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RESEARCH ARTICLE

Solving Fuzzy Delay Differential Equations using Fuzzy Elzaki Transform and Fuzzy Elzaki Decomposition Method

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Abstract

Objective: In this work, the fuzzy Elzaki transform and fuzzy Elzaki decomposition method are used to solve fuzzy delay differential equations. The fuzzy Elzaki decomposition method is a combination of the fuzzy Elzaki transform and Adomian decomposition method in a fuzzy environment. The use of transforms in solving differential equations makes the solution process simpler. **Methodology:** The fuzzy Elzaki transform is applied in the fuzzy delay differential equation. The nonlinear terms are decomposed using Adomian polynomials in a nonlinear fuzzy delay differential equation. In the transformed domain, the resulting algebraic equation is solved. The solution in the original time domain is obtained by applying the inverse Elzaki transform. **Findings:** The fuzzy Elzaki transform and fuzzy Elzaki decomposition method are useful for handling the complexities associated with fuzzy functions and delays in differential equations. The technique used to solve the fuzzy delay differential equation gives results with better accuracy compared to the exact solution. The applicability of the method is illustrated with numerical examples. **Novelty:** The Elzaki transform is a helpful tool for solving delay differential equations with discontinuities. For the resilient solution of complicated differential equations, especially those including nonlinearity and fuzzy logic, the fuzzy Elzaki decomposition approach is useful. It gives an organized way to find the solution to fuzzy delay differential equation by utilizing the advantages of both the Elzaki transform and the Adomian decomposition method in a fuzzy situation.

Keywords: Triangular Fuzzy Number; Fuzzy Elzaki Transform; Adomian decomposition method; Fuzzy Elzaki decomposition method; Fuzzy Delay Differential Equations (FDDE)

1 Introduction

The fuzzy delay differential equation (FDDE) is a differential equation where some of the parameters or variables are characterized by fuzzy sets instead of precise numerical values. Hamza et.al[1] and Jameel A.F et. al[2] have developed numerical methods to solve linear fuzzy delay differential equations. Yookesh et.al[3] utilized the variational iteration method to solve time delay differential equations under uncertainity conditions. Pantograph type equation is a kind of delay differential equation in which the delay term is of the form y(αt) where 0 < α < 1. Jafari H Mahmoudi et.al[4] proposed a new numerical method to solve pantograph delay differential equations with convergence analysis. Transforms are powerful tools used to solve differential equations by converting them into algebraic equations. The combination of the Sumudu transform and the Adomian Decomposition method creates the Sumudu Decomposition Method, which is used to solve fuzzy delay differential equations. Abdul Rahman[5] proposed fuzzy Sumudu decomposition method for fuzzy delay differential equations with strongly generalised differentiability. The Elzaki transform has similarities to the other transforms but provides different benefits depending on the specific problem being addressed. Rehab Ali Khudair et.al[6] used fuzzy Elzaki transform to solve the vibrating string equation. Archana et.al[7] utilized Elzaki decomposition method for the Mathematical modelling of the atmospheric internal waves phenomenon and its solution. Emad et.al[8] obtained an approximate solution to Non-Linear Schrodinger Equation with Harmonic Oscillator by the Elzaki Decomposition method. Savla SL and Sharmila RG[9,10] proposed the Shehu Adomian Decomposition method to solve linear and nonlinear fuzzy fractional Volterra -Fredholm integro differential equations and to solve fuzzy fractional biological population model. Ali, L., Zou, G., Li, N. et al.[11] proposed Analytical treatments of time-fractional seventh-order nonlinear equations via Elzaki transform.

In this study, the fuzzy Elzaki transform is employed to solve linear fuzzy delay differential equations, and the Fuzzy Elzaki decomposition method is proposed for solving nonlinear fuzzy delay differential equations. The numerical algorithm for these methods is developed, and numerical examples are provided to verify their accuracy in solving FDDEs.

The structure of the paper is as follows: Section 2 is methodology which consists of definitions and basic concepts required for the solution process and the numerical algorithm to solve linear and nonlinear fuzzy delay differential equations. Section 3 illustrates the numerical examples to show the results and accuracy of the method. Section 4 gives the conclusion.

2 Methodology

2.1. Fuzzy Number

A fuzzy number u in parametric form is a pair $\underline{u}(\alpha), \bar{u}(\alpha)$ of functions $\underline{u}(\alpha), \bar{u}(\alpha), 0 \leq \alpha \leq 1$, that meets the requirements as follows:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in (0,1] while right continuous at 0.
2. $\bar{u}(\alpha)$ is a bounded non-increasing left continuous function in (0,1] and right continuous at 0.
3. $\underline{u}(\alpha) \leq \bar{u}(\alpha), \ 0 \leq \alpha \leq 1$

2.2 Triangular Fuzzy Number

A fuzzy set is $\bar{A}$ the triangular fuzzy number with peak a, left width $\alpha > 0$ and right $\beta > 0$ if the membership function has the form as follows:

$$\mu(x) = \begin{cases} 1 - \frac{a-x}{a}, & if \quad a-\alpha < x < a, \\ 1 - \frac{x-a}{\beta}, & if \quad a < x < a+\beta, \\ 0, & otherwise \end{cases}$$

2.3 Fuzzy Elzaki Transform

The Elzaki Transform is denoted by the operator E(.) defined by the integral equations

$$E[f(x)] = T(v) = v\int_0^\infty f(x)\, e^{-x/v}\, dx, x \geq 0, k_1 \leq v \leq k_2$$

Some properties of Elzaki Transform

$a.\ E[1] = r^2$
$b.\ E[x^n] = n!\, r^{n+2}$
$c.\ E^{-1}[r^{n+2}] = \frac{x^n}{n!}$

Table 1. Standard Elzaki Transform for some special functions

Special functions	Elzaki Transform Equivalent
f(x)	E[f(x)]=T(r)
$\frac{x^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha>0$	$r^{\alpha+1}$
$\frac{x^{n-1}e^{\alpha x}}{(n-1)!}$, n=1,2,3…	$\frac{r^{n+1}}{(1-\alpha r)^{n}}$
$sin\alpha x$	$\frac{\alpha r^3}{1+\alpha^2 r^2}$
$cos\alpha x$	$\frac{r^2}{1+\alpha^2 r^2}$
$sinh\alpha x$	$\frac{\alpha r^3}{1-\alpha^2 r^2}$
$Cos\, h\alpha x$	$\frac{\alpha r^2}{1-\alpha^2 r^2}$
$e^{\alpha x}$	$\frac{-(1-\alpha r)}{r}$

The fuzzy Elzaki Transformation is denoted by

$$E\left(f\left(x\right)\right]=\left(e\left[\underline{f}\left(x\right)\right],\, e\left[\overline{f}\left(x\right)\right]\right)$$

where,

$$e\left[\underline{f}\left(x\right)\right]=v\int_{0}^{\infty}\underline{f}\left(x\right)e^{-\frac{x}{v}}\,dx$$

$$e\left(\overline{f}\left(x\right)\right]=v\int_{0}^{\infty}\overline{f}\left(x\right)e^{-\frac{x}{v}}\,dx$$

2.3.1. *Linearity Property*

Let $u,v:R\to E(R)$ be two continuous fuzzy functions, Assume that b_1 and b_2 are random constants then $E\left[b_1u\left(x\right)+b_2u\left(x\right)\right]=b_1E(u\left(x\right))+\,b_2E(u\left(x\right))$

2.3.2. *Change of Scale Property*

Consider $f:R\to E(R)$ be a continuous fuzzy function and b is a random constant then $E\left[f\left(bx\right)\right]=y(bx)$

2.4 The derivatives to fuzzy Elzaki Transform

$$E\left[\tilde{f}'\left(x\right)\right]=E\left[\underline{f}'\left(x\right),\overline{f}'\left(x\right)\right]=(e[\underline{f}'\left(x\right)],e[\overline{f}'\left(x\right)])=(\frac{1}{r}T\left(r\right)-r\underline{f}\left(0\right),\frac{1}{r}T\left(r\right)-r\overline{f}\left(0\right))$$

$$E\left[\tilde{f}"\left(x\right)\right]=E\left[\underline{f}''\left(x\right),\overline{f}''\left(x\right)\right]=e[\underline{f}''\left(x\right)],e[\overline{f}''\left(x\right)])=(\frac{1}{r^2}T\left(r\right)-r\underline{f}'^{(0)}-\underline{f}\left(0\right),\frac{1}{r^2}T\left(r\right)-r\overline{f'}\left(0\right)-r\overline{f}\left(0\right))$$

$$E\left[\widetilde{f}^{n}\left(x\right)\right]=E\left[\underline{f}^{n}\left(x\right),\overline{f}^{n}\left(x\right)\right]=(e[\underline{f}^{n}\left(x\right)],e[\overline{f}^{n}\left(x\right)])=(\tfrac{T(r)}{r^n}-\textstyle\sum_{k=0}^{n-1}r^{2-n+m}\underline{f}^{(m)}\left(0\right)),$$
$$(\tfrac{T(r)}{r^n}-\textstyle\sum_{k=0}^{n-1}r^{2-n+m}\overline{f}^{(m)}\left(0\right)$$

2.5 Numerical Algorithm for solving nonlinear Fuzzy Delay Differential Equations

In this section, the required procedure is discussed to solve FDDE using non-linear Fuzzy Elzaki Decomposition method. This method is a combination of the Fuzzy Elzaki Transform and the Adomian Decomposition method. Consider

$$\frac{d\widetilde{f}}{dx}+\widetilde{R}\left(f\right)+\widetilde{N}\left(x-\tau\right)=\tilde{y}(x) \tag{1}$$

with initial condition

$$\tilde{f}(0) = [\underline{f}_0, \overline{f}_0] \tag{2}$$

where $\tilde{f} = f(x)$, $\widetilde{R}$ is linear Fuzzy operator, $\tilde{y}(x)$ a continuous function, $\widetilde{N}$ is a nonlinear fuzzy operator and $\frac{d\tilde{f}}{dx}$ the first order fuzzy derivative. Taking Fuzzy Elzaki Transform on both sides,

$$E\left[\frac{d\tilde{f}}{dx}\right] + E\left[\widetilde{R}(f)\right] + E\left[\widetilde{N}(x-r)\right] = E[y(x)] \tag{3}$$

The parametric representation of the above equation is

$$\begin{gathered}\frac{E\left[\underline{f}(x;\alpha)\right] - r^2\underline{f}_0}{r} + E\left[\underline{R}(f)\right] + E\left[\underline{N}(x-r)\right] = E\left[\underline{y}(x);\alpha\right] \\ \frac{E\left[\overline{f}(x;\alpha)\right] - r^2\overline{f}_0}{r} + E\left[\overline{R}(f)\right] + E\left[\overline{N}(x-r)\right] = E\left[\overline{y}(x);\alpha\right]\end{gathered} \tag{4}$$

and we obtain

$$\begin{gathered}E\left[\underline{f}(x)\right] - r^2\underline{f}_0 + rE\left[\underline{R}(f)\right] + rE\left[\underline{N}(x-r)\right] = rE\left[\underline{y}(x)\right] \\ E\left[\overline{f}(x)\right] - r^2\overline{f}_0 + rE\left[\overline{R}(f)\right] + rE\left[\overline{N}(x-r)\right] = rE\left[\overline{y}(x)\right]\end{gathered} \tag{5}$$

Then we define

$$\begin{gathered}\underline{f}(x) = \sum_{n=0}^{\infty} \underline{f}_n(x) \\ \overline{f}(x) = \sum_{n=0}^{\infty} \overline{f}_n(x)\end{gathered} \tag{6}$$

The fuzzy non-linear operator can be further decomposed as follows

$$\begin{gathered}\underline{N}(x-\tau) = \sum_{n=0}^{\infty} \underline{A}_n(x) \\ \overline{N}(x-\tau) = \sum_{n=0}^{\infty} \overline{A}_n(x)\end{gathered} \tag{7}$$

Where $[\underline{A}_n, \overline{A}_n]$ is the fuzzy Adomain polynomial of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \ldots. \tilde{f}_n$

$$\begin{gathered}\underline{A}_n = \frac{1}{n!}\frac{d}{d\lambda}\left[\underline{N}\sum_{n=0}^{\infty} \lambda^n \underline{f}_n\right] \\ \overline{A}_n = \frac{1}{n!}\frac{d}{d\lambda}\left[\overline{N}\sum_{n=0}^{\infty} \lambda^n \overline{f}_n\right]\end{gathered} \tag{8}$$

At instance,

$$\begin{gathered}\underline{A}_0 = \underline{y}(f_0) \\ \overline{A}_0 = \overline{y}(f_0) \\ \underline{A}_1 = \underline{y}'(f_0) \\ \overline{A}_1 = \overline{y}'(f_0)\end{gathered} \tag{9}$$

Substituting Equations (6), (7), (8) and (9) in Equation (5)

$$E\left[\sum_{n=0}^{\infty} \underline{f}_n(x;\alpha)\right] = r^2\underline{f}_0 - rE\left[\underline{R}\sum_{n=0}^{\infty} \underline{f}_n(x;\alpha)\right] - rE\left[\sum_{n=0}^{\infty} \underline{A}_n(x;\alpha)\right] + rE\left[\underline{y}(x;\alpha)\right],$$

$$E\left[\sum_{n=0}^{\infty}\overline{f}_n(x;\alpha)\right]=r^2\overline{f}_0-rE\left[\overline{R}\sum_{n=0}^{\infty}\underline{f}_n(x;\alpha)\right]-rE\left[\sum_{n=0}^{\infty}\overline{A}_n(x;\alpha)\right]+rE\left[\overline{y}(x;\alpha)\right],$$

Comparing coefficients of $\overline{f}$, we obtain

$$E\left[\underline{f}_0(x;\alpha)\right]=r^2\underline{f}_0(x;\alpha)+rE\left[\underline{y}(x;\alpha)\right]$$

$$E\left[\overline{f}_0(x;\alpha)\right]=r^2\overline{f}_0(x;\alpha)+rE\left[\overline{y}(x;\alpha)\right]$$

$$E\left[\underline{y}_n(x;\alpha)\right]=-rE\left[\underline{R}f_{n-1}(x;\alpha)\right]-rE\left[\underline{A}_{n-1}(x;\alpha)\right]$$

$$E\left[\overline{y}_n(x;\alpha)\right]=-rE\left[\overline{R}f_{n-1}(x;\alpha)\right]-rE[\overline{A}_{n-1}(x;\alpha)]$$

Applying inverse Fuzzy Elzaki Transform

$$\underline{f}_0(x;\alpha)=E^{-1}\left[r^2\underline{f}_0(x;\alpha)\right]+E^{-1}\left[rE\underline{y}(x;\alpha)\right]$$

$$\overline{f}_0(x;\alpha)=E^{-1}[r^2\overline{f}_0(x;\alpha)]+E^{-1}\left[rE\overline{y}(x;\alpha)\right]$$

$$\underline{f}_n=-E^{-1}rE\left[\underline{R}f_{n-1}(x;\alpha)\right]+rE\left[\underline{A}_{n-1}(x;\alpha)\right]$$

$$\overline{f}_n=-E^{-1}rE\left[\overline{R}f_{n-1}(x;\alpha)\right]+rE\left[\overline{A}_{n-1}(x;\alpha)\right]$$

Hence for $n>0$ the series of solutions will be obtained.

3 Results and Discussion

Numerical Example

In this section, numerical examples are solved using the Fuzzy Elzaki transform method for linear and non-linear fuzzy delay differential equations.

Example 1: Consider the linear delay differential equation

$$\widetilde{f}'(x)=\frac{1}{2}e^{\frac{x}{2}}\widetilde{f}\left(\frac{x}{2}\right)+\frac{1}{2}\widetilde{f}(x) \tag{11}$$

Subject to initial conditions $y(0)=(\alpha,2-\alpha)$

On using a triangular fuzzy number, the Exact solution of Equation (11)

$$\underline{f}(x,\alpha)=\alpha e^x$$

$$\overline{f}(x,\alpha) = (2-\alpha)e^{x}$$

Applying the Elzaki transform on both sides of Equation (11)

$$\begin{aligned} &E[\underline{f}(x) = r^2\underline{f}(0) + \frac{1}{2}rE[e^{0.5x}\underline{f}\left(\frac{x}{2}\right) + \underline{f}(x)] \\ &E[\underline{f}(x) = r^2\alpha + \frac{1}{2}rE[e^{0.5x}\underline{f}\left(\frac{x}{2}\right) + \underline{f}(x)] \end{aligned} \tag{12}$$

$$\underline{f}(x) = \alpha + \frac{1}{2}E^{-1}\left[rE\left[e^{0.5x}\underline{f}\left(\frac{x}{2}\right) + \underline{f}(x)\right]\right] \tag{13}$$

$$\begin{aligned} &E\left(\overline{f}(x)\right] = r^2\overline{f}(0) + \frac{1}{2}rE\left[e^{0.5x}\overline{f}\left(\frac{x}{2}\right) + \overline{f}(x)\right] \\ &E[\overline{f}(x) = r^2\alpha + \frac{1}{2}rE[e^{0.5x}\overline{f}\left(\frac{x}{2}\right) + \overline{f}(x)] \end{aligned} \tag{14}$$

$$\overline{f}(x) = \alpha + \frac{1}{2}E^{-1}\left[rE\left[e^{0.5x}\overline{f}\left(\frac{x}{2}\right) + \overline{f}(x)\right]\right] \tag{15}$$

By the Elzaki transform method,

$$\underline{f}_0(x) = \alpha,\ \overline{f}_0(x) = (2-\alpha)$$

$$\begin{aligned} &\underline{f}_{n+1}(x) = \frac{1}{2}E^{-1}\left[rE\left[e^{0.5x}\underline{f}_n\left(\frac{x}{2}\right) + \underline{f}_n(x)\right]\right] n \geq 0 \\ &\overline{f}_{n+1}(x) = \frac{1}{2}E^{-1}\left[rE\left[e^{0.5x}\overline{f}_n\left(\frac{x}{2}\right) + \overline{f}_n(x)\right]\right] n \geq 0 \end{aligned} \tag{16}$$

for n = 0,

$$\underline{f}_1(x) = \frac{1}{2}E^{-1}\left[ar\left[e^{0.5x}\right] + \alpha.r\right] \tag{17}$$

$$\begin{aligned} &\overline{f}_1(x) = \frac{1}{2}E^{-1}\left[rE\left[(2-\alpha)e^{0.5x} + (2-\alpha)\right]\right] \\ &= \frac{1}{2}E^{-1}\left[\left[(2-\alpha)rE[e^{0.5x}] + (2-\alpha)\right].r\right] \end{aligned} \tag{18}$$

But,

$$E\left[e^{0.5x}\right] = \frac{r^2}{1-0.5r} = r^2 + 0.5r^3 + 0.25r^4 + 0.125r^5 + \ldots$$

Hence

$$\underline{f}_1 = \frac{1}{2}E^{-1}[\alpha\left[r^3 + 0.5r^4 + 0.25r^5 + 0.125r^6 + \cdots + r^2\right]]$$

$$\overline{f}_1(x) = \frac{1}{2}E^{-1}[(2-\alpha)\left[r^3 + 0.5r^4 + 0.25r^5 + 0.125r^6 + \cdots + r^2\right]]$$

By definition, we have

$$\underset{-1}{f}(x)=\left[\alpha\left(x+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\ldots\right)\right]$$

$$\overline{f}_1(x)=[(2-\alpha)(x+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\ldots)]$$

$$\underset{-2}{f}(x)=\left[\alpha\left(\frac{3}{8}x^2+\frac{13}{192}x^3+\frac{13}{1024}x^4+\ldots\right)\right]$$

$$\overline{f}_2(x)=\left[(2-\alpha)\left(\frac{3}{8}x^2+\frac{13}{192}x^3+\frac{13}{1024}x^4+\ldots\right)\right]$$

$$\underset{-3}{f}(x)=\left[\alpha\left(\frac{5}{64}x^3+\frac{63}{4096}x^4+\ldots\right)\right]$$

$$\overline{f}_3(x)=\left[(2-\alpha)\left(\frac{5}{64}x^3+\frac{63}{4096}x^4+\ldots\right)\right]$$

Thus the approximate becomes

$$\underset{-}{f}(x)=\sum_{n=0}^{\infty}\underset{-n}{f}(x)=\underset{-0}{f}(x)+\underset{-1}{f}(x)+\underset{-2}{f}(x)+\underset{-3}{f}(x)+\ldots$$

$$=\left[\alpha\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)\right]=\alpha e^x \tag{19}$$

$$\overline{f}(x)=\sum_{n=0}^{\infty}\overline{f}_n(x)=\overline{f}_0(x)+\overline{f}_1(x)+\overline{f}_2(x)+\overline{f}_3(x)+\ldots$$

$$=\left[(2-\alpha)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)\right]=(2-\alpha)e^x \tag{20}$$

Hence for $n\geq 0$ the solution converges rapidly to the exact solution on using triangular fuzzy number. The Figure 1 gives the graphical representation of the solution obtained by the fuzzy. Elzaki transform to solve linear fuzzy delay differential equation. The Equations (19) and (20) shows that the solution converges to the exact solution.

Example 2: consider a nonlinear fuzzy delay differential equation

$$\tilde{f}'(x)=1-2\tilde{f}^2\left(\frac{x}{2}\right) \tag{21}$$

Subject to initial conditions $\tilde{f}(0)=(\alpha-1,1-\alpha)$

Applying fuzzy Elzaki Transform on both sides,

$$E[\tilde{f}'(x])=E\left[1-2\tilde{f}^2\left(\frac{x}{2}\right)\right] \tag{22}$$

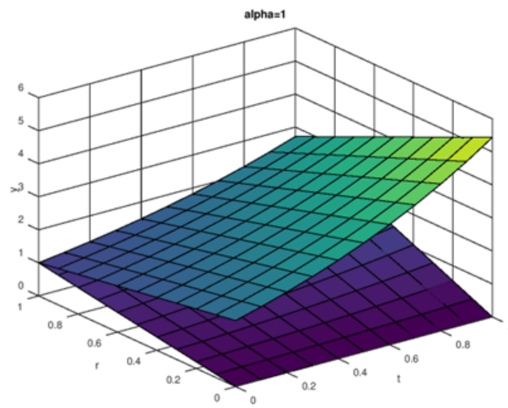

Fig 1. Solution of Linear FDDE

$$E\left[\widetilde{f'}(x)\right] = E(1) - 2E\left[\tilde{f}^2\left(\frac{x}{2}\right)\right]$$

$$E\left[\underline{f}'(x)\right] = E(1) - 2E\left[\underline{f}^2\left(\frac{x}{2}\right)\right]$$

$$E\left[\overline{f}'(x)\right] = E(1) - 2E\left[\overline{f}^2\left(\frac{x}{2}\right)\right]$$

$$\frac{E\left[\underline{f}(x)\right] - r^2\underline{f}(0)}{r} = E(1) - 2E\left[\underline{f}^2\left(\frac{x}{2}\right)\right]$$

$$\frac{E\left[\overline{f}(x)\right] - r^2\overline{f}(0)}{r} = E(1) - 2E\left[\overline{f}^2\left(\frac{x}{2}\right)\right]$$

Rearranging the terms

$$E\left[\underline{f}(x)\right] = r^2(\alpha - 1) + rE(1) - 2rE\left[\underline{f}^2\left(\frac{x}{2}\right)\right]$$

$$E\left[\overline{f}(x)\right] = r^2(1-\alpha) + rE(1) - 2rE\left[\overline{f}^2\left(\frac{x}{2}\right)\right]$$

Taking inverse transform on both sides,

$$\underline{f}(x) = (\alpha - 1) + x - 2rE\left[\underline{f}^2\left(\frac{x}{2}\right)\right]$$

$$\overline{f}(x) = (1-\alpha) + x - 2rE\left[\overline{f}^2\left(\frac{x}{2}\right)\right] \quad (23)$$

In general,

$$\underset{-0}{f}(x)=(\alpha-1)+x,\ \overline{f}(x)=(1-\alpha)+x$$

$$\underset{-n-1}{f}(x)=-2E^{-1}\left[rE\left[A_n\left(\frac{x}{2}\right)\right)\right]$$

where,

$$\underset{-n}{A}\left(\frac{x}{2}\right)=\sum_{r=0}^{n}\underset{-r}{f}\left(\frac{x}{2}\right)+\underset{-n-r}{f}\left(\frac{x}{2}\right) \tag{24}$$

$$\overline{f}_{n-1}(x)=-2E^{-1}\left[rE\left[A_n\left(\frac{x}{2}\right)\right)\right]$$

where,

$$\overline{A}_n\left(\frac{x}{2}\right)=\sum_{r=0}^{n}\overline{f}_r\left(\frac{x}{2}\right)+\overline{f}_{n-r}\left(\frac{x}{2}\right) \tag{24}$$

For $n=0$,

$$\underset{-0}{A}\left(\frac{x}{2}\right)=(\alpha-1)^2+\frac{x^2}{4}+2(\alpha-1)\left(\frac{x}{2}\right)$$

$$\overline{A}_0\left(\frac{x}{2}\right)=(1-\alpha)^2+\frac{x^2}{4}+2(1-\alpha)\left(\frac{x}{2}\right) \tag{25}$$

$$\underset{-1}{f}(x)=-2\left[(\alpha-1)^2x+(\alpha-1)\frac{x^2}{2}+\frac{x^3}{12}\right]$$

$$\overline{f}_1(x)=-2\left[(1-\alpha)^2x+(1-\alpha)\frac{x^2}{2}+\frac{x^3}{12}\right]$$

For $n=1$,

$$\underset{-1}{A}\left(\frac{x}{2}\right)=-\left[(\alpha-1)^3 2x-(\alpha-1)^2\frac{3x^2}{4}-(\alpha-1)\left(\frac{7x^3}{24}\right)-\frac{x^4}{48}\right]$$

$$\overline{A}_1\left(\frac{x}{2}\right)=-\left[(1-\alpha)^3 2x-(1-\alpha)^2\frac{3x^2}{4}-(1-\alpha)\left(\frac{7x^3}{24}\right)-\frac{x^4}{48}\right] \tag{26}$$

$$\underset{-2}{f}(x)=2(\alpha-1)^3x^2+(\alpha-1)^2x^3+7(\alpha-1)\frac{x^4}{48}+\frac{x^5}{120}$$

$$\overline{f}_2(x)=2(1-\alpha)^3x^2+(1-\alpha)^2x^3+7(1-\alpha)\frac{x^4}{48}+\frac{x^5}{120}$$

For $n=2$,

$$\underset{-2}{A}\left(\frac{x}{2}\right)=-2(\alpha-1)^4x^2+\frac{5}{4}(\alpha-1)^3x^3+\frac{95}{384}(\alpha-1)^2x^4+\frac{77}{3840}(\alpha-1)^5+\frac{8x^6}{11520}$$

$$\overline{A}_2\left(\frac{x}{2}\right) = -2(1-\alpha)^4x^2 + \frac{5}{4}(1-\alpha)^3x^3 + \frac{95}{384}(1-\alpha)^2x^4 + \frac{77}{3840}(1-\alpha)^5 + \frac{8x^6}{11520} \tag{27}$$

$$\underline{f}_3(x) = -\left[4(\alpha-1)^4\frac{x^3}{3!} + 15(\alpha-1)^3\frac{x^4}{4!} + \frac{95}{12}(\alpha-1)^2\frac{x^5}{5!} + \frac{77}{16}(\alpha-1)\frac{x^6}{6!} + \frac{x^7}{7!}\right]$$

$$\overline{f}_3(x) = -\left[4(1-\alpha)^4\frac{x^3}{3!} + 15(1-\alpha)^3\frac{x^4}{4!} + \frac{95}{12}(1-\alpha)^2\frac{x^5}{5!} + \frac{77}{16}(1-\alpha)\frac{x^6}{6!} + \frac{x^7}{7!}\right]$$

The Fuzzy series solution is as follows

$$\underline{f}(x) = \sum_{n=0}^{\infty}\underline{f}_n(x) = \underline{f}_0(x) + \underline{f}_1(x) + \underline{f}_2(x) + \underline{f}_3(x) + \ldots$$

$$\underline{f}(x) = (\alpha-1) + x - 2\left[(\alpha-1)^2x + (\alpha-1)\frac{x^2}{2} + \frac{x^3}{12}\right] + 2(\alpha-1)^3x^2 + (\alpha-1)^2x^3$$

$$+7(\alpha-1)\frac{x^4}{48} + \frac{x^5}{120} - \left[4(\alpha-1)^4\frac{x^3}{3!} + 15(\alpha-1)^3\frac{x^4}{4!} + \frac{95}{12}(\alpha-1)^2\frac{x^5}{5!} + \frac{77}{16}(\alpha-1)\frac{x^6}{6!} + \frac{x^7}{7!}\right] + \ldots \tag{28}$$

$$\overline{f}(x) = \sum_{n=0}^{\infty}\overline{f}_n(x) = \overline{f}_0(x) + \overline{f}_1(x) + \overline{f}_2(x) + \overline{f}_3(x) + \ldots$$

$$\overline{f}(x) = (1-\alpha) + x - 2\left[(1-\alpha)^2x + ((1-\alpha))\frac{x^2}{2} + \frac{x^3}{12}\right] + 2(1-\alpha)^3x^2 + (1-\alpha)^2x^3$$

$$+7(1-\alpha)\frac{x^4}{48} + \frac{x^5}{120} - \left[4(1-\alpha)^4\frac{x^3}{3!} + 15(1-\alpha)^3\frac{x^4}{4!} + \frac{95}{12}(1-\alpha)^2\frac{x^5}{5!} + \frac{77}{16}(1-\alpha)\frac{x^6}{6!} + \frac{x^7}{7!}\right] + \ldots \tag{29}$$

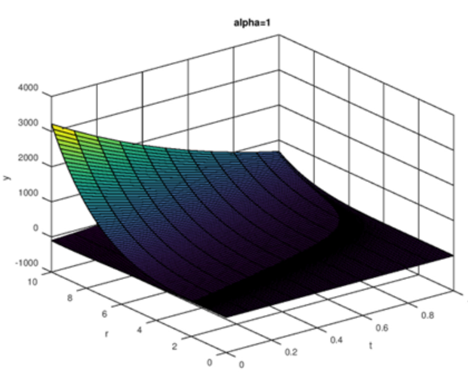

Fig 2. Nonlinear FDDE

Equations (28) and (29) show the series of solutions obtained by the fuzzy Elzaki decomposition method to solve the nonlinear fuzzy delay differential equation.

The solution for nonlinear fuzzy delay differential equation obtained by the fuzzy Elzaki decomposition method is verified with the solution obtained using the fuzzy Sumudu decomposition method proposed by Abdul Rahman [5].

4 Conclusion

In this study, the fuzzy Elzaki transform and the fuzzy Elzaki decomposition method are proposed to find the solution of linear and nonlinear fuzzy delay differential equations respectively. The fuzzy Elzaki transform is a novel approach to handle discontinuities in a delay differential equation with fuzzy initial conditions. Figures 1 and 2 illustrate the graphical representation of the solutions derived from the proposed method for linear and nonlinear FDDE respectively. The solution obtained by the proposed method rapidly converges to the exact solution. In future, the proposed methods can be used to solve different kinds of delay differential equations with fuzzy initial conditions and it can also be used to solve the problems on applications of FDDE.

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