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RESEARCH ARTICLE

Coupled Fixed Point Theorems In G_b - Metric Space

K Varalakshmi[1]*, G Upender Reddy[2]

1 Research Scholar, Department of Mathematics, University College of Science. Osmania University, Telangana, Hyderabad, India
2 Associate Professor, Department of Mathematics, Nizam College(A), Osmania University, Hyderabad, India

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* **Corresponding author**.

varak2121@gmail.com

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Abstract

Objective: The purpose of this research paper is to establish coupled common fixed-point theorems in G_b metric space. **Methodology**: By using integral type contractive conditions and weakly compatible mappings. **Findings**: Unique Common coupled fixed-point theorems are proved and two corollaries are stated. Also discussed is an example that supports our main result. **Novelty**: In this research paper we obtained common coupled fixed points by using integral type contractive conditions for given mappings in G_b metric space. These results extend the results of E. L. Ghasaba, H. Majania, G. Soleimani Rad'.

Keywords: $G_b$metric space; Lebesgue Integral; Compatible mappings; Coupled fixed point; Coupled coincidence point

1 Introduction

Banach contraction principle plays an important role in the development of fixed-point theory. Its significance lies in its vast applicability to a great number of branches of mathematical sciences. The generalizations of this principle are made either by using different contractive conditions or by imposing some additional conditions on the ambient spaces. One of the contractions is Integral type contraction introduced by Branciari in 2002. He used the idea of Lebesgue integral in metric fixed point theory and proved the existence and uniqueness of fixed point for integral contraction whenever the metric space is complete. On the other hand, a number of generalizations of metric spaces have been done and one such generalization is a G_b-metric space introduced by Aghajani et al in 2014 by combining both the concepts of b-metric space and G metric space. From the recent literature authors[1–11] investigated the existence and uniqueness of fixed points and coupled fixed points in various metric spaces by using various versions of integral-type contractions. The purpose of this research paper is to prove coupled common fixed-point theorems in G_b metric space by utilizing w-compatible mappings which are weaker conditions and integral-type contractions. We also provided corollary and example to validate our results.

2 Methodology

In this paper, we use a few definitions and examples which are helpful to prove our results.

Definition 2.1:[7]

Let H be any non-void set. If a function G:$H^3 \to [0,\infty)$ satisfies the following possessions such that

(G1) G (ϱ, ϖ, $\check{\mathfrak{z}}$) = 0 implies $\varrho = \varpi = \check{\mathfrak{z}}$;
(G2) G (ϱ,,ϖ) > 0 for all, ϱ, ϖ ϵ H with $\varrho \neq \varpi$;

(G3) G (ϱ,,ϖ) $\leq$ G (ϱ,ϖ,$\check{\mathfrak{z}}$), $\forall$ ϱ,ϖ,$\check{\mathfrak{z}}$ ϵ H with $\varpi \neq \check{\mathfrak{z}}$;

(G4) G (ϱ,,$\check{\mathfrak{z}}$) = G (ϱ, $\check{\mathfrak{z}}$, ϖ) = G (ϖ,ϱ, $\check{\mathfrak{z}}$) = _ _ _ _ _ (symmetry)

(G5) G (ϱ,,$\check{\mathfrak{z}}$)) $\leq$ G (ϱ, ꝑ, ꝑ) + G (ꝑ, ϖ, $\check{\mathfrak{z}}$) for all ϱ,ϖ,$\check{\mathfrak{z}}$, ꝑ ϵ H

Then the function G together with H, that is (H, G) is called a G-metric space.

Definition 2.2[8]

Let $\widehat{U}$ be a non-void set, $\tau \geq 1$ is a real number.
A function d: $\widehat{U}^2 \to$ [0, ∞) is said to be a b-metric on $\widehat{U}$ $\forall$ $\lambda, \mu, \nu \epsilon \widehat{U}$ if
the following properties hold.
1: $d\,(\lambda,\,\mu) = 0 \leftrightarrow \lambda = \mu$
2: $d\,(\lambda,\,\mu) = d\,(\mu,\,\lambda)$
3: $d\,(\lambda,\,\nu) \leq \tau\,[(d\,(\lambda,\mu) + d\,(\mu,\,\nu))]$
The pair ($\widehat{U}$, d) is called a b-metric space, and the number $\tau \geq 1$ is called the coefficient of ($\widehat{U}$, d).

Definition 2.3[9]

Consider $\dot{S}$ to be a non-void set and $\tau \geq$ 1be a real number. Assume
that $G_b : \dot{S}^3 \to [0,\infty)$ fulfills.

(G_b1) G_b (ꝑ,ꝗ,ϖ) = 0 implies ꝑ=ꝗ=ϖ;

(G_b2) G_b (ꝑ,ꝑ,ꝗ) > 0 for all ꝑ, ꝗ ϵ Ṡ with ꝑ ≠ ꝗ

(G_b3) G_b (ꝑ,ꝑ,ꝗ) $\leq$ G_b (ꝑ,ꝗ,ϖ) for all ꝑ,ꝗ,ϖ ϵ Ṡ with ꝗ ≠ ϖ

(G_b4) G_b (ꝑ,ꝗ,ϖ) = G_b (ꝗ,ϖ,ꝑ) = G_b (ϖ,ꝑ, ꝗ) = _ _ _ _ _ (symmetry)

(G_b5) G_b (ꝑ,ꝗ,ϖ) $\leq$ τ [G_b (ꝑ, σ, σ) + G_b (σ, ꝗ, ϖ)] for all ꝑ,ꝗ,ϖ ϵ Ṡ

Then ($\dot{S}$, G_b) is called a G_b metric space and G_b is called a G_b metric on $\dot{S}$.

Definition 2.4 :[10]

Assume M be a nonvoid set. The mappings
$\dot{\eta} : \mathrm{M} \times \mathrm{M} \to \mathrm{M}$ and $\chi : \mathrm{M} \to \mathrm{M}$are commutative if
$\chi\dot{\eta}(6,\vartheta,) = \dot{\eta}(\chi 6, \chi\vartheta) \forall 6, \vartheta \in \mathrm{M}$.

Definition 2.5:[10]

Let $\mathfrak{E}$ be any non-void set,$\grave{\eta} : \mathfrak{E}^2 \to \mathfrak{E}$ & $\chi : \mathfrak{E} \to \mathfrak{E}$ are said to be w - compatible iff $\chi\dot{\eta}(6,\vartheta) = \dot{\eta}(\chi 6, \chi\vartheta)$whenever $\dot{\eta}(6,\vartheta) = \chi 6 \& \dot{\eta}(\vartheta, 6) = \chi\vartheta, \forall 6, \vartheta \in \mathfrak{G}$

Definition 2.6:[11]

A component $(\sigma,\tau)\in \overset{\vee}{U}\times\overset{\vee}{U}$ is called a coupled fixed point of a mapping $M:\overset{\vee}{U}\times\overset{\vee}{U}\rightarrow\overset{\vee}{U}\, if\, M(\sigma,\tau)=\sigma\, and\, M(\tau,\sigma)=\tau$

Definition 2.7 :[11]

A component $(\sigma,\tau)\in\mathfrak{E}\times\mathfrak{E}$ is said to be a coupled coincidence point of a mapping $M:\ \mathfrak{E}\, x\, \mathfrak{E}\rightarrow\mathfrak{E}\, and\, t:\ \mathfrak{E}\rightarrow\mathfrak{E}\, if\ \ M(\sigma,\tau)= t\sigma\, and\, M(\tau,\sigma)=\ t\tau.$

Definition 2.8

Let $\varphi,\ \psi:\ [0,\ \infty)\ \rightarrow\ [0,\ \infty)$ be two functions, we consider the following

($ø_1$) ø is non-increasing on (0, ∞);

($ø_2$) ø is Lebesgue integrable;

($ø_3$) for any $\epsilon>0, \int_0^{\epsilon}\ \varphi(t)\,dt>0$;

($ø_4$) ø is continuous;

and

(ψ_1) ψ is non-decreasing;

$(\psi 2)\ \psi\,()\ \leq\ \ \forall,\ \ >\ 0;$

(ψ_3) ψ is additive function;

$(\psi 4)\ \sum_{i=0}^{\infty}2^{i+n-1}s^{i+1}\psi^{i+n}<\infty,\forall\, s\ >\ 0,\, n\ >\ 0.$

3 Results and Discussion

Theorem 3.1:

Assume (U, G_b) is a Complete G_b- metric space. $\ddot{\gamma}:\ U^2\rightarrow U,\ \dot{f}:\ U\rightarrow U$ are two mappings and $\psi\ \geq\ 0$ such a way

$$\int_0^{G_b(\ddot{\gamma}(,),\ddot{\gamma}\left(\overset{\vee}{\alpha},\overset{\vee}{\eta}\right),\ \ddot{\gamma}(\mathfrak{M},\mathfrak{N}))}\phi(t)\,dt\ \leq\ \ \psi\int_0^{G_b\left(\dot{f},\dot{f}\overset{\vee}{\alpha},\dot{f}\mathfrak{M},\right)+G_b\left(\dot{f}\phi,\dot{f}\overset{\vee}{\eta},\dot{f}\mathfrak{N},\right)}\phi(t)\,dt \tag{1}$$

$\forall,\phi,\overset{\vee}{\alpha},\overset{\vee}{\eta}\mathfrak{M},\mathfrak{N},\in R\ with\ \ s\geq 1\ and$

a) $\ddot{\gamma}(U^2)\ \subseteq\ \dot{f}(U)$

b) $\dot{f}(U)\ is\ complete$

c) $(\ddot{Y},\ \dot{f})$

Then one can find unique coupled common fixed points of $\ddot{\gamma}$ and $\dot{f}$ on U.

Proof:

Consider, $\{\vartheta_p\}$,$\{\varpi_p\}$and$\{\overset{\vee}{3}_p\}$, $\{\ \ddot{\eta}_p\ \}$ be sequences on U such that $\ddot{\gamma}(\vartheta_p,\ \varpi_p)\ =\ \dot{f}\vartheta_{p+1}\ =\ \overset{\vee}{3}_p$ and $\ddot{\gamma}(\varpi_p,\ \vartheta_p)\ =\ \dot{f}\varpi_{p+1}= \overset{\vee}{\eta}_p$

also, from Equation (1)

$\int_0^{G_b(\overset{\vee}{3}_p,\overset{\vee}{3}_{p+1},\overset{\vee}{3}_{p+1})}\ \emptyset(t)\,dt=\int_0^{G_b((\vartheta_p,\ \varpi_p),\ (\vartheta_{p+1},\ \varpi_{p+1}),\ (\vartheta_{p+1},\varpi_{p+1}))}\emptyset(t)\,dt$

$\leq\ \psi\int_0^{G_b\left(\dot{f}\vartheta_p,\dot{f}\vartheta_{p+1},\dot{f}\vartheta_{p+1}\right)+G_b\left(\dot{f}\varpi_p,\dot{f}\varpi_{p+1},\dot{f}\varpi_{p+1}\right)}\emptyset(t)\,dt$

$$\int_0^{G_b(\overset{\vee}{3}_p,\ \overset{\vee}{3}_{p+1},\ \overset{\vee}{3}_{p+1})}\emptyset(t)\,dt\leq\psi\int_0^{G_b\left(\overset{\vee}{3}_{p-1},\overset{\vee}{3}_p,\overset{\vee}{3}_p\right)+G_b(\ddot{\eta}_{p-1},\ddot{\eta}_p,\ddot{\eta}_p)}\emptyset(t)\,dt \tag{2}$$

Similarly,

$\int_0^{G_b(\ddot{\eta}_p,\ddot{\eta}_{p+1},\ddot{\eta}_{p+1})}\emptyset(t)\,dt=\int_0^{G_b(\ddot{\gamma}(\varpi_p,\vartheta_p),\ddot{\gamma}(\varpi_{p+1},\vartheta_{p+1}),\ddot{\gamma}(\varpi_{p+1},\vartheta_{p+1}))}\emptyset(t)\,dt$

$\leq\ \psi\int_0^{G_b\left(\dot{f}\varpi_p,\dot{f}\varpi_{p+1},\dot{f}\varpi_{p+1}\right)+G_b\left(\dot{f}\vartheta_p,\dot{f}\vartheta_{p+1},\dot{f}\vartheta_{p+1}\right)}\emptyset(t)\,dt$

$\leq\psi\int_0^{G_b(\ddot{\eta}_{p-1},\ddot{\eta}_p,\ \ddot{\eta}_p)+G_b(\overset{\vee}{3}_{p-1},\ \overset{\vee}{3}_p,\ \overset{\vee}{3}_p)}\emptyset(t)\,dt$

$\leq \psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_p,\check{\mathfrak{z}}_p\right)+G_b(\left(\check{\eta}_{p-1},\check{\eta}_p,\check{\eta}_p\right)} \emptyset(t)\,dt$

We can write for non-decreasing function

$\int_0^{a+b} \emptyset(t)\,dt \leq \int_0^{a} \emptyset(t)\,dt + \int_0^{b} \emptyset(t)\,dt$

from Equation (2) we can write,

$$\int_0^{G_b(\check{\mathfrak{z}}_p,\, \check{\mathfrak{z}}_{p+1},\, \check{\mathfrak{z}}_{p+1})} \emptyset(t)\,dt \leq \psi \int_0^{G_b(\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_p,\check{\mathfrak{z}}_p)} \emptyset(t)\,dt + \psi \int_0^{G_b(\ddot{n}_{p-1},\ddot{n}_p,\ddot{n}_p)} \emptyset(t)\,dt \tag{3}$$

$\int_0^{G_b(\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_p,\check{\mathfrak{z}}_p)} \emptyset(t)\,dt \leq \psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1}\right)+G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})} \emptyset(t)\,dt$

$\int_0^{G_b(\ddot{n}_{p-1},\ddot{n}_p,\ddot{n}_p)} \emptyset(t)\,dt \leq \psi \int_0^{G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})+G_b(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1})} \emptyset(t)\,dt$

$\leq \psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1}\right)+G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})} \emptyset(t)\,dt$

From Equation (3)

$\int_0^{G_b(\check{\mathfrak{z}}_p,\, \check{\mathfrak{z}}_{p+1},\, \check{\mathfrak{z}}_{p+1})} \emptyset(t)\,dt \leq \psi[\psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1}\right)+G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})} \emptyset(t)\,dt]$

$+\psi[\psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1}\right)+G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})} \emptyset(t)\,dt]$

$\leq \psi[2\psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1}\right)+G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})} \emptyset(t)\,dt]$

$\leq 2\psi^2[\int_0^{G_b\left(\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-1},\check{\mathfrak{z}}_{p-1}\right)+G_b(\ddot{n}_{p-2},\ddot{n}_{p-1},\ddot{n}_{p-1})} \emptyset(t)\,dt]$

$\leq 2\psi^2[2\psi \int_0^{G_b\left(\check{\mathfrak{z}}_{p-3},\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-2}\right)+G_b(\ddot{n}_{p-3},\ddot{n}_{p-2},\ddot{n}_{p-2})} \emptyset(t)\,dt]$

$\leq 4\psi^3 \int_0^{G_b\left(\check{\mathfrak{z}}_{p-3},\check{\mathfrak{z}}_{p-2},\check{\mathfrak{z}}_{p-2}\right)+G_b(\ddot{n}_{p-3},\ddot{n}_{p-2},\ddot{n}_{p-2})} \emptyset(t)\,dt]$

$\leq 8\psi^4 \int_0^{G_b\left(\check{\mathfrak{z}}_{p-4},\check{\mathfrak{z}}_{p-3},\check{\mathfrak{z}}_{p-3}\right)+G_b(\ddot{n}_{p-4},\ddot{n}_{p-3},\ddot{n}_{p-3})} \emptyset(t)\,dt]$

$$\int_0^{G_b(\check{\mathfrak{z}}_p,\, \check{\mathfrak{z}}_{p+1},\, \check{\mathfrak{z}}_{p+1})} \emptyset(t)\,dt \leq 2^{p-1}\psi^p \int_0^{G_b\left(\check{\mathfrak{z}}_0,\check{\mathfrak{z}}_1,\check{\mathfrak{z}}_1\right)+G_b(\ddot{n}_0,\ddot{n}_1,\ddot{n}_1)} \emptyset(t)\,dt \tag{4}$$

Similarly,

$$\int_0^{G_b(\ddot{n}_p,\ddot{n}_{p+1},\ddot{n}_{p+1})} \emptyset(t)\,dt \leq 2^{p-1}\psi^p \int_0^{G_b\left(\check{\mathfrak{z}}_0,\check{\mathfrak{z}}_1,\check{\mathfrak{z}}_1\right)+G_b(\ddot{n}_0,\ddot{n}_1,\ddot{n}_1)} \emptyset(t)\,dt \tag{5}$$

for m>n

$\int_0^{G_b(\check{\mathfrak{z}}_n,\check{\mathfrak{z}}_m,\check{\mathfrak{z}}_m)} \emptyset(t)\,dt \leq \int_0^{s[G_b\left(\check{\mathfrak{z}}_n,\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_{n+1}\right)+G_b\left(\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_m,\check{\mathfrak{z}}_m\right)]} \emptyset(t)\,dt$

$\leq \int_0^{sG_b\left(\check{\mathfrak{z}}_n,\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_{n+1}\right)} \emptyset(t)\,dt + \int_0^{sG_b\left(\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_m,\check{\mathfrak{z}}_m\right)} \emptyset(t)\,dt$

$\leq \int_0^{sG_b\left(\check{\mathfrak{z}}_n,\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_{n+1}\right)} \emptyset(t)\,dt +$

$\int_0^{s^2G_b\left(\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_{n+2},\check{\mathfrak{z}}_{n+2}\right)+s^3G_b\left(\check{\mathfrak{z}}_{n+2},\check{\mathfrak{z}}_{n+3},\check{\mathfrak{z}}_{n+3}\right)+s^4G_b\left(\check{\mathfrak{z}}_{n+3},\check{\mathfrak{z}}_{n+4},\check{\mathfrak{z}}_{n+4}\right)\pm-\mp s^{m-n}G_b\left(\check{\mathfrak{z}}_{m-1},\check{\mathfrak{z}}_m,\check{\mathfrak{z}}_m\right)} \emptyset(t)\,dt$

$\leq \int_0^{sG_b(\check{\mathfrak{z}}_n,\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_{n+1})} \emptyset(t)\,dt + \int_0^{s^2G_b\left(\check{\mathfrak{z}}_{n+1},\check{\mathfrak{z}}_{n+2},\check{\mathfrak{z}}_{n+2}\right)} \emptyset(t)\,dt + \int_0^{s^3G_b\left(\check{\mathfrak{z}}_{n+2},\check{\mathfrak{z}}_{n+3},\check{\mathfrak{z}}_{n+3}\right)} \emptyset(t)\,dt + - - - +$
$\int_0^{s^{m-n}G_b(\check{\mathfrak{z}}_{m-1},\check{\mathfrak{z}}_m,\check{\mathfrak{z}}_m)} \emptyset(t)\,dt$

$$\leq s\int_0^{G_b((\check{3}_n,\check{3}_{n+1},\check{3}_{n+1})}\emptyset(t)\,dt+s^2\int_0^{G_b(\check{3}_{n+1},\check{3}_{n+2},\check{3}_{n+2})}\emptyset(t)\,dt+$$
$$s^3\int_0^{G_b(\left(\check{3}_{n+2},\check{3}_{n+3},\check{3}_{n+3}\right))}\emptyset(t)\,dt+s^4\int_0^{G_b(\left(\check{3}_{n+3},\check{3}_{n+4},\check{3}_{n+4}\right)}\emptyset(t)\,dt+--$$
$$+s^{m-n}\int_0^{G_b(\check{3}_{m-1},\check{3}_m,\check{3}_m)}\emptyset(t)\,dt$$

From Equation (4)

$$\leq s2^{n-1}\psi^n\int_0^{G_b\left(\check{3}_0,\check{3}_1,\check{3}_1\right)+G_b(\check{\eta}_0,\check{\eta}_1,\check{\eta}_1)}\emptyset(t)\,dt+s^2 2^n\psi^{n+1}\int_0^{G_b\left(\check{3}_0,\check{3}_1,\check{3}_1\right)+G_b(\check{\eta}_0,\check{\eta}_1,\check{\eta}_1)}\emptyset(t)\,dt+$$

$$s^3.2^{n+1}\psi^{n+2}\int_0^{G_b\left(\check{3}_0,\check{3}_1,\check{3}_1\right)+G_b(\check{\eta}_0,\check{\eta}_1,\check{\eta}_1)}\emptyset(t)\,dt+--$$

$$+s^{m-n}2^{m-1}\psi^m\int_0^{G_b\left(\check{3}_0,\check{3}_1,\check{3}_1\right)+G_b(\check{\eta}_0,\check{\eta}_1,\check{\eta}_1)}\emptyset(t)\,dt$$

$$\int_0^{G_b(\check{3}_n,\check{3}_m,\check{3}_m)}\emptyset(t)\,dt\ \leq\ \sum_{i=0}^{m-n}2^{i+n-1}s^{i+1}\psi^{i+n}\int_0^{G_b\left(\check{3}_0,\check{3}_1,\check{3}_1\right)+G_b(\check{\eta}_0,\check{\eta}_1,\check{\eta}_1)}\emptyset(t)\,dt$$
$$<\sum_{i=0}^{\infty}2^{i+n-1}s^{i+1}\psi^{i+n}\int_0^{G_b\left(\check{3}_0,\check{3}_1,\check{3}_1\right)+G_b(\ddot{n}_0,\ddot{n}_1,\ddot{n}_1)}\emptyset(t)\,dt<\infty$$
$$\therefore as\ m,n\ \to\infty,\int_0^{G_b(\check{3}_n,\check{3}_m,\check{3}_m)}\emptyset(t)\,dt\ \to 0$$
$$G_b\left(\check{3}_n,\check{3}_m,\check{3}_m\right)\to 0$$

$\{\check{3}_n\}$ *is a cauchy sequence* in U
Similarly, $\{\ddot{n}_n\}$ *is* a cauchy sequence in U.
But $\dot{f}$(U) is a complete subspace of (U, G_b) then there exist the sequences
$\{\check{3}_n\},\{\ddot{n}_n\}$ *are in* $\dot{f}(U)$ *such that they converges to* $\check{3}$, $\ddot{n}\varepsilon$ $\dot{f}$(U)
in complete G_b metric space ($\dot{f}$(U),G_b).
i.e, $\lim_{n\to\infty}\check{3}=\check{3}$ *and* $\lim_{n\to\infty}\check{\eta}_n=\ddot{n}$
Since $\dot{f}:U\to U and\ \check{3}$ $\ddot{n}\varepsilon$ $\dot{f}$(U), there exists , ε U such that $\dot{f}$ = $\check{3}$ and $\dot{f}$ = $\ddot{n}$
Therefore $\lim_{n\to\infty}\check{3}_n=\check{3}=\dot{f}\psi$ *and* $\lim_{n\to\infty}\check{\eta}_n=\ddot{n}=\ \dot{f}\psi$
Now we claim that $\ddot{\gamma}(,)=\ \check{3}$ *and* $\ddot{\gamma}(,)=\ddot{n}$

$$\int_0^{G_b(\ddot{\gamma}(,)),\check{3}_{n+1},\check{3}_{n+1})}\emptyset(t)\,dt=\int_0^{G_b(\ddot{\gamma}(,),\ddot{\gamma}(\vartheta_{n+1},\varpi_{n+1}),\ddot{\gamma}(\vartheta_{n+1},\varpi_{n+1})}\emptyset(t)\,dt$$
$$\leq\psi\int_0^{G_b\left(\dot{f},\dot{f}\vartheta_{n+1},\dot{f}\vartheta_{n+1}\right)+G_b\left(\dot{f}\phi,\dot{f}\varpi_{n+1},\dot{f}\varpi_{n+1}\right)}\emptyset(t)\,dt$$
$$\leq\psi\int_0^{G_b\left(\dot{f},\check{3}_n,\check{3}_n\right)+G_b\left(\dot{f},\ddot{n}_n,\ddot{n}_n\right)}\emptyset(t)\,dt$$
$$\lim_{n\to\infty}G_b\left(\dot{f},\check{3}_n,\check{3}_n\right)\Rightarrow G_b\left(\dot{f},\check{3},\check{3}\right)\to 0$$

Similarly, $\lim_{n\to\infty}G_b\left(\dot{f},\ddot{n}_n,\ddot{n}_n\right)\Rightarrow G_b\left(\dot{f},\ddot{n},\ddot{n}\right)\to 0$

$$\int_0^{G_b(\ddot{\gamma}(,),\check{3}_{n+1},\check{3}_{n+1})}\emptyset(t)\,dt=\int_0^{G_b(\ddot{\gamma}(,),\check{3},\check{3})}\emptyset(t)\,dt=0\ as\ n\to\infty$$

Therefore $\ddot{\gamma}(,)=\ \check{3}\ =\ \dot{f}$ *and* $\ddot{\gamma}(,)\ =\ \check{\eta}=\ \dot{f}$
$\ddot{\gamma}(,)=\ \dot{f}$ *and* $\ddot{\gamma}(,\ )\ =\ \dot{f}$

$(,)$ is coupled coincidence point of $\ddot{\gamma}$ and $\dot{f}$.
Since $(\ddot{\gamma}, \dot{f})$ is $\omega-$compatible, then we have
$\ddot{Y}(\check{3}, \check{\eta}) = \dot{f}\check{3}$ *and* $\ddot{Y}(\check{\eta}, \check{3}) = \dot{f}\check{\eta}$

Now we have to prove $\dot{f}\check{3} = \check{3}$ *and* $\dot{f}\check{\eta} = \check{\eta}\check{3}$ and $\dot{f}\ddot{n} = \ddot{n}$

Since $\int_0^{G_b(\dot{f}\check{3}, \check{3}_{n+1}, \check{3}_{n+1})} \emptyset(t)\,dt = \int_0^{G_b(\ddot{\gamma}(\check{3}, \ddot{n}), \ddot{\gamma}(\vartheta_{n+1}, \varpi_{n+1}), \ddot{\gamma}(\vartheta_{n+1}, \varpi_{n+1})} \emptyset(t)\,dt$
$\leq \psi \int_0^{G_b(\dot{f}\check{3},\ \dot{f}\vartheta_{n+1}, \dot{f}\vartheta_{n+1}) + G_b(\dot{f}\ddot{n},\ \dot{f}\varpi_{n+1}, \dot{f}\varpi_{n+1})} \emptyset(t)\,dt$
$\leq \psi \int_0^{G_b(\dot{f}\check{3}, \check{3}_n, \check{3}_n) + G_b(\dot{f}\ddot{n}, '\ddot{n}_n, '\ddot{n}_n)} \emptyset(t)\,dt$
This inequality holds if as n$\rightarrow\infty$,$G_b\left(\dot{f}\check{3}, \check{3}, \check{3}\right) \rightarrow 0 \Rightarrow \check{3} = \dot{f}\check{3}$
and $G_b\left(\dot{f}\ddot{n}, \ddot{n}, \ddot{n}\right) \rightarrow 0\ \Rightarrow \dot{f}\ddot{n} = \ddot{n}$
$\Rightarrow (\ddot{\gamma}\left(\check{3}, '\check{\eta}\right) = \dot{f}\check{3} = \check{3}$*and* $\ddot{\gamma}(\ddot{n}, \check{\alpha}) = \dot{f}\ddot{n} = \ddot{n}$

$\therefore (\check{3}, \ddot{n})$ *is common coupled fixed point* of $\ddot{\gamma}$ and $\dot{f}$.

If possible, let assume that $(\check{3}^1, \ddot{n}^1)$ is another coupled fixed point of $\ddot{\gamma}$ and $\ddot{n}$.
Now,

$$\int_0^{G_b\left(\check{3}^1, \check{3}_{n+1}, \check{3}_{n+1}\right)} \phi(t)\,dt = \int_0^{G_b\left(\left(\check{3}^1, \ddot{n}^1\right),\ \ddot{Y}(\vartheta_{n+1}, \varpi_{n+1}),\ \ddot{Y}(\vartheta_{n+1}, \varpi_{n+1})\right)} \phi(t)\,dt$$

$\leq \psi \int_0^{G_b\left(\dot{f}\check{3}^1, \dot{f}\vartheta_{n+1}, \dot{f}\vartheta_{n+1}\right) + G_b\left(\dot{f}'\check{\eta}^1, \dot{f}\varpi_{n+1}, \dot{f}\varpi_{n+1}\right)} \emptyset(t)\,dt$
$\leq \psi \int_0^{G_b\left(\dot{f}\check{3}^1, \check{3}_n, \check{3}_n\right) + G_b\left(\dot{f}\ddot{n}^1, \ddot{n}n, \ddot{n}_n\right)} \emptyset(t)\,dt$
This inequality holds if $G_b\left(\check{\eta}^1, \check{\eta}, \check{\eta}\right) \rightarrow 0\ as\ n \rightarrow \infty$,
and $G_b\left(\ddot{n}^1, \ddot{n}, \ddot{n}\right) \rightarrow 0\ as\ n \rightarrow \infty$,
$\Rightarrow \check{3}^1 = \check{3}$ and $\ddot{n}^1 = \ddot{n}$

Corollary 1.1:
Let (U, G_b) *be a complete* G_b-*metric space*. $\ddot{\gamma} : U^2 \rightarrow U$ *and* $\Omega :\ U\ \rightarrow\ U$ are two mappings such that

$$\int_0^{G_b(\ddot{Y}(,), \ddot{y}(\vartheta,\rho), \ddot{y}(\sigma,\tau))} \emptyset(t)\,dt\ \leq\ \int_0^{max\{G_b\left(\Omega, \Omega\vartheta, \Omega\sigma\right), G_b\left(\Omega q, \Omega\rho, \Omega\tau\right)\}} \emptyset(t)\,dt$$

and
a) $\ddot{Y}(U^2) \subseteq \Omega(U)$ *and* $\Omega(U)$ *is complete*
b) The pair $(\ddot{Y}, \Omega)$ is $\omega-$compatible then one can exist a unique common coupled fixed point of $\ddot{\gamma}$ *and* Ω.

Corollary 1.2:
Let (U, G_b) *be a complete* G_b-*metric space*
$\ddot{n} : U^2 \rightarrow U$ be a mapping such that

$$\int_0^{G_b(\ddot{n}(,), \ddot{n}(\vartheta,\rho), \ddot{n}(\sigma,\tau))} \emptyset(t)\,dt\ \leq\ \psi\ \int_0^{G_b(,\vartheta,\sigma)+(,\rho,\tau)} \emptyset(t)\,dt$$

then $\grave{\eta}$ has a unique coupled fixed point in U.

Example:

Let U = [0, 1] and $G_b : U^2 \to [0, \infty)$ be a mapping
defined as $G_b(\kappa, \lambda, \mu) = |\kappa - \lambda| + |\lambda - \mu| + |\mu - \kappa|, \ \kappa \forall, \lambda, \mu \epsilon U$
then (U, G_b) is complete $G_b - metric\ space.$
Now a mapping $\ddot{Y} : U^2 \to U$ defined as $\ddot{Y}(\kappa, \lambda,) = \frac{\kappa\lambda}{10} \quad and$

$$\psi(t) = \frac{t}{20}\ \forall\ t\ \epsilon\ [0.\infty]$$

We can write $|\kappa p - \lambda q| \leq |\kappa - \lambda| + |p - q|$ holds $\forall \kappa, \lambda, p, q\ \epsilon\ U$
and $\Omega : U \to U$ be a mapping defined as $\Omega(\kappa) = 2k forall\ k \epsilon\ U$
Now

$$\int_0^{G_b((\ddot{Y}(,q), \ddot{Y}(\vartheta, \rho), \ddot{Y}(\sigma, \tau))} \emptyset(t)\, dt$$

$$= \int_0^{G_b(|(\ddot{Y}(,q) - \ddot{Y}(\vartheta, \rho)| + |(\ddot{Y}(\vartheta, \rho) - \ddot{Y}(\sigma, \tau)| + (\ddot{Y}(\sigma, \tau) - \ddot{Y}(,q)|)} \emptyset(t)\, dt$$

$= \int_0^{\left|\frac{\psi q}{10} - \frac{\vartheta\rho}{10}\right| + \left|\frac{\vartheta\rho}{10} - \frac{\sigma\tau}{10}\right| + \left|\frac{\sigma\tau}{10} - \frac{\psi q}{10}\right|} \emptyset(t)\, dt$
$= \frac{1}{10} \int_0^{|q - \vartheta\rho| + |\vartheta\rho - \sigma\tau| + |\sigma\tau - q|} \emptyset(t)\, dt$
$\leq \frac{1}{10} \int_0^{|-\vartheta| + |q - \rho| + |\vartheta - \sigma| + |\rho - \tau| + |\sigma - | + |\tau - q|} \emptyset(t)\, dt$
$\leq \frac{1}{10} \int_0^{|-\vartheta| + |\vartheta - \sigma| + |\sigma - | + |q - \rho| + |\rho - \tau| + |\tau - q|} \emptyset(t)\, dt$
$\leq \frac{1}{20} \int_0^{|2 - 2\vartheta| + |2\vartheta - 2\sigma| + |2\sigma - 2| + |2q - 2\rho| + |2\rho - 2\tau| + |2\tau - 2q|} \emptyset(t)\, dt$
$\leq \frac{1}{20} \int_0^{G_b\left|\Omega - \Omega\vartheta\right| + \left|\Omega\vartheta - \Omega\sigma\right| + \left|\Omega\sigma - \Omega\right| + \left|\Omega q - \Omega\rho\right| + \left|\Omega\rho - \Omega\tau\right| + \left|\Omega\tau - \Omega q\right|} \emptyset(t)\, dt$
$\leq \psi \int_0^{G_b\left(\Omega, \Omega\vartheta, \Omega\sigma\right) + \left(\Omega q, \Omega\rho, \Omega\tau\right)} \emptyset(t)\, dt$
From the main theorem, we can conclude that $\ddot{\gamma}$ and Ω have coupled common fixed points (0,0) in U.

This example also supports corollary 1.2 if $\psi(t) = \frac{t}{10}\ \forall\ t\ \epsilon\ (0, \infty)$.

4 Conclusion

In this research paper, we proved common coupled fixed-point theorems by using integral type contraction in the context of G_b metric space and also discussed an example that supports our results.

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