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RESEARCH ARTICLE

Distance Degree Sequence and Distance Neighborhood Degree Sequence of Vertices in Fuzzy Graphs and Some of Their Properties

K Radha[1]*, **S R Gayathri**[2]

1 Associate Professor, PG and Research Department of Mathematics, Thanthai Periyar Government Arts and Science College (Affiliated to Bharathidasan university), Tiruchirapalli, 620023, Tamil Nadu, India

2 Research scholar, PG and Research Department of Mathematics, Thanthai Periyar Government Arts and Science College (Affiliated to Bharathidasan university), Tiruchirapalli, 620023, Tamil Nadu, India

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* **Corresponding author**.

radhagac@yahoo.com

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Abstract

Objectives: The objective of this study is to explore new types of sequences in fuzzy graphs namely distance degree sequence and distance neighborhood degree sequence of fuzzy graphs which may have various applications in the real world. **Methods:** In this paper, two unique sequences for each vertex of fuzzy graphs are given by means of its degree and neighborhood degree which are called as the distance degree sequence and distance neighborhood degree sequence of vertices of fuzzy graphs respectively. The n-tuple of the distance degree sequence of vertices of a fuzzy graph arranged in lexicographic order is called the distance degree sequence of the fuzzy graph and the n-tuple of distance neighborhood degree sequence of vertices of a fuzzy graph arranged in lexicographic order is called the distance neighborhood degree sequence of the fuzzy graph. **Findings:** The distance degree sequence and distance neighborhood degree sequence of fuzzy graphs are introduced. The properties of these sequences and the general forms of these sequences of various fuzzy graphs are given. The relationship between the distance degree sequence and distance neighborhood degree sequences is discussed. **Novelty:** The distance degree sequence of graphs has already been introduced. In fuzzy graph theory, the distance degree sequence and distance neighborhood degree sequence are introduced in this paper.

Mathematics Subject Classification (202 0): 03E72, 05C72, 05C12.

Keywords: Fuzzy graphs; Distance degree sequence; Distance neighborhood-degree sequence; Regular fuzzy graphs

1 Introduction

Introducing sequences within the realm of fuzzy graphs opens a captivating avenue for understanding the nuanced relationships between vertices. Unlike traditional graphs, fuzzy graphs embrace the inherent uncertainty and gradual transitions between connectivity states, offering a rich tapestry for exploring sequences. Within this framework, sequences provide a structured lens to unravel the intricacies of connectivity patterns, allowing us to discern subtle gradations and variations in relationships among vertices. Embarking on this exploration promises not only to deepen our comprehension of fuzzy graphs but also to unveil novel insights into the dynamic interplay of uncertainty and connectivity in complex networks.

The study of distance degree sequences helps to build the theoretical foundations of fuzzy graph theory. It improves our understanding of fuzzy graphs and their features, laying the framework for future advances in both theoretical and applied graph theory. In social networks, biological networks, and communication networks, analyzing distance degrees can reveal insights about connectivity, clustering, and community structures. Understanding distance degree sequences can lead to the development of efficient algorithms for problems such as network routing, clustering, and optimization. So based on the distance degree in graphs, distance degree and distance neighbourhood degree in fuzzy graphs are introduced and their properties are developed.

The distance degree sequence in graph theory is introduced by Medha Itagi Huilgol[1] and some of their properties and applications are discussed by S. Meenakshi and K. Deepika[2]. Algorithms designed for crisp graphs frequently require adaption for fuzzy graphs due to the nature of fuzzy sets and operations. Analyzing distance degree sequences in fuzzy graphs can lead to the creation of new algorithms designed specifically for fuzzy networks, hence advancing computational approaches such as optimization, clustering, and pattern recognition. The edge degree sequence of intuitionistic fuzzy graphs is studied by K.Radha and P.Pandian[3]. Some of the recent trends and properties in fuzzy graphs is discussed by Elumalai[4] and Nagadurga Sathavalli and Subashish Biswas[5]. The distance degree sequence of fuzzy graph G is denoted by DDS(G) and the distance neighborhood degree sequence of fuzzy graph G is denoted by DnDS(G)

2 Methods

Let G:(σ, μ) be a fuzzy graph, where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ, that is σ:V$\rightarrow (0,1]$ and μ: $VXV \rightarrow (0,1]$ are such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)\, \forall u, v \in V$. The underlying crisp graph of G:(σ, μ) is denoted by G*:(V, E) where E $\subseteq$ V$\times$V. Two vertices of G:(σ, μ) are adjacent if $\mu(uv) > 0$.

The degree of a vertex u is given by d(u) = $\sum_{u\neq v} \mu(uv)$. The neighborhood degree of a vertex u is given by $nd_G(u)$ or $nd(u)$ = $\sum_{v\in N(u)} \sigma(v)$ where N(u) is the set of all vertices adjacent to in G (or in G*). The j^{th} neighbourhood of u, denoted by N_j(u), is the set of all vertices at distance j from u in G*. A fuzzy graph G is complete if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in$ V where uv denotes the edge between u and v. Given a fuzzy graph G:(σ, μ) with the underlying vertex set V, the order of G is defined and denoted as $O(G) = \sum_{v\in V} \sigma(v)$ and the size of G is defined and denoted as $S(G) = \sum_{u,v\in S} \mu(uv)$.

A path P_n is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $v_i v_{i+1}$ where i = 1, 2, $\cdots$, n – 1. A cycle C_n is a closed path $v_1 v_2 \cdots v_{n-1} v_n$.

A star graph $K_{1,n}$ with n+1 vertices, n$\geq$ 3, is a graph in which the apex vertex v is connected to all the vertices $\{v_1, v_2, \ldots, v_n\}$where v_i is not connected to v_j for $i, j = 1,\ 2,\ \ldots,\ n$.

A bistar graph is the graph obtained by joining the apex vertices of two copies of $K_{1.n}$ by an edge and it is denoted by $B_{n.n}$. The vertex set of $B_{n.n}$ is V($B_{n.n}$) = $\{v_1,\ v_2,\ \cdots, v_n, v, u, u_1, u_2,\ \cdots u_n\}$, where u, v are apex vertices and $v_1,\ v_2, v_3 \ldots v_n, u_1, u_2 \ldots u_n$ are pendent vertices. The edge set of $B_{n.n}$ is E ($B_{n.n}$) = $\{vv_1,\ vv_2, vv_3 \ldots vv_n, vu, uu_1, uu_2 \ldots uu_n\ \}$. The wheel graph W_n, (n $\geq$ 3) is a (n + 1)-vertices graph obtained by connecting all the vertices $v_1, v_2, v_3 \ldots, v_n$ of cycle C_n to the centre vertex v. Here, he centre vertex v is called as apex vertex and other vertices $v_1, v_2, v_3 \ldots, v_n$ in the cycle are called as rim vertices.

The distance degree sequence (dds) of a vertex v in a graph G* is a list of the number of vertices at distances 1,2,3,….$e(v)$ in that order, where $e(v)$ denotes the eccentricity of v, the length of the farthest vertex from v in G*. Thus the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j} \ldots)$ is the distance degree sequence of a vertex v_i in G* where d_{i_j} denotes the number of vertices at distance j from v_i. The n-tuple of distance degree sequences of the vertices of G with entries arranged in lexicographic order is the distance degree sequence (DDS) of G.

The Preliminaries in fuzzy graph theory are referred from[6–8], and the preliminaries in graph theory are referred from[9–12].

Definition 2.1

Consider a fuzzy graph G:(σ, μ) where G*(V, E) is the underlying crisp graph and V = $\{v_1, v_2, \cdots, v_n\}$. The DDS of a vertex v_i in G is given by the sequence $(d_{i_0}, d_{i_1}, d_{i_2} \ldots, d_{i_j} \ldots, d_{i_{e(v_i)}})$ where d_{i_0} is given by $\sigma(v_i)$, d_{i_1} is given by $\sum_{u\in N_1(v_i)} \mu(uv_i)$, d_{i_j} is given by $\sum_{u\in N_j(v_i)} \mu(wu)$ where wu is the edge adjacent to u in the $v_i - u$ path of length j in G*. The DDS(G) with n vertices is given by the n-tuple of DDS of the n vertices of G where the entries are sequenced numerically and is denoted by DDS$_G(v_1, v_2, v_3, \cdots, v_n)$.

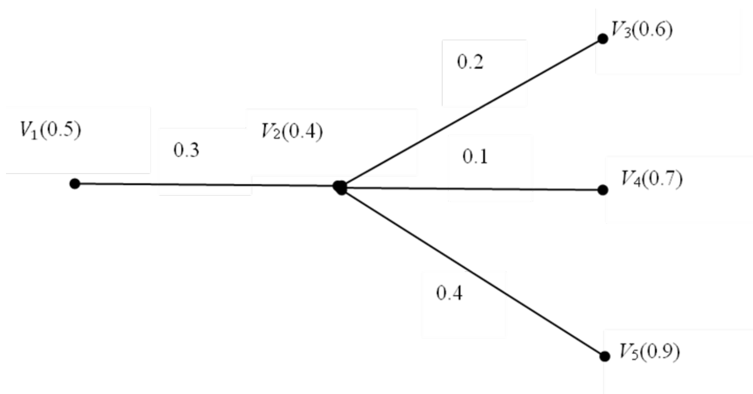

Fig 1. DDS of Fuzzy graph *G*

The DDS of the fuzzy graph G in Figure 1 is DDS$(v_1, v_2, v_3, v_4, v_5)$ = ((0.5,0.3,0.7), (0.4,1), (0.6,0.2,0.8), (0.7,0.1,0.9), (0.9,0.4,0.6).

Remark 2.1

When σ(u) = 1 for each vertex u and μ(e) = 1 for each edge e, then the DDS of a vertex v in fuzzy graphs gives the number of edges on paths of length j incident at vertices which are j distance from v. When the underlying graph G* is a tree, σ(u) = 1 for all the vertices u and μ(e) = 1 for all the vertices e, then the DDS in fuzzy graphs coincides with the DDS in crisp graphs.

Definition 2.2

Let G denote a fuzzy graph with membership and similarity functions (σ, μ) where G* is the underlying crisp graph with vertex set V = $\{v_1, v_2, \cdots, v_n\}$. The DnDS of vertex v_i in G is given by the sequence $(nd_{i_0}, nd_{i_1}, nd_{i_2}, \cdots, nd_{i_j} \cdots, nd_{i_{e(v)}})$ where nd_{i_0} is given by $\sigma(v_i)$, nd_{i_j} is given by $\sum_{u\in N_j(v_i)} \sigma(u)$, where i = 1, 2, 3, $\cdots$, n, j = 2, 3, $\cdots$, e(v_i) and $N_j(v_i)$ is the j^{th} neighborhood of v_i in G* that is the distance between u and v_i is j. The DnDS(G) with n vertices is given by n-tuple of DnDS of the n vertices of G and is denoted by DnDS(v_1, v_2, v_3, $\cdots$, v_n).

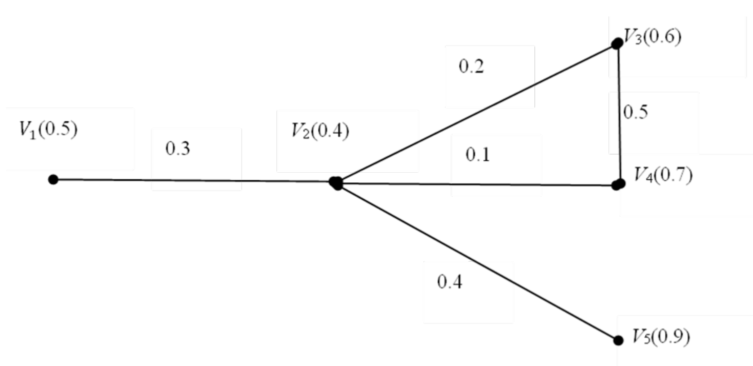

Fig 2. DnDS of Fuzzy graph *G*

The DnDS of the fuzzy graph G in Figure 2 is DnDS(v_1, v_2, v_3, v_4, v_5) = ((0.5, 0.4, 2.2), (0.4, 2.7), (0.6, 1.1, 1.4), (0.7, 1, 1.4), (0.9, 0.4, 1.8)).

Remark 2.2

When σ(u) = 1 for all the vertices u and μ(e) = 1 for all the vertices e, then the DnDS in fuzzy graphs coincide with the DDS in crisp graphs.

3 Results and Discussions

3.1 DDS of fuzzy graphs

The DDS of a fuzzy graph provides a systematic way to analyze the distribution of distances between vertices while considering their degrees. This sequence records the frequencies of vertices having certain degrees at certain distances from a given vertex or set of vertices.

We have developed several theorems to explore properties and relationships within the distance degree sequence of fuzzy graphs.

Theorem 3.1.1

Let G:(σ, μ) be a fuzzy graph where V = {$v_1, v_2, \cdots, v_n$}. Let ($d_{i_0},\ d_{i_1},\ d_{i_2}, \cdots,\ d_{i_j},\ \cdots, d_{i_{e(v_i)}}$) be the DDS of the vertex v_i in G. Then each $d_{i_0} \leq 1$ and $d_{i_j} \leq$ n-1 for $j \geq 1$, i = 1, 2, 3, ⋯, n, where n is the total number of vertices.

Proof

$d_{i_0} = \sigma(v_i) \leq 1$. d_{i_1} is given by $\sum_{\substack{u \in N_1(v_i) \\ wu\epsilon v_i - u\,path}} \mu(v_i u) \leq \sum_{u \in N_1(v_i)} 1 = d_{G^*}(v_i) \leq n-1$. Since the path connecting any vertex at distance j, j≥2, must pass through a vertex in $N_1(v_j)$, vertices that are j distance away is less than or equal to $d_{G^*}(v_i)$ ≤n-1. Therefore, $\sum_{\substack{u \in N_j(v_i) \\ wu\epsilon v_i - u\,path}} \mu(wu) \leq \sum_{u \in N_j(v_i)} 1 \leq d_{G^*}(v_i) \leq n-1$ where j = 2,3, ⋯,e(v_i)

Theorem 3.1.2

Suppose G:(σ, μ) represents a fuzzy graph on a tree G*: (V, E), then the sum of all elements in the DDS of any vertex v in G is equal to S(G) + $\sigma(u)$.

Proof

Let V = {$v_1, v_2, \cdots,\ v_n$}. Consider any vertex v_i. Let $v_j v_k$ be any edge in G. If either i = j or i = k, then the edge is $v_j v_k$ adjacent to v_i. Therefore $\mu(v_j v_k)$ appears in d_{i_1}. Assume that i ≠ j and i ≠ k. Due to the fact that G* forms a tree, there exists only one path P from v_i to v_j and a unique path Q between v_i and v_k. For the sake of convenience, let's assume that the length of the v_i-v_j path P in G* is less than the length of the v_i-v_k path Q in G*. Suppose the edge $v_j v_k$ is not in Q. Then there will be a cycle in P U Q U {$v_j v_k$} which is a contradiction. Therefore, the edge $v_j v_k$ is in Q. Since v_i-v_j path in G* is unique, Q = P U {$v_j v_k$}. If the length of Q in G* is r, then the edge $v_j v_k$ is in the v_i-v_k path of length r and incident on $v_k \in N_r(v_i)$. Therefore every edge is incident on a vertex in $N_j(v_i)$ for some j=1,2, ⋯,$e(v_i)$. Therefore for any vertex v_i, $\sum_{j=0}^{e(v_i)} d_{i_j} = \sigma(v_i)$ + — = $\sigma(v_i)$ + $\sum_{wu\epsilon E} \mu(wu)$ = $\sigma(v_i)$ + S(G).

Remark 3.1.2

Theorem 3.1.2 need not be true if G* is not a tree. For example, in the fuzzy graph of Figure 2, $\sum_{j=0}^{e(v_2)} d_{2_j}(v_2)$ = 0.4 + (0.3 + 0.2 + 0.1 + 0.4) = $\sigma(v_2)$ + 1 ≠ $\sigma(v_2)$ + S(G) since S(G) = 1.5.

But if we consider the fuzzy graph on a tree in Figure 1, $\sum_{j=0}^{e(v_i)} d_{i_j}(v_i)$ = $\sigma(v_i)$ + S(G) for all the vertices v_i.

Theorem 3.1.3

Assume G to be a fuzzy graph. In the degree sequence of vertices v_i on G, the first two terms i.e, d_{i_0} and d_{i_1} are given by $\sigma(v_i)$ and $d_G(v_i)$ where $d_G(v_i)$ is the degree of v_i in G.

Proof

By the definition of DDS, d_{i_0} = $\sigma(v_i)$ and d_{i_1} = $\sum_{u \in N_1(v_i)} \mu(v_i u)$ = $d_G\ (v_i)$.

3.2 DnDS of fuzzy graphs

The DnDS of fuzzy graphs extends the concept of the distance degree sequence by incorporating neighborhood information. In fuzzy graphs, the neighborhood of a vertex encompasses not only its adjacent vertices but also those within a certain distance. The distance neighborhood degree sequence captures the distribution of degrees among vertices within specified neighborhoods at different distances.

Theorem 3.2.1

Suppose G:(σ, μ) represents a fuzzy graph where V = $\{v_1, v_2, \cdots, v_n\}$. Let $(nd_{i_0},\ nd_{i_1},\ nd_{i_2}, \cdots,\ nd_{i_j},\ \cdots, nd_{i_{e(v_i)}})$ be the DnDS of vertex v_i in G. Then, each $nd_{i_0} \leq 1$ and $nd_{i_j} \leq$ n-1 for $j \geq 1$, i = 1, 2, 3, ⋯, n, where n is the total number of vertices.

Proof

$nd_{i_0} = \sigma(v_i) \leq 1.$ nd_{i_1} is given by $\sum_{u \in N_1(v_i)} \sigma(u) \leq \sum_{u \in N_1(v_i)} 1 = d_{G^*}(v_i) \leq$ n-1. Since the path connecting any vertex at distance j, j≥2, must pass through a vertex in $N_1(v_j)$, the count of vertices at a distance of j is less than or equal to $d_{G^*}(v_i) \leq$n-1. Therefore $nd_{i_j} = \sum_{u \in N_j(v_i)} \sigma(u) \leq \sum_{u \in N_j(v_i)} 1 \leq d_{G^*}(v_i) \leq$ n-1 where j = 2,3, ⋯, e(v_i).

Theorem 3.2.2

Suppose G:(σ, μ) represents a connected fuzzy graph. Then the aggregate of all elements in the DnDS of any vertex in G is the order of G.

Proof

Let V = $\{v_1, v_2, \cdots,\ v_n\}$. Since G is connected, each vertex $u \neq v_i$ is in some $N_j(v_i)$, j=1,2, ⋯,$e(v_i)$. Therefore, for any vertex v_i, $\sum_{j=0}^{e(v_i)} nd_{i_j} = \sigma(v_i) + \sum_{j=1}^{e(v_i)} \sum_{u \in N_j(v_i)} \sigma(u) = \sigma(v_i) + \sum_{j=1,\ j \neq i} \sigma(v_j) = \sum_{j=1}^{n} \sigma(v_j) = O(G)$.

Remark 3.2.3

If G is not connected with components $G_1, G_2, \cdots,\ G_n$ and if a vertex v_i is in some G_k, then $\sum_{j=0}^{e(v_i)} nd_{i_j} = O(G_k)$.

Theorem 3.2.4

Suppose G:(σ, μ) represents a fuzzy graph. In the DnDS of vertices v_i in G, the first two terms i.e, nd_{i_0} and nd_{i_1} are given by $\sigma(v_i)$ and $nd_G(v_i)$ where $nd_G(v_i)$ is the neighborhood degree of v_i.

Proof

By the definition 4.1, $\sigma(v_i) = nd_{i_0}$ and $nd_{i_1} = \sum_{u \in N_1(v_i)} \sigma(u) = nd_G\ (v_i)$.

3.3 DnDS of some fuzzy graphs

In this division, the DnDS of vertices in the fuzzy graph on a complete graph, a star graph, wheel graph, bistar graph, and path graph is derived

Theorem 3.3.1

If vertex v is adjacent to all other vertices in the fuzzy graph, then DnDS(v) = $(\sigma(v)$, O(G)$-\sigma(v)$).

Proof

Consider a vertex v in G, since every other vertices in G is at a distance one from v, DnDS(v) =$(nd_{i_o},\ nd_G(v))$ = $(nd_{i_o},\ nd_{i_1})$. By the definition, $nd_{i_0} = \sigma(v_i)$. Since all the other vertices are adjacent to v, nd$_{i_1}$ $=\sum_{u \in N_j(v)} \sigma(u)$ = $\sum_{u \in V} \sigma(u) - \sigma(v)$ = O(G)$-\sigma(v)$. Therefore, the DnDS of any vertex v_i in G is given by $(\sigma(v)$, O(G)$-\sigma(v)$).

Theorem 3.3.2

Let G represent a complete fuzzy graph. Then the DnDS of any vertex v_i in G is given by $(\sigma(v_i)$, O(G)$-\sigma(v_i)$).

Proof

Consider a vertex v_i in G. Since G is a complete fuzzy graph, every other vertex in G is at a distance 1 from v_i in G^*. By the above theorem 5.1, the DnDS of any vertex v_i in G is given by $(\sigma(v_i)$, O(G)$-\sigma(v_i)$).

Corollary 3.3.3

Let G be a complete fuzzy graph such that σ has constant value c. Then the DnDS of any vertex v_i in G is given by (c, (n-1)c).

Proof

From the above theorem 5.2, $d_{i_0} = \sigma(v_i)$ = c and $d_{i_1} = \sum_{u\in N_j(v_i)} \sigma(u)$ = c d(v_i) = c(n-1). Therefore, the DnDS of any vertex v_i in G is (c, (n-1)c).

Remark 3.3.4

If the fuzzy graph G has all the σ values and all the μ values to be 1, then the DnDS and DDS of any vertex in G is the same as the DDS of G*.

Theorem 3.3.5

Let G represent a fuzzy graph where the underlying graph G^* is a star on n+1 vertices $v, v_1, v_2, \cdots, v_n$ with v as the apex vertex. Then the DnDS of the apex vertex v is given by $(\sigma(v), \ O(G) - \sigma(v))$ and for every other vertices, DnDS(v_i) = $(\sigma(v_i), \sigma(v), \ O(G) - \sigma(v_i) - \sigma(v))$, i = 1 to n.

Proof

According to the definition of the DnDS of any vertex of a fuzzy graph: $nd_{i_0} = \sigma(v_i)$, $nd_{i_j} = \sum_{u\in N_j(v_i)} \sigma(u)$.

All the rim vertices v_i are at the distance one in $G^*$from the apex vertex. Therefore, the DnDS of the apex vertex is DnDS(v) = $(\sigma(v), O(G) - \sigma(v))$.

For all the rim vertices v_i, the apex vertex is at distance one and every other vertex $v_j, \ j \neq i$, is at distance two. Therefore $nd_{i_0} = \sigma(v_i), nd_{i_1} = \sigma(v)$. Also from theorem 4.5, $nd_{i_0} + nd_{i_1} + nd_{i_2} = O(G)$ which gives $d_{i_2} = O(G) - \sigma(v_i) - \sigma(v)$. Hence, the DnDS of each v_i is given by DnDS(v_i) = $(\sigma(v_i), \sigma(v), \ O(G) - \sigma(v_i) - \sigma(v))$.

Corollary 3.3.6

Let G represent a fuzzy star graph with n + 1 vertices denoted by $v, v_1, v_2, \cdots, v_n$ where v is the apex vertex. Let σ be a constant function of constant value c. Then, the DnDS of v in G is given by $(c, \ (n-1)c)$ and for v_i, it is given by $(c, \ c, \ (n-1)c)$.

Theorem 3.3.7

Let G be a fuzzy graph with underlying crisp graph G^* where G^* is a wheel graph. Let v be the apex vertex in G^* and the rim vertices be denoted by v_i, i = 1 to n. The DnDS of v is expressed as $(\sigma(v), O(G) - \sigma(v))$ and the DnDS of v_i is expressed as $(\sigma(v_i), nd_G(v_i)$, O(G) $- \sigma(v_i) - nd_G(v_i))$.

Proof

According to the definition of wheel graph, the apex vertex is at distance 1 from all the other vertices. Therefore, the distance neighborhood degree of apex vertex v is expressed as $(\sigma(v), \ nd_G(v)) = (\sigma(v), O(G) - \sigma(v))$.

If we consider the rim vertices, the vertex v_i has three vertices including the apex vertex at distance one and the remaining vertices at distance two. Therefore, the DnDS of v_i is expressed as $(\sigma(v_i), nd_G(v_i)$, O(G) $- \sigma(v_i) - nd_G(v_i))$.

Theorem 3.3.8

Let G be a fuzzy graph and G^* be a bistar with 2n+2 vertices $u, u_1, u_2, \cdots, u_n; v, v_1, v_2, \cdots, v_n$ where u and v are the apex vertices. The DnDS of apex vertex u of G is expressed as $(\sigma(u), nd_G(u), nd_G(v) - \sigma(u))$. The DnDS of vertices u_i is expressed as $(\sigma(u_i), \sigma(u), nd_G(u) - \sigma(u_i), \ nd_G(v) - \sigma(u))$. The result is similar for $v, v_1, v_2, \cdots, v_n$.

Proof

Let us consider only the star with apex vertex u. Since the graph is symmetric the result is also true for $v, v_1, v_2, \cdots, v_n$.

For the vertex u, the maximum distance from u is 2. $nd_0(u) = \sigma(u)$, $nd_1(u) = nd_G(u)$, $nd_2(u) = \sum_{i=1}^{n} \sigma(v_i) = \sum_{i=1}^{n} \sigma(v_i) + \sigma(u) - \sigma(u) = nd_G(v) - \sigma(u)$. Therefore, the DnDS of the apex vertex u of G is expressed as $(\sigma(u), nd_G(u), nd_G(v) - \sigma(u))$.

For the vertices u_i, the maximum distance from u_i is 3. u is at distance 1, $nd_{i_0} = \sigma(u_i)$, $nd_{i_1} = \sigma(u)$, $nd_{i_2} = \sum_{j=1, j\neq i}^{n} \sigma(v_j) + \sigma(v) = nd_G(u) - \sigma(u_i)$ and $nd_{i_3} = \sum_{i=1}^{n} \sigma(v_i) = nd_G(v) - \sigma(u)$. Therefore, the DnDS of vertices u_i is given by $(\sigma(u_i), \sigma(u), nd_G(u) - \sigma(u_i), \ nd_G(v) - \sigma(u))$. The result is the same for $v, v_1, v_2, \cdots, v_n$.

Theorem 3.3.9

Let $P_n : (\sigma, \mu)$ be a fuzzy graph on a path with n vertices $v_1, v_2, v_3, \ldots, v_n$ where n is even. Then the DnDS of the vertices are as follows:

1. DnDs(v_1) = $(\sigma(v_1), nd(v_i), \sigma(v_3), \sigma(v_4), \ldots, \sigma(v_n))$
2. DnDs(v_2) = $(\sigma(v_2), nd(v_2), \sigma(v_4), \sigma(v_5) \ldots, \sigma(v_n))$
3. For r =3,4,… $\frac{n}{2}$,

DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r+2})$+ $\sigma(v_{r-2})$ … , $\sigma(v_1)$+ $\sigma(v_{2r-1})$, $\sigma(v_{2r})$,…, $\sigma(v_n)$)

4. For r= $\frac{n}{2}+1$, …, n-2,

DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r+2})$+ $\sigma(v_{r-2})$ … , $\sigma(v_n)$+ $\sigma(v_{2r-n})$, $\sigma(v_{2r-n-1})$,…, $\sigma(v_1)$)

5. DnDs(v_{n-1})= ($\sigma(v_{n-1})$, $nd(v_{n-1})$, $\sigma(v_{n-3})$, $\sigma(v_{n-2})$ … , $\sigma(v_1)$)

6. DnDs(v_n)= ($\sigma(v_n)$, $nd(v_n)$, $\sigma(v_{n-2})$, $\sigma(v_{n-3})$ … , $\sigma(v_1)$)

Proof

1. For r = 1, the vertices $v_2, v_3, \ldots, v_n$ are at distance 1,2,3… n-1 respectively. So, the maximum distance from vertex v_1 is n-1. From the theorem 3.3, $nd_{i_1} = nd_G(v_i)$. Therefore, the DnDS of v_1 is given by DnDS(v_1)= ($\sigma(v_1)$, $nd(v_i)$, $\sigma(v_3)$, $\sigma(v_4)$, …, $\sigma(v_n)$).

2. For r = 2, the maximum distance of a vertex from v_2 is n-2. Only the vertices v_1 and v_3 are at distance 1 from v_2. Therefore, $d_{2_1} = \sigma(v_1) + \sigma(v_3)$ The vertices $v_4, v_5, \ldots, v_n$ are at distance 2, 3, …, n-2 respectively. In general, for j = 2, 3, …, n-2, v_{j+2} is the only vertex at distance j from v_2 and d_{2_j} =$\sigma(v_{j+2})$. Therefore, the DnDS of v_2 is given by DnDS(v_2)= ($\sigma(v_2)$, $nd(v_2)$, $\sigma(v_4)$, $\sigma(v_5)$, … , $\sigma(v_n)$).

3. For r = 3 ,4,… $\frac{n}{2}$, the maximum distance of a vertex from v_r is n-r.

For j= 2,3,$\cdots, r-1$, v_{r+j} and v_{r-j} are the only vertices at distance j from v_r. Therefore, d_{r_j}= $\sigma(v_{r-j})$+ $\sigma(v_{r+j})$, j = 2,3,…,r-1.

For j = r, r+1,…, n-r, v_{r+j} is the only vertex at distance j from v_r. Therefore, d_{r_j}= $\sigma(v_{r+j})$ for j = r, r+1, …, n-r.

Hence the DnDS for r = 3, 4, …, $\frac{n}{2}$ is given by,

DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r+2})$+ $\sigma(v_{r-2})$, … , $\sigma\left(v_{r-(r-1)}\right)$+ $\sigma\left(v_{r+(r-1)}\right)$, $\sigma(v_{2r})$,…, $\sigma(v_n)$) = ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r+2})$+ $\sigma(v_{r-2})$, … , $\sigma\left(v_{1)}\right)$+ $\sigma\left(v_{2r-1)}\right)$, $\sigma(v_{2r})$,…, $\sigma(v_n)$)

4. For r = $\frac{n}{2}$ +1, …, n-2, the maximum distance of a vertex from v_r is r-1.

For j= 2, 3, $\cdots, n-r$,v_{r+j} and v_{r-j} are the only vertices at distance j from v_r. Therefore d_{r_j}= $\sigma(v_{r-j})$ + $\sigma(v_{r+j})$, j = 2, 3, …, n - r.

For j = n – r + 1, …, r - 1, v_{r+j} is the only vertex at distance j from v_r. Therefore d_{r_j}= $\sigma(v_{r-j})$ for j = n-r+1, … r-1.

So DnDS(v_r) = ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$ … , $\sigma\left(v_{r-(n-r)}\right)$ + $\sigma\left(v_{r+(n-r)}\right)$, $\sigma(v_{2r-n-1})$,…, $\sigma(v_1)$) = ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$ … , $\sigma(v_n)$ + $\sigma(v_{2r-n})$, $\sigma(v_{2r-n-1})$,…, $\sigma(v_1)$).

5. The maximum distance of a vertex from v_{n-1} is n-2. For j=2,3..n-2, v_{n-1-j} is the only vertex at distance j from v_{n-1}. Therefore, nd_{n-1_j}= $\sigma(v_{n-1-j})$, j = 2,3,…, n-2. Hence, DnDs(v_{n-1}) = ($\sigma(v_{n-1})$, $nd(v_{n-1})$, $\sigma(v_{n-3})$, $\sigma(v_{n-2})$ … , $\sigma(v_1)$).

6. The maximum distance of a vertex from v_n is n-1. For j=2,3..n-2, v_{n-j} is the only vertex at distance j from v_n. Therefore, nd_{n_j}= $\sigma(v_{n-j})$, j = 2,3,…, n-2. Hence, DnDs(v_n)= ($\sigma(v_n)$, $nd(v_n)$, $\sigma(v_{n-2})$, $\sigma(v_{n-3})$ … , $\sigma(v_1)$).

Theorem 3.3.10

Let $P_n : (\sigma, \mu)$ be a fuzzy graph on a path with n vertices $v_1, v_2, v_3, \ldots, v_n$ where n is odd. Then the DnDS of the vertices are as follows:

1. DnDs(v_1)= ($\sigma(v_1)$, $nd(v_i)$, $\sigma(v_3)$, $\sigma(v_4)$, … , $\sigma(v_n)$)

2. DnDs(v_2)= ($\sigma(v_2)$, $nd(v_2)$, $\sigma(v_4)$, $\sigma(v_5)$ … , $\sigma(v_n)$)

3. For r=3,4,… $\frac{n-1}{2}$

DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$ … , $\sigma\left(v_{r-(r-1)}\right)$+$\sigma\left(v_{r+(r-1)}\right)$, $\sigma(v_{2r})$,…, $\sigma(v_{n-1})$).

4. For r= $\frac{n+1}{2}$, DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$ … , $\sigma(v_1)$ + $\sigma(v_{2r-1})$)

5. For r= $\frac{n+3}{2}$, …, n-2,

DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$ + $\sigma(v_{r+2})$ … , $\sigma(v_{2r-n})$ + $\sigma(v_n)$, $\sigma(v_{2r-n-1})$,…, $\sigma(v_1)$)

6. DnDs(v_{n-1})= ($\sigma(v_{n-1})$, $nd(v_{n-1})$, $\sigma(v_{n-3})$, $\sigma(v_{n-2})$ … , $\sigma(v_1)$)

7. DnDs(v_n)= ($\sigma(v_n)$, $nd(v_n)$, $\sigma(v_{n-2})$, $\sigma(v_{n-3})$ … , $\sigma(v_1)$)

Proof

The proofs of (1), (2), (6), and (7) are as in the proof of (1), (2), (5) and (6) of the theorem 5.10.

For r =3,4,… \frac{n-1}2

The maximum distance of a vertex from v_r is n-r. j=2,3,…, r-1, v_{r+j} and v_{r-j} are the only vertices at distance j from v_r. Therefore, d_{r_j}= $\sigma(v_{r-j})$+ $\sigma(v_{r+j})$, j = 2,3,…,r-1.

For j = r, r+1,…, n-r, v_{r+j} is the only vertex at distance j from v_r. Therefore, d_{r_j}= $\sigma(v_{r+j})$ for j = r, r+1,…, n-r.

The DnDS for r = 3, 4, …, $\frac{n+1}{2}$, is given by, DnDS(v_r)= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+ $\sigma(v_{r+2})$, … , $\sigma\left(v_{r-(r-1)}\right)$+ $\sigma\left(v_{r+(r-1)}\right)$, $\sigma(v_{2r})$,…, $\sigma(v_{n-1})$) = ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$, … , $\sigma\left(v_{1)}\right)$+$\sigma\left(v_{2r-1)}\right)$, $\sigma(v_{2r})$,…, $\sigma(v_{n-1})$)

4. For r= $\frac{n+1}{2}$, since r^{th} vertex is the middle vertex of the path graph, the maximum distance of a vertex from v_r is r - 1. v_{r+j} and v_{r-j} are the only vertices at distance j from v_r. Therefore, d_{r_j} = $\sigma(v_{r-j})$ + $\sigma(v_{r+j})$, j = 2, 3, …, r – 1.

So DnDS(v_r) = ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$ + $\sigma(v_{r+2})$, … , $\sigma\left(v_{r-(r-1)}\right)$ + $\sigma\left(v_{r+(r-1)}\right)$).

= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$ … , $\sigma(v_{2r-1})$ + $\sigma(v_1)$).

5. For r = $\frac{n+3}{2}$, …, n-2, the maximum distance of a vertex from v_r is r-1.

For j = 2,3,…, n – r, v_{r+j} and v_{r-j} are the only vertices at distance j from v_r.

Therefore d_{r_j} = $\sigma(v_{r-j})$ + $\sigma(v_{r+j})$, j = 2,3,…, n - r.

For j = n - r +1,…, r -1, v_{r+j} is the only vertex at distance j from v_r.

Therefore d_{r_j} = $\sigma(v_{r+j})$ for j = n - r +1,…, r -1.

So DnDS(v_r) = ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$+$\sigma(v_{r+2})$ … , $\sigma\left(v_{r-(n-r)}\right)$+ $\sigma\left(v_{r+(n-r)}\right)$, $\sigma(v_{2r-n-1})$,…, $\sigma(v_1)$)

= ($\sigma(v_r)$, $nd(v_r)$, $\sigma(v_{r-2})$ + $\sigma(v_{r+2})$, … , $\sigma\left(v_{2r-n)}\right)$+ $\sigma\left(v_{n)}\right)$, $\sigma(v_{2r-n-1})$,…, $\sigma(v_1)$).

6.For r = n - 1, the maximum distance of a vertex from v_{n-1} is n - 2. For j = 2, 3, ..., n - 2, v_{n-1-j} is the only vertex at distance j from v_{n-1}. Therefore, DnDs(v_{n-1}) = ($\sigma(v_{n-1})$, $nd(v_{n-1})$, $\sigma(v_{n-3})$, $\sigma(v_{n-2})$ … , $\sigma(v_1)$).

7. For r = n, the maximum distance of a vertex from v_n is n - 1. For j = 2, 3, ..., n - 2, v_{n-j} is the only vertex at distance j from v_n. Therefore, DnDs(v_n) = ($\sigma(v_n)$, $nd(v_n)$, $\sigma(v_{n-2})$, $\sigma(v_{n-3})$ … , $\sigma(v_1)$).

3.4 Relationship between DDS and DnDS of fuzzy graphs

The relationship between the DDSand the DnDS in fuzzy graphs provides valuable insights into both local and global connectivity patterns within the graph structure.

Theorem 3.4.1

Let G be a fuzzy graph with vertex set $\{v_1, v_2, v_3 \ldots, v_n\}$. Let the DDS of vertex v_i in G be given by $(d_{i_0}, d_{i_1}, d_{i_2} \ldots, d_{i_{e(v_i)}})$ and the distance neighborhood degree of v_i in G be given by $(nd_{i_0}, nd_{i_1}, nd_{i_2} \ldots, nd_{i_j} \ldots, nd_{i_{e(v_i)}})$. For each vertex of the fuzzy graph, d_{i_0} and nd_{i_0} are the same and are equal to $\sigma(v_i)$ and $d_{i_j} \leq nd_{i_j}$ where i=1 to n and j= 1 to e(v_i).

Proof

By the definition of DDS and DnDS, $d_{i_0} = \sigma(v_i) = nd_{i_0}$.

$d_{i_1} = \sum_{u\in N_1(v_i)} \mu(uv_i) \leq \sum_{u\in N_1(v_i)} (\sigma(u) \wedge \sigma(v_i)) \leq \sum_{u\in N_1(v_i)} \sigma(u) = nd_{i_1}$

$d_{i_j} = \sum_{u\in N_j(v_i)} \mu(uw) \leq \sum_{u\in N_j(v_i)} \sigma(u) = nd_{i_j}$, i= 1 to n and j= 1 to e(v_i).

Theorem 3.4.2

Let G: (σ, μ) be a fuzzy graph on a tree with vertex set $\{v_1, v_2, v_3 \ldots, v_n\}$. Let the DDS of the vertex v_i in G be given by $(d_{i_0}, d_{i_1}, d_{i_2} \ldots, d_{i_{e(v_i)}})$ and the distance neighborhood degree of v_i in G be given by $(nd_{i_0}, nd_{i_1}, nd_{i_2} \ldots, nd_{i_j} \ldots, nd_{i_{e(v_i)}})$. If σ and μ are constant functions of the same constant value, then $d_{i_j} = nd_{i_j}$ where i= 1 to n and j = 1 to e(v_i). Hence DDS of the vertex v_i is the same as the DnDS of the vertex v_i, for all i= 1 to n.

Proof

Let σ and μ be constant functions of the same constant value c.

By the definition of DDS and DnDS, $d_{i_0} = \sigma(v_i)$ = c = nd_{i_0}.

$d_{i_1} = \sum_{u\in N_1(v_i)} \mu(uv_i) = \sum_{u\in N_1(v_i)} c = \sum_{u\in N_1(v_i)} \sigma(u) = nd_{i_1}$

$d_{i_j} = \sum_{u\in N_j(v_i)} \mu(uw) = \sum_{u\in N_j(v_i)} c = \sum_{u\in N_j(v_i)} \sigma(u) = nd_{i_j}$, i=1 to n and j= 1 to e(v_i).

Hence DDS of the vertex v_i is the same as the DnDS of the vertex v_i, for all i= 1 to n.

4 Conclusion

In this study, the DDS and the DnDS of fuzzy graphs are introduced and illustrated with examples. The DnDS of some fuzzy graphs is obtained. The relationship between distance degree sequence and DnDS is discussed. Based on these results, distance degree sequences and distance neighborhood degree sequences in fuzzy graphs on ladder graphs, fan graphs, butterfly graphs, etc. can be obtained in the future.

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