INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Chromatic M-polynomial of Mobius Function Graphs of Product Groups

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OPEN ACCESS

Received: 14-09-2024

Accepted: 16-10-2024

Published: 16-11-2024

Citation: Jose R, Susha D (2024) Chromatic M-polynomial of Mobius Function Graphs of Product Groups. Indian Journal of Science and Technology 17(41): 4332-4337. https://doi.org/10.17485/IJST/v17i41.2959

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objective: To investigate the chromatic M-polynomials of the Mobius function graph of product groups, specifically focusing on the direct product of finite groups. **Methods**: We used algebraic and number theoretical techniques to construct a Mobius function graph of product groups and derived the chromatic M-polynomials using graph theoretic methods. **Findings**: Chromatic M polynomials of Mobius function graphs are introduced with a focus on deriving their significant properties and providing a practical approach for computation. **Novelty**: This research introduces a novel approach to studying the chromatic M-polynomials of the Mobius function graph of product groups, providing insights into the interplay between group theory, graph theory, and number theory. Our findings contribute connection between chromatic M polynomials and coloring problems, followed by a discussion and understanding of graph polynomials.

Keywords: Mobius function graph of groups; Chromatic number; Chromatic M-polynomials; Product group; Dihedral Groups

1 Introduction

Let G be a graph on n vertices, V(G) be the vertex set of G and $\{c_1, c_2, c_3, \ldots, c_m\} \in C$. A proper coloring of G is an assignment ϕ: V(G) $\rightarrow$ C such that any two adjacent vertices have distinct colors. The minimum number of colors needed to properly colour its vertices is called its chromatic number and is denoted as $\chi(G)$. Let χ (G) = m, we define a function ς:V(G) $\rightarrow$\{1, 2, 3, …, m\} such that ς (v_i) = s if and only if ϕ (v_i) = c_s; $c_s \in C$.

The scheme of coloring the vertices of G, where c_1 is assigned to the maximum possible number of vertices, then c_2 is assigned to the maximum possible number of remaining uncolored vertices, and proceeding in this manner until all vertices are colored, is called a χ^- coloring of G. In a similar manner, if c_m is assigned to the maximum possible number of remaining uncolored vertices, then c_{m-1} is assigned to the maximum possible number of remaining uncolored vertices and proceeds in this manner until all vertices are colored, then such a coloring is called a χ^+ coloring of G.

For graph coloring and for all terms and definitions not defined specifically in this paper, we refer to [1,2]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite, and connected.

2 Methodology

The concept of chromatic polynomials of graphs together with the minimal and maximal parameter coloring was introduced in [3]. The chromatic M-polynomial is defined in [4] and a study of the chromatic M-polynomial of graph products is found in [5]. We begin with the definition of the chromatic M-polynomial of a graph G.

Definition 2.1: [4] Let G be a connected graph with a chromatic number χ (G). Then, the chromatic M- polynomial of G, denoted by Mχ (G, x, y) is defined as Mχ (G, x, y) = $\sum_{u,v\in V(G)} M_{i,j}x^iy^j$, where $M_{i,j}$ is the number of edges in G whose one end vertex has the colour c_i and the other end vertex has the colour c_j with respect to the given χ coloring of G.

Definition 2.2: [4] Let G be a connected graph with a chromatic number χ. Then,

1. the χ-chromatic M- polynomial of G, denoted by Mχ-(G, x, y) is defined as

Mχ-(G, x, y) = $\sum_{u,v\in V(G)} {M^-}_{i,j}\, x^iy^j$, where ${M^-}_{i,j}$ is the number of edges in G whose one end vertex has the colour c and the other end vertex has the colour c with respect to the given χ-coloring of G.

2. the $\chi+$ chromatic M- polynomial of G, denoted by Mχ^+(G, x, y) is defined as

Mχ^+(G, x, y) = $\sum_{u,v\in V(G)} {M^+}_{i,j}\, x^iy^j$, where ${M^+}_{i,j}$ is the number of edges in G whose one end vertex has the colour c and the other end vertex has the colour c with respect to the given $\chi+$coloring of G.

The Mobius function graph of a finite group is defined in [6,7]. Some results, which are relevant to our study, are stated below:

Definition 2.3: [6,7] The Mobius function graph of a finite group G (denoted by M(G)) is a simple graph whose vertex set is the same as the elements of G and any two distinct vertices a and b are adjacent in M(G) if and only if $\mu(|a||b|) = \mu(|a|)\mu(|b|)$.

3 Results and Discussion

Here in this section, we study about the Mobius function graph of various product groups. For any group G and H, its product group is G $\times$H= {(g, h): g $\in$G, h $\in$ H}. The order of G $\times$H is the product of orders of G and H. Also, the order of element (g, h) is the least multiple (LCM) of orders of g and h.

In the upcoming theorems, we study the graph M(G) obtained from various product groups. Also, we discuss their χ-chromatic M- polynomial as well as their $\chi+$ chromatic M- polynomial of M(G).

Theorem 3.1; The Mobius function graph of $Z_p \times Z_p \times \ldots \times Z_p$ (n times) where p prime is isomorphic to a star graph.

Proof: Let G =$Z_p \times Z_p \times \ldots \times Z_p$ (n times) where p is prime.

The vertex set of M(G) is V(M(G)) = {(u_1, u_2, …, u_n): $u_i \in Z_p$} with |V(M(G))| = p

In M(G) the identity element e = (0, 0, …, 0) has order 1. All other vertices have order p. By using the definition of the Mobius Function graph, the identity element is adjacent to all other vertices.

For any two vertices u, v $\in$V(M(G))-{e}, $\mu(|u||v|) \neq \mu(|u|)\mu(|v|)$ see Definition 2.3. Thus, we can identify that no two vertices other than identity are adjacent.

Hence, we get M(G) $\cong K_{1,p^n-1}$ Therefore, the Mobius function graph of $Z_p \times Z_p \times \ldots \times Z_p$ is isomorphic to a star graph.

Theorem 3.2: Let G = $Z_p \times Z_p \times \ldots \times Z_p$ (n times) where p prime, then

Mχ - (G, x, y) = (p^n -1) xy^2.

Proof: Given |G| = p^n, with p being a prime number and n>1. By Theorem 3.1, M(G) is isomorphic to a star graph K_{1,p^n-1}. Therefore, the chromatic number χ (M(G)) = 2. Thus, the vertices take two colors say c_1 and c_2 respectively. c_1 is assigned to all the pendant vertices and c_2 to the central vertex of the star.

Table 1. Calculating ${M^-}_{i,j}$ of M ($Z_p \times Z_p \times \ldots \times Z_p$)

Color Pairs (c_i, c_j)	Number of pairs, ${M^-}_{i,j}$
(c_1, c_2)	p^n-1

Consider the Table 1. Suppose there is an edge between two vertices having colour c_i and c_j, then in the table (c_i, c_j) is the colour pairs. And ${M^-}_{i,j}$ denotes the number of such pairs in χ- coloring. Therefore

Mχ - (G, x, y) = ${M^-}_{i,j}$xy = (p -1)xy.

Reversing the order of colors given in the Theorem 3.2, we get:

Corollary 3.3: For a group G = $Z_p \times Z_p \times \ldots \times Z_p$, p is prime then M$\chi^+$(G, x, y) = ($p^n$ -1) x^2 y.

Theorem 3.4: The Mobius function graph of $Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$ (n times) where p prime and k $\in$ Z^+ is isomorphic to a split graph $\overline{K}_{\varphi(p)^n} + K_{p^{nk}-\varphi(p)^n}$.

Proof: Consider G =$Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$ (n times) where p prime and k $\in$ Z^+. The vertex set V(M(G)) = {(u_1, u_2, …, u_n): $u_i \in Z_{p^k}$}. Hence total number of vertices in M(G) is p^{nk}. We partitioned the vertex set of M(G) into two sets say

Δ_1= {u $\in$V(M(G)): |u| = p} and Δ_2= {v $\in$V(M(G)): |v|$\neq$ p}.

Then $|\Delta_1|= \varphi(p)^n$ and $|\Delta_2|= p^{nk}-\varphi(p)^n$.

By the definition of Mobius function graph for any u_1, $u_2 \in \Delta_1$, $\mu(|u_1||u_2|) \neq \mu(|u_1|)\mu(|u_2|)$. That is no two vertices of Δ_1 are adjacent. Hence Δ_1 is an independent set, say $\overline{K}_{\varphi(p)^n}$.

Consider Δ_2, let v_1, $v_2 \in \Delta_2$, by Definition 2.3, $\mu(|v_1||v_2|) = \mu(|v_1|)\mu(|v_2|) = 0$. That is all the vertices of Δ_2 is adjacent to each other. That is Δ_2 form a clique in M(G), say $K_{p^{nk}-\varphi(p)^n}$.

Also, if we take $u_1 \in \Delta_1$ and $v_1 \in \Delta_2$, then also $\mu(|u_1||v_1|) = \mu(|u_1|)\mu(|v_1|)$. Thus, we can say that the Mobius function graph M(G) forms a split graph which is isomorphic to $\overline{K}_{\varphi(p)^n} + K_{p^{nk}-\varphi(p)^n}$.

Theorem 3.5: For the group G =$Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$ (n times) where p prime and k $\in$ Z^+, then

Mχ^-(G, x, y) = $\varphi(p)^n x \sum_{i=2}^{t+1} y^i + \sum_{i=2}^{t} \sum_{j=i+1}^{t+1} x^i y^j$

Proof: Given G = G =$Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$ (n times), then |G| = p^{nk} with p being a prime number and k > 1. By Theorem 3.4, M(G) is isomorphic to $\overline{K}_{\varphi(p)^n} + K_{p^{nk}-\varphi(p)^n}$.

Let us take t =$p^{nk}-\varphi(p)^n$. All the vertices of Δ_1 is given the colour c_1. And since Δ_2 forms a complete graph in M(G), all the t vertices of it are given t distinct colours other than c_1. Therefore, the chromatic number χ (M(G)) = t+1. Thus, the vertices take t+1 colors say $c_1, c_2, \ldots, c_{t+1}$ respectively. We can calculate χ-chromatic M polynomial by using Table 2 shown below.

Table 2. Calculating $M^-{}_{i,j}$ of M ($Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$)

Color Pairs	Number of pairs
(c_1, c_i), 2$\leq$ i $\leq$ t+1	$\varphi(p)^n$
(c_i, c_j), 2$\leq$ i < j$\leq$ t+1	1

Therefore, from the table Mχ^-(G, x, y) = $\varphi(p)^n x \sum_{i=2}^{t+1} y^i + \sum_{i=2}^{t} \sum_{j=i+1}^{t+1} x^i y^j$

Corollary 3.6: For the group G =$Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$ (n times) where p prime and k $\in$ Z^+, then

Mχ^+(G, x, y) = $\varphi(p)^n y^{t+1} \sum_{i=1}^{t} x^i + \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} x^i y^j$

Proof: Given G =$Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$ (n times), |G| = p^{nk} with p being a prime number and k > 1. By Theorem 3.4 and Theorem 3.5, we can colour the vertices as follows.

All the vertices of Δ_1 is given the colour c_{t+1}. And since Δ_2 forms a complete graph in M(G), all the t vertices of it are given t distinct colours, say $c_1, c_2, \ldots, c_t$. Therefore, the chromatic number χ (M(G)) = t+1. Thus, the vertices take t+1 colors say $c_1, c_2, \ldots, c_{t+1}$ respectively. We can calculate χ^+chromatic M polynomial by using the Table 3.

Table 3. Calculating $M^+{}_{i,j}$ of M ($Z_{p^k} \times Z_{p^k} \times \ldots \times Z_{p^k}$)

Color Pairs	Number of pairs
(c_i, c_j) , 1$\leq$ i< j $\leq$ t+1	1
(c_i, c_{t+1}) , 1$\leq$ i $\leq$ t	$\varphi(p)^n$

Therefore, from the table Mχ^+(G, x, y) = $\varphi(p)^n y^{t+1} \sum_{i=1}^{t} x^i + \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} x^i y^j$.

Theorem 3.7: For G = $Z_p \times Z_{pq}$ where p and q are prime numbers, then

M(G) $\cong$($\overline{K}_{p^2-1} + \overline{K}_{q-1} + K_1) \cup (K_1 + \overline{K}_{(p^2-1)(q-1)})$.

Proof: Let us consider the group G = $Z_p \times Z_{pq}$, where p and q are prime numbers with |G|= p^2q. We can partition the vertex set of the Mobius function graph as

$$\Delta_1 = \{e\}$$
$$\begin{aligned}\Delta_2 &= \{(u, v) : LCM\,(|u|, |v|) = p\} \\ &= \{(u, v) : |u| = p \;\&\; v = e\} \cup \\ &\quad \{(u, v) : u = e \;\&\; |v| = p\} \cup \\ &\quad \{(u, v) : |u| = p \;\&\; v = p\}\end{aligned}$$
$$\begin{aligned}\Delta_3 &= \{(u, v) : LCM\,(|u|, |v|) = q\} \\ &= \{(u, v) : u = e \;\&\; |v| = q\}\end{aligned}$$
$$\begin{aligned}\Delta_4 &= \{(u, v) : LCM\,(|u|, |v|) = pq\} \\ &= \{(u, v) : |u| = p \;\&\; |v| = pq\} \cup \\ &\quad \{(u, v) : u = e \;\&\; |v| = pq\} \cup \\ &\quad \{(u, v) : |u| = p \;\&\; |v| = q\}\end{aligned}$$

Then $|\Delta_1| = 1$, $|\Delta_2| = p^2-1$, $|\Delta_3| = q-1$, $|\Delta_4| = (p^2-1)(q-1)$

By the definition of Mobius function graph, we get the identity vertex in Δ_1 is adjacent to all vertices of M(G). And sets Δ_2, Δ_3, Δ_4 are independent sets. Also, all the vertices of Δ_2 are adjacent to all the vertices of Δ_3 in M $(Z_p \times Z_{pq})$. Corresponding graph M(G) is shown in Figure 1. Thus, we can say that

M(G) $\cong (\overline{K}_{p^2-1} + \overline{K}_{q-1} + K_1) \cup (K_1 + \overline{K}_{(p^2-1)(q-1)})$.

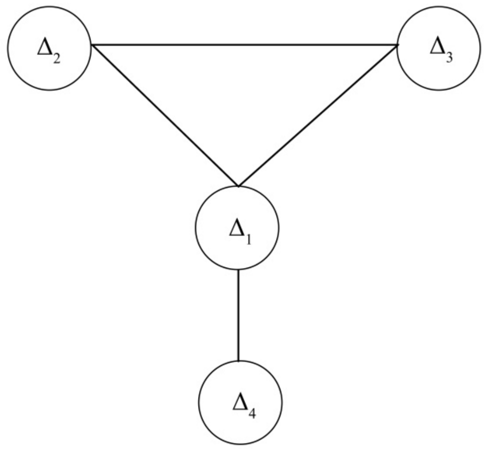

Fig 1. Figure of M $(Z_p \times Z_{pq})$

Example 3.8: Consider $Z_2 \times Z_6$.Then the vertex set of Mobius function graph is {(x, y):x $\in Z_2$, y$\in Z_6$}. By the proof of above Theorem 3.7, we can partition the vertex set as

Δ_1 = {(0,0)}

Δ_2= {(0,3), (1,0), (1,3)}

Δ_3= {(0,2), (0,4)}

Δ_4 = {(0,1), (0,5), (1,1), (1,2), (1,4), (1,6)}

The colouring of $Z_2 \times Z_6$ is shown in Figure 2.

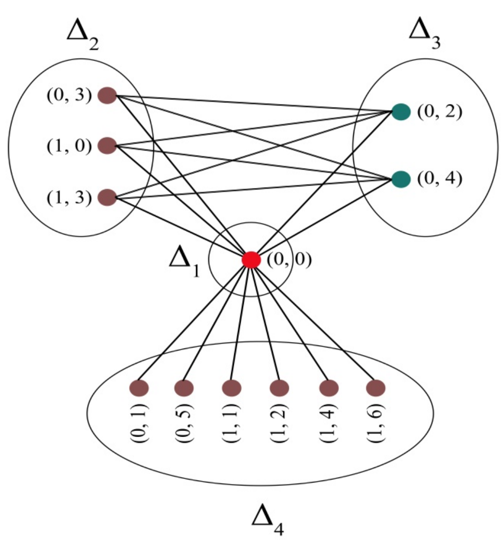

Fig 2. Mobius function graph of $Z_2 \times Z_6$

Theorem 3.9: For the group G = $Z_p \times Z_{pq}$ where p and q are prime numbers, then Mχ - (G, x, y) = (p^2-1) [q(y+1) -1] xy^2 + (q-1) $x^2 y^3$.

Proof: Given G = $Z_p \times Z_{pq}$, where p and q are prime numbers.

By Theorem 3.7, M(G) is isomorphic to $(\overline{K}_{p^2-1} + \overline{K}_{q-1} + K_1) \cup (K_1 + \overline{K}_{(p^2-1)(q-1)})$. By using the proof of Theorem 3.7 we can partition the vertex set as Δ_1 , Δ_2, Δ_3, Δ_4. All the vertices of $\Delta_2 \cup \Delta_4$ is given the colour c_1. And all the vertices in Δ_3 are given the same colour c_2. At last, the identity element in Δ_1 is given the colour c_3. Therefore, the chromatic number χ (M(G)) = 3. Thus, the vertices take 3 colors say c_1, c_2, c_3 respectively. We can calculate χ-chromatic M polynomial by using the Table 4.

Therefore, from the Table 4

$$\begin{aligned} M\chi-(G,\,x,\,y) &= (p^2-1)\,(q-1)\,xy^2 + q(p^2-1)\,xy^3 + (q-1)\,x^2y^3 \\ &= (p^2-1)\,[q(y+1)-1]\,xy^2 + (q-1)\,x^2y^3. \end{aligned}$$

Table 4. Calculating $M^-{}_{i,j}$ of M ($Z_p \times Z_{pq}$)

Color Pairs	Number of pairs
(c_1, c_2)	(p^2-1) (q-1)
(c_1, c_3)	q(p^2-1)
(c_2, c_3)	(q-1)

Corollary 3.10: For the group G = $Z_p \times Z_{pq}$ where p and q are prime numbers, then

Mχ^+(G, x, y) = (p^2-1) [q(x+1) -1] xy^3 + (q-1) x y^2.

Proof: Given G = $Z_p \times Z_{pq}$, where p and q are prime numbers.

By Theorem 3.7, M(G) is isomorphic to $(\overline{K}_{p^2-1} + \overline{K}_{q-1} + K_1) \cup (K_1 + \overline{K}_{(p^2-1)(q-1)})$. By using the proof of Theorem 3.7 we can partition the vertex set as Δ_1 , Δ_2, Δ_3, Δ_4. The identity element in Δ_1 is given the colour c_1. All the vertices in Δ_3 are given the same colour c_2. And at last, all the vertices of $\Delta_2 \cup \Delta_4$ is given the colour c_3. Therefore, the chromatic number χ (M(G)) = 3. Thus, the vertices take 3 colors say c_1, c_2, c_3 respectively. We can calculate χ+chromatic M polynomial by using the Table 5.

Table 5. Calculating $M^+{}_{i,j}$ of M ($Z_p \times Z_{pq}$)

Color Pairs	Number of pairs
(c_1, c_2)	(q-1)
(c_1, c_3)	q(p^2-1)
(c_2, c_3)	(p^2-1) (q-1)

Therefore, from the table

$$\begin{aligned} M\chi^+(\,G,\,x,\,y) &= (q-1)\,xy^2 + q(p^2-1)\,xy^3 + (p^2-1)\,(q-1)\,x^2y^3 \\ &= (p^2-1)\,[q(x+1)-1]\,xy^3 + (q-1)\,xy^2. \end{aligned}$$

Theorem 3.11: For G = $Z_p \times D_{2p}$ where p is a prime number, then

M(G) $\cong (\overline{K}_{p^2-1} + \overline{K}_p + K_1) \cup (K_1 + \overline{K}_{p(p-1)})$.

Proof: Let us consider the group G = $Z_p \times D_{2p}$ where p is prime numbers with |G|=p^2 q. We can partition the vertex set of the Mobius function graph as

$$\begin{aligned} \Delta_1 &= \{e\} \\ \Delta_2 &= \{(u,\,v):\,LCM\,(|u|,\,|v|)\,=p\} \\ &= \{(u,\,v):\,|u|=p\,\&\,v=e\}\cup \\ &\quad \{(u,\,v):\,u=e\,\&\,|v|=\,p\}\cup \\ &\quad \{(u,\,v):\,|u|=p\,\&\,|v|=p\} \\ \Delta_3 &= \{(u,\,v):\,LCM\,(|u|,\,|v|)\,=2\} \\ &= \{(u,\,v):\,u=e\,\&\,|v|=\,2\} \\ \Delta_4 &= \{(u,\,v):\,LCM\,(|u|,\,|v|)\,=2p\} \\ &= \{(u,\,v):\,|u|=p\,\&\,|v|=2\} \end{aligned}$$

Then |Δ_1 | = 1, |Δ_2| = p^2-1, | Δ_3|= p, | Δ_4|= p(p-1)

By the definition of Mobius function graph, we get Δ_2, Δ_3, Δ_4 are independent sets and the identity vertex is adjacent to all the vertices of M(G). Also, all the vertices of $\Delta_2$are adjacent to all the vertices of Δ_3 in M ($Z_p \times D_{2p}$). Thus, we can say that M(G) $\cong (\overline{K}_{p^2-1} + \overline{K}_p + K_1) \cup (K_1 + \overline{K}_{p(p-1)})$.

Theorem 3.12: For the group G = $Z_p \times D_{2p}$ where p is prime numbers, then

$M\chi^-(G, x, y) = p(p^2-1)\, xy^2 + (2p+1)(p-1)\, xy^3 + p\, x^2y^3$.

Proof: Given G = $Z_p \times D_{2p}$, where p and q are prime numbers.

By Theorem 3.11, M(G) is isomorphic to $(\overline{K}_{p^2-1} + \overline{K}_p + K_1) \cup (K_1 + \overline{K}_{p(p-1)})$. By using the proof of Theorem 3.11 we can partition the vertex set as Δ_1 , Δ_2, Δ_3, Δ_4. All the vertices of $\Delta_2 \cup \Delta_4$ is given the colour c_1. And all the vertices in Δ_3 are given the same colour c_2. At last, the identity element in Δ_1 is given the colour c_3. Therefore, the chromatic number χ (M(G)) = 3. Thus, the vertices take 3 colors say c_1, c_2, c_3 respectively. We can calculate χ-chromatic M polynomial by using the Table 6.

Therefore, from the table $M\chi^-(G, x, y) = p(p^2-1)\, xy^2 + (2p+1)(p-1)\, xy^3 + p\, x^2y^3$

Table 6. Calculating $M^-{}_{i,j}$ of M ($Z_p \times D_{2p}$)

Color Pairs	Number of pairs
(c_1, c_2)	$p(p^2-1)$
(c_1, c_3)	$(2p+1)(p-1)$
(c_2, c_3)	p

Reversing the order of colors given in the Theorem 3.12, we get $M\chi^+(G, x, y)$

Theorem 3.13: For G =$D_{2p} \times D_{2p}$, where p is a prime number, then M(G) $\cong (\overline{K}_{p^2-1} + \overline{K}_{p^2+2p} + K_1) \cup (K_1 + \overline{K}_{2p(p-1)})$.

Proof: Let us consider the group G = $D_{2p} \times D_{2p}$, where p is prime numbers with |G|=$4p^2$. We can partition the vertex set of the Mobius function graph as

$$\Delta_1 = \{e\}$$
$$\begin{aligned}\Delta_2 &= \{(u, v) : LCM(|u|, |v|) = p\} \\ &= \{(u, v) : |u| = p \,\&\, v = e\} \cup \\ &\quad \{(u, v) : u = e \,\&\, |v| = p\} \cup \\ &\quad \{(u, v) : |u| = p \,\&\, |v| = p\}\end{aligned}$$
$$\begin{aligned}\Delta_3 &= \{(u, v) : LCM(|u|, |v|) = 2\} \\ &= \{(u, v) : |u| = 2 \,\&\, v = e\} \cup \\ &\quad \{(u, v) : u = e \,\&\, |v| = 2\} \cup \\ &\quad \{(u, v) : |u| = 2 \,\&\, |v| = 2\}\end{aligned}$$
$$\begin{aligned}\Delta_4 &= \{(u, v) : LCM(|u|, |v|) = 2p\} \\ &= \{(u, v) : |u| = p \,\&\, |v| = 2\} \cup \\ &\quad \{(u, v) : |u| = 2 \,\&\, |v| = p\}\end{aligned}$$

Then $|\Delta_1| = 1$, $|\Delta_2| = p^2-1$, $|\Delta_3| = p^2+2p$, $|\Delta_4| = 2p(p-1)$

By the definition of Mobius function graph, we get Δ_2, Δ_3, Δ_4 are independent sets. Also, all the vertices of Δ_2 are adjacent to all the vertices of Δ_3 in M ($D_{2p} \times D_{2p}$). Thus, we can say that M(G) $\cong (\overline{K}_{p^2-1} + \overline{K}_{p^2+2p} + K_1) \cup (K_1 + \overline{K}_{2p(p-1)})$.

4 Conclusion

In recent years a lot of work has been done in the area of colouring of graphs and chromatic polynomials. This study is an attempt to find the chromatic M polynomial of the Mobius function graph of product groups. This study may provide a new opening to both proper and improper colouring.

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