INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS

Received: 28-09-2024

Accepted: 21-10-2024

Published: 13-11-2024

Citation: Shanthi J, Gopalan MA (2024) Formulation of Special Pythagorean Triangles through Integer Solutions of the Hyperbola $y^2 = (k^2 + 2k)x^2 + 1$. Indian Journal of Science and Technology 17(41): 4307-4312. https://doi.org/10.17485/IJST/v17i41.3132

* **Corresponding author**.

shanthitharshi@gmail.com

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Formulation of Special Pythagorean Triangles through Integer Solutions of the Hyperbola $y^2 = (k^2 + 2k)x^2 + 1$

J Shanthi[1]*, M A Gopalan[2]

1 Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Bharathidasan University, Trichy, 620002, Tamil Nadu, India
2 Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy, 620 002, Tamil Nadu, India

Abstract

Objectives: The objective of this research paper is to formulate Pythagorean triangles, each of which satisfies the relation: Hypotenuse $= (k + 1)$ times the leg with even values added with unity, through employing the integer solutions to the Hyperbola $y^2 = (k^2 + 2k)x^2 + 1$. **Methods:** The sides of the Pythagorean triangle satisfying the requirement are obtained by suitably choosing its generators consisting of the integer solutions to the considered hyperbola. **Findings :** There are plenty of Pythagorean triangles satisfying the given characterization for each value of k > 0 in the binary quadratic equation given by $y^2 = (k^2 + 2k)x^2 + 1$.The characterizations of special pattern of the Pythagorean triangle for $k = 2$is illustrated. **Novelty:** From the integer solutions of a binary quadratic equation, namely, a hyperbola (Two-dimensional geometrical representation), it is possible to obtain integer solutions to a ternary quadratic equation, namely, Pythagorean triangle (Three-dimensional geometrical representation).

Keywords: Binary quadratic equation; Ternary quadratic equation; Hyperbola; Pythagorean triangle; Integer solutions

1 Introduction

The fascinating branch of mathematics is the theory of numbers wherein Pythagorean triangles have been a matter of interest to various mathematicians and to lovers of mathematics because it is a treasure house in which the search for many hidden connections is a treasure hunt. No wonder that Pythagorean triangles have a wealth of historical significance. It pushes the boundaries of unresolved mathematical problems presenting challenges to mathematicians over the course of centuries. A careful observer of patterns may note that there is a one-to-one correspondence between the polygonal numbers and the number of sides of the polygon. Apart from the above patterns we have some more fascinating patterns of numbers namely Jarasandha numbers, Nasty numbers, and Dhuruva numbers. For a rich variety of fascinating problems, one may refer to[1–10].

This communication exhibits Pythagorean triangles, each of which satisfies the relation: Hypotenuse= $(k+1)$ times the leg with even values added with unity, through employing the integer solutions to the Hyperbola $y^2=(k^2+2k)x^2+1$. The characterizations of a special pattern of the Pythagorean triangle for k=2 are illustrated.

2 Methodology

Consider the hyperbola given by

$$y^2=(k^2+2k)x^2+1, k>0 \tag{1}$$

The smallest positive integer solution to Equation (1) is

$$x_0=1, y_0=k+1$$

The general solution (x_n, y_n) to Equation (1) is given by

$$x_n=x_n(k)=\frac{1}{2\sqrt{k^2+2k}}g(n,k),$$

$$y_n=y_n(k)=\frac{1}{2}f(n,k), n=0,1,2,... \tag{2}$$

where

$$f(n,k)=[(k+1+\sqrt{k^2+2k})^{n+1}+(k+1-\sqrt{k^2+2k})^{n+1}],$$

$$g(n,k)=[(k+1+\sqrt{k^2+2k})^{n+1}-(k+1-\sqrt{k^2+2k})^{n+1}].$$

Observe that

$$y_n(k)+(k+1)x_n(k)>x_n(k)>0$$

Treat $y_n(k)+(k+1)x_n(k), x_n(k)$ as the generators ofthe Pythagorean triangle, whose legs are $X_n(k), Y_n(k)$and hypotenuse is $Z_n(k)$. Now, consider

$$X_n(k)=2x_n(k)[y_n(k)+(k+1)x_n(k)]$$

$$=2x_n(k)y_n(k)+2(k+1)x_n^2(k)$$

$$=2x_n(k)y_n(k)+2(k+1)x_n^2(k)$$

$$=\frac{1}{2\sqrt{k^2+2k}}f(n,k)g(n,k)+\frac{(k+1)}{2(k^2+2k)}g^2(n,k)$$

$$=\frac{1}{2(k^2+2k)}[\sqrt{k^2+2k}f(n,k)g(n,k)+(k+1)g^2(n,k)]$$

$$Y_n(k)=[y_n(k)+(k+1)x_n(k)]^2-x_n^2(k)$$

$$= [\frac{f(n,k)}{2} + \frac{(k+1)}{2\sqrt{k^2+2k}} g(n,k)]^2 - \frac{1}{4(k^2+2k)} g^2(n,k)$$

$$= \frac{1}{4(k^2+2k)} \left[(k^2+2k) f^2(n,k) + (k^2+2k+2) g^2(n,k) + 2(k+1)\sqrt{k^2} + 2k f(n,k) g(n,k) \right]$$

$$Z_n(k) = [y_n(k) + (k+1)x_n(k)]^2 + x_n^2(k)$$

$$= \frac{1}{4(k^2+2k)} [(k^2+2k) f^2(n,k) + (k^2+2k+2) g^2(n,k) + 2(k+1)\sqrt{k^2+2k} f(n,k) g(n,k)]$$

It is seen that

$$f^2(n,k) = g^2(n,k) + 4 \tag{3}$$

In view of Equation (3), we have

$$Z_n(k) = \frac{1}{4(k^2+2k)} [(k^2+2k)(g^2(n,k)+4) + (k^2+2k+2) g^2(n,k) + 2(k+1)\sqrt{k^2+2k} f(n,k) g(n,k)]$$

$$= \frac{1}{4(k^2+2k)} [(2k^2+4k+2) g^2(n,k) + 2(k+1)\sqrt{k^2+2k} f(n,k) g(n,k)] + 1$$

$$= \frac{1}{2(k^2+2k)} [(k^2+2k+1) g^2(n,k) + (k+1)\sqrt{k^2+2k} f(n,k) g(n,k)] + 1$$

Thus, the above Pythagorean triangle satisfies the relation

$$Z_n(k) = (k+1) X_n(k) + 1$$

For the benefit of the readers, the legs and hypotenuse of first three Pythagorean triangles are given below:

First Pythagorean triangle

$$X_0(k) = 4k + 4,$$

$$Y_0(k) = 4k^2 + 8k + 3,$$

$$Z_0(k) = 4k^2 + 8k + 5.$$

Second Pythagorean triangle

$$X_1(k) = 16k^3 + 48k^2 + 44k + 12,$$

$$Y_1(k) = 16k^4 + 64k^3 + 84k^2 + 40k + 5,$$

$$Z_1(k) = 16k^4 + 64k^3 + 92k^2 + 56k + 13.$$

Third Pythagorean triangle

$$X_2(k) = (16k^3 + 48k^2 + 40k + 8)(4k^2 + 8k + 3),$$

$$Y_2(k) = 64k^6 + 384k^5 + 880k^4 + 960k^3 + 504k^2 + 112k + 7,$$

$$Z_2(k) = 64k^6 + 384k^5 + 912k^4 + 1088k^3 + 680k^2 + 208k + 25.$$

The recurrence relations satisfied by $X_n(k), Y_n(k), Z_n(k)$ are respectively

$$X_{n+2}(k) - (4k^2 + 8k + 2)X_{n+1}(k) + X_n(k) = 4k + 4$$

$$Y_{n+2}(k) - (4k^2 + 8k + 2)Y_{n+1}(k) + Y_n(k) = 0,$$

$$Z_{n+2}(k) - (4k^2 + 8k + 2)Z_{n+1}(k) + Z_n(k) = 4, n = 0, 1, 2, ...$$

To analyse the characterizations of the above Pythagorean triangle, one has to take particular values to k in (1). For simplicity and clear understanding, we present below the sides of the Pythagorean triangle when $k = 2$ and exhibit a few interesting relations satisfied by the sides of the Pythagorean triangle for $k = 2$.

Substituting $k = 2$ in Equation (1), the corresponding hyperbola is

$$y^2 = 8x^2 + 1$$

whose general solution $(x_n(2), y_n(2))$is given by

$$x_n(2) = \frac{1}{2\sqrt{8}} g(n, 2),$$

$$y_n(2) = \frac{1}{2} f(n, 2), n = 0, 1, 2, ...$$

where

$$f(n, 2) = [(3 + \sqrt{8})^{n+1} + (3 - \sqrt{8})^{n+1}],$$

$$g(n, 2) = [(3 + \sqrt{8})^{n+1} - (3 - \sqrt{8})^{n+1}]$$

The legs $X_n(2), Y_n(2)$and hypotenuse $Z_n(2)$ of corresponding Pythagorean triangle are given by

$$X_n(2) = \frac{1}{16}[3g^2(n, 2) + \sqrt{8} f(n, 2) g(n, 2)],$$

$$Y_n(2) = \frac{1}{32}[8g^2(n, 2) + 8f^2(n, 2) + 6\sqrt{8} f(n, 2) g(n, 2)],$$

$$Z_n(2) = \frac{1}{32}[10g^2(n, 2) + 8f^2(n, 2) + 6\sqrt{8} f(n, 2) g(n, 2)].$$

The following Table 1 exhibits a few numerical examples of Pythagorean triangles:

Table 1. Pythagorean triangles

n	$X_n(2)$	$Y_n(2)$	$Z_n(2)$
0	12	35	37
1	420	1189	1261
2	14280	40391	42841
3	485112	1372105	1455337
4	16479540	46611179	49438621
5	559819260	1583407981	1679457781

The recurrence relations satisfied by $X_n(2), Y_n(2), Z_n(2)$ are respectively

$$X_{n+2}(2) - 34X_{n+1}(2) + X_n(2) = 12$$

$$Y_{n+2}(2) - 34Y_{n+1}(2) + Y_n(2) = 0,$$

$$Z_{n+2}(2) - 34Z_{n+1}(2) + Z_n(2) = 4, n = 0, 1, 2, ...$$

A few interesting relations among the sides of each of the above Pythagorean triangles are presented:

1. $96[Z_n(2) - Y_n(2)]$ $is\,six\,times\,a\,square\,multiple\,of\,an\,irrational\,number$
2. $96[Z_n(2) - Y_n(2)] + 24$ $is\,six\,times\,a\,square\,multiple\,of\,an\,integer$
3. $32[X_n(2) + 3Y_n(2) - 3Z_n(2)]^2 - 64[Z_n(2) - Y_n(2)] is\,fourth\,power\,of\,an\,irrational\,number$
4. $32[X_n(2) + 3Y_n(2) - 3Z_n(2)]^2 + 64[Z_n(2) - Y_n(2)] + 16\,is\,fourth\,power\,of\,an\,integer$
5. $9[8Z_n(2) - 8Y_n(2) + 1]^2 - 8[9Y_n(2) - 8Z_n(2) - 1]^2 = 9$
6. $9[48X_n(2) - 16Y_n(2) + 18]^2 - 2[36Y_n(2) - 32Z_n(2) - 4]^2 = 36$
7. $[8Z_n(2) - 8Y_n(2) + 1]^2 - 8[3Y_n(2) - 8X_n(2) - 3]^2 = 1$
8. $[24X_n(2) - 8Y_n(2) + 9]^2 - 2[16X_n(2) - 6Y_n(2) + 6]^2 = 1$
9. $72Y_n(2) - 56Z_n(2) + 24X_n(2) \equiv 16(mod 3)$
10. $20Z_n(2) - 16Y_n(2) - 12X_n(2) is\,a\,perfect\,square.$
11. $88Z_n(2) - 80Y_n(2) - 24X_n(2) + 12\,is\,a\,square\,multiple\,of\,5.$

In a similar manner, by taking different k values in Equation (1), one may obtain plenty of Pythagorean triangles with their corresponding relations among their sides.

3 Conclusion

The main thrust of this research paper is to formulate Pythagorean triangles, each of which satisfies the relation: Hypotenuse $= (k+1)$ times the leg with even values added with unity, through employing the integer solutions to the Hyperbola $y^2 = (k^2 + 2k)x^2 + 1$. The sides of the Pythagorean triangle satisfying the requirement are obtained by suitably choosing its generators consisting of the integer solutions to the considered hyperbola**.** As there are plenty of hyperbolas, one may consider other forms of hyperbolas, from whose integer solutions, obtain different patterns of Pythagorean triangles, each of which satisfies other types of characterizations. One may also perform similar analysis by taking pairs of Pythagorean triangles.

References

1) Vidhyalaksmi S, Mahalakshmi T, Gopalan MA. Pythagorean triangles with 2*A/P as Gopa- Vidh number. *International Journal of Research and Analytic Reviews*. 2019;6(2):59–63. Available from: https://ijrar.com/upload_issue/ijrar_issue_20543689.pdf.
2) Vidhyalaksmi S, Mahalakshmi T, Gopalan MA. Special Pythagorean triangle in Relation with Pronic Numbers. *International Journal of Mathematics Trends and Technology*. 2019;65(8):166–169. Available from: https://doi.org/10.14445/22315373/IJMTT-V65I8P519.
3) Vidhyalaksmi S, Mahalakshmi T, Gopalan MA. Special Pythagorean Triangle with 2*Area/ Perimeter + Hypotenuse- a leg as a pronic number. *Adalya Journal*. 2019;8(9):1052–1055. Available from: https://doi.org/10.16.10089.AJ.2019.V8I9.285311.6224.
4) Vidhyalaksmi S, Shanthi J, Mahalakshmi T, Gopalan MA. A Classification of Rectangles in Connection with Fascinating Number Patterns. *International Journal of Science and Research Methodology*. 2020;15(2):295–303. Available from: https://ijsrm.humanjournals.com/wp-content/uploads/2020/05/21.S.Vidhyalakshmi-J.Shanthi-T.Mahalakshmi-M.A.Gopalan.pdf.
5) Mallika S. Pythagorean triangle with 2A/P+H-Leg as a Narcisstic number of orders 3,4 and 5, . *Global Journal of Engineering Science And Researches (GJESR), .* 2019;6(3):1–4. Available from: https://www.gjesr.com/Issues%20PDF/Archive-2019/March-2019/1.pdf.

6) Mallika S. A connection between Pythagorean triangle and SphenicNumbers. *International Journal for Research in Applied Science & Engineering Technology*. 2019;7(3). Available from: https://www.ijraset.com/fileserve.php?FID=20176.
7) Devibala S, Gopalan MA. Pythagorean triangle with 2A/P as Gopa numbers of first kind . *International Journal of Research Publication and Reviews*. 2022;3(5):2472–2477. Available from: https://ijrpr.com/uploads/V3ISSUE5/IJRPR4138.pdf.
8) Devibala S, Gopalan MA. Pythagorean Triangle with 2*A/P as Gopa Numbers of the First Kind. *International Journal of Research Publication and Reviews*. 2022;3(5):2474–2477. Available from: https://ijrpr.com/uploads/V3ISSUE5/IJRPR4138.pdf.
9) Rajput C, Manjunath H. Pythagorean triangle with 2A/P as Gopanumbers of second kind. *International Research Journal of Engineering and Technology*. 2022;9(5):2040–2042. Available from: https://www.irjet.net/archives/V9/i5/IRJET-V9I5508.pdf.
10) Kavitha A, Triangle HCBP, Numbers. ,A connection between Pythagorean triangle and Harshad Numbers. *International Journal for Research in Applied Science & Engineering Technology*. 2019;7(3):91–101. Available from: https://www.ijraset.com/fileserve.php?FID=20219.