+y)
         if  (Gm)**x+(Gn)**y==(n+1)**y :
             print(x,'\t',y,'\t',n,'\t', Gm,'\t',Gn,'\t',(l,b))
m =int(input("Enter the maximum range:"))
#m is the maximum range
rectangle()
```

Fig 1. Coding 1 (F or the equation $G_{n+1}^{x}+G_{n}^{x}=(n+1)^{y}$)

```
Enter the maximum range:500
x         y         n         Gm         Gn         (l,b)
0         1         1         3          1           (5, 3)
1         2         1         3          1           (9, 5)
>>>
```

Fig 2. Output for Coding 1

```
#ED rectangle over Gn
import math
def rectangle():
 print('x\ty\tn\tGm\tGn\t(l,b)')
 for x in range(0,m+1):
    for y in range (0,m+1):
        for n in range (0,m+1):
         Gm=2*n+1
         Gn=2*n-1
         l=3*n+2*(x+y)
         b=2*n+(x+y)
         if  (Gm)**x-(Gn)**y==(n+1)**y :
             print(x,'\t',y,'\t',n,'\t', Gm,'\t',Gn,'\t',(l,b))
m =int(input("Enter the maximum range:"))
#m is the maximum range
rectangle()
```

Fig 3. Coding 2 (For the equation $G_{n+1}^{x}-G_{n}^{x}=(n+1)^{y}$

```
Enter the maximum range:100
x         y         n         Gm         Gn         (l,b)
1         1         1         3          1           (7, 4)
2         3         1         3          1           (13, 7)
>>>
```

Fig 4. Output for Coding 3

4 Conclusion

In this work, we solved the exponential Diophantine equation with the help of Mihailescu's Theorem and Binomial expansion. Finally, we infer that there are only four Exponential Diophantine rectangles over G_n with sides $(5,3)$, $(9,5)$, $(7,4)$ and $(13,7)$. One may perform this type of work on different shapes and different type of exponential Diophantine equation. We are working on the application of the above concept in cryptography.

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