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On Radio k – Chromatic Number for Mycielski of Some Graphs

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Abstract

Objectives: In radio networks, the main difficulty is managing the radio spectrum, assigning radio frequencies to transmitters optimally without any interferences. This study aims to find the smallest span of Mycielski of some graphs, the maximum color assigned to any node is called span. **Methods :** This study focused on the problem of reducing interference by modeling it with a radio k - coloring problem on graphs. Where transmitters are modeled as nodes in a graph, with edges connecting nodes that represent transmitters in close proximity to one another. For a graph G, with node set V, edge set E and an integer $1 \leq k \leq diam(G)$, a radio k - coloring of G is a function $f : V(G) \rightarrow 0, 1, 2, \ldots, n$ satisfying the condition $|f(x) - f(y)| \geq 1 + k - d(x, y)$ for any two nodes x and y, where $d(x, y)$ is the distance between x and y in G. The radio k - coloring of f, is the maximum color assigned to any node of G and it is denoted by $rc_k(f)$. The radio k -chromatic number of G is the minimum value of $rc_k(f)$ taken over all radio k - coloring of G and it is denoted by $rc_k(G)$. **Findings:** This study obtained the radio k-chromatic number for Mycielski of some graphs for $k = 2$ and 3. **Novelty:** To solve the channel assignment problem in radio transmitters, the interference graph is developed, and the channel assignment has been converted into a graph coloring. Reducing the interference by a radio k - coloring problem will motivate many researchers to find the radio k - coloring in various graphs.

Keywords: Radio k – Coloring; Mycielski graph; Double Star graph; Triple Star graph; Sunlet graph; Helm graph and Closed Helm graph

1 Introduction:

All the graphs considered in this paper are non-trivial, simple, and undirected. Let G be a graph with a node set V and edge set E. A proper k – coloring of a graph $G = (V, E)$ is a partition $p = \{v_1, v_2, \ldots . v_k\}$ of V into independent sets. The chromatic number $\chi(G)$, of G is the minimum integer k such that G has a proper k-coloring. The distance $d(x, y)$ between two nodes x and y of a graph G is the length of the shortest path connecting x and y. The diameter $diam(G)$ of a graph G, is the maximum $d(x, y)$ taken over every pair of nodes x, y of G.

The frequency assignment problem is a problem of getting an optimal allocation of frequencies to transmitters. The significance of the frequency assignment problem is increasing significantly due to the fast increase in radio networks and the relatively restricted radio spectrum. The frequency assignment problem (FAP) is to assign radio frequencies to transmitters at dissimilar positions without causing interference and reducing the span. In graph theory, graph coloring plays a major part in FAP. Frequencies are assigned to nodes with the aim of either reducing the spectrum used or reducing interference. That is, if V is the set of all transmitters with $v \in V, f(v)$ denotes the continuum of frequencies assigned to v by the assignment rule f. The problem of channel assignment to FM radio stations is one of the most recent frequency assignment problems. Chartrand et al.[1] have introduced radio k-coloring of graphs as a variation of the channel assignment problem.

For any positive integer k, $1 \leq k \leq diam(G)$, a radio k – coloring is an assignment of non-negative integers to the nodes of G such that $|f(x) - f(y)| \geq 1 + k - d(x,y)$ for every two distinct nodes x and y in G. The span of f is the largest integer assigned by f and it is denoted by $rc_k(f)$. The radio k – chromatic number $rc_k(G)$ of G is the minimum among the span of all radio k – coloring of G. Chartrand[1] have introduced new graph coloring when $k = diam(G)$, $k = diam(G) - 1$, $k = diam(G) - 2$ of radio k – chromatic number $rc_k(G)$ is called the radio number $rn(G)$, radio antipodal number $ac(G)$ and nearly antipodal number $ac^{'}(G)$ respectively. The radio k-chromatic number has been studied by several researchers[2,3].

In 2019, Niranjan P. K. and Srinivasa Rao Kola obtained the radio number for the corona of paths and cycles[4] and also proved the radio k-chromatic number of paths in 2022[5]. In 2019, Laxman Saha gave an upper bound for the radio k-chromatic number of graphs in connection with the partition of the vertex set[6]. In 2019, Elsayed M. Badr and Mahmoud I. Moussa has given the upper bound of the radio k-coloring problem and its integer linear programming model[7].

In this paper, we obtained the radio k-chromatic number for the integer $k = 2$ and 3 for Mycielski graphs of Double Star graph, Triple Star graph, Helm graph, Closed Helm graph, and Sunlet graph. As a consequence of a span of f is accompanied by the total number of colors assigned and the missing colors of the graph.

2 Preliminaries:

Definition 2.1:[1] A radio k – coloring of a graph G is an assignment f of non-negative integers to the nodes of G such that $|f(x) - f(y)| \geq 1 + k - d(x,y)$ for every pair of x and y nodes in G. The span $rc_k(f)$ of f is $max\{f(v) : v \in V(G)\}$. The radio k – chromatic number $rc_k(G)$ of G is $min\{rc_k(f) : f$ is a radio k – coloring of $G\}$.

Definition 2.2: [8] The Mycielski graph $\mu(G)$ of graph G with node set $\{x_i : 1 \leq i \leq n\}$ is a graph G obtained from G by adding $n+1$ new nodes $\{y, y_i : 1 \leq i \leq n\}$ joining y to each node $\{y_i : 1 \leq i \leq n\}$ and joining $\{y_i : 1 \leq i \leq n\}$ to each neighbor of $\{x_i : 1 \leq i \leq n\}$ in G.

Definition 2.3:[9] The Helm graph (H_n) is obtained from the Wheel graph $W_{1,n}$ by adjoining a pendant edge at each node of the cycle.

Definition 2.4:[10] The Closed Helm graph CH_n is obtained from the Helm graph H_n by connecting the edges between all the pendant nodes.

Definition 2.5: [11] The Sunlet graph S_n is obtained from the Cycle graph C_n by joining pendant edges at each node.

Definition 2.6:[12] The Double Star Graph $k_{1,n,n}$ is obtained from a Star Graph $k_{1,n}$ by joining a new pendant edge to each existing pendant node.

Definition 2.7:[13] The Triple Star Graph $k_{1,n,n,n}$ is obtained from Double Star Graph $k_{1,n,n}$ by joining a new pendant edge to each existing pendant node.

3 Results and Discussion

The radio k-chromatic for Mycielski of Double Star graph, Triple Star graph, Sunlet graph, Helm graph, and Closed Helm graph are obtained in this section.

Lemma 3.1:[4] For any graph G and $1 \leq k \leq diam(G)$, we have

$$rc_k(G) \geq \begin{cases} |D_k| - 2p + 2\sum_{i=0}^{p} L_i(p-i) \; if \; k = 2p+1 \\ |D_k| - 2p + 2\sum_{i=0}^{p} L_i(p-i) + 1 \; if \; k = 2p \end{cases}$$

Theorem 3.2: For $n \geq 2$, the radio k -chromatic number for Mycielski of Double Star graph $K_{1,n,n}$ is given by

$$rc_k[\mu(K_{1,n,n})] = \begin{cases} 7, & if \; k = 2, \; n = 2 \\ 2n+2, & if \; k = 2, \; n \geq 3 \\ 4n+8, & if \; k = 3, \; n \geq 2 \end{cases}$$

Proof: The node and edge set of $K_{1,n,n}$ are denoted by

$V\left(K_{1,\,n,\,n}\right)=\{p_{i}:1\leq i\leq n\}\cup\{q_{i}:1\leq i\leq n\}\cup\{t\}$ and $E\left(K_{1,n,n}\right)\ =\ \{pp_{i}:1\leq i\leq n\}\cup\{p_{i}q_{i}:1\leq i\leq n\}$

$\therefore\left|V\left(K_{1,n,n}\right)\right|=2n+1$ and $\left|E\left(K_{1,n,n}\right)\right|=2n$

By the construction of the Mycielski graph for $K_{1,n,n}$, then the vertex set of $K_{1,n,n}$ is given by $V\left[\mu\left(K_{1,n,n}\right)\right]\ =\ \{p_{i}\ :\ 1\ \leq i\ \leq n\}\cup\{q_{i}\ :\ 1\ \leq i\ \leq n\}\cup\{t\}\cup\{t^{'}\}\cup\{p_{i}^{'}:1\leq i\leq n\}\cup\{q_{i}^{'}:1\leq i\leq n\}\cup\{z\}$

$\therefore\left|V\left[\mu\left(K_{1,n,n}\right)\right]\right|=4n+3$

Here, we consider the following cases, based on the value of k

Case 1: When $k=2$

Subcase 1: For $n\ =\ 2$

From the lemma 3.1, it is observed that,

$rc_{2}\left[\mu\left(K_{1,2,2}\right)\right]\geq|D_{2}|-2+2\sum_{i=0}^{1}L_{i}\left(1-i\right)+1$

Therefore, $rc_{2}\left[\mu\left(K_{1,2,2}\right)\right]\geq\ 7$

To prove the upper bound, $rc_{2}\left[\mu\left(K_{1,n,n}\right)\right]\leq7$, Consider a function f_{1}, such that $f_{1}:V\left[\mu\left(K_{1,2,2}\right)\right]\rightarrow\{0,1,2,\ldots,7\}$

Allocate the following 7 - colors to $\mu\left(K_{1,2,2}\right)$ as radio 2 - chromatic:

$f_{1}\left(z\right)\rightarrow0$, $f_{1}\left(t^{'}\right)\rightarrow2$, $f_{1}\left(t\right)\rightarrow1$, $f_{1}\left(p_{1}^{'}\right)\ =f_{1}\left(q_{2}\right)\rightarrow3$, $f\left(q_{1}\right)\rightarrow7$, $f\left(q_{1}^{'}\right)=f\left(p_{2}\right)\rightarrow6$,

$f\left(q_{2}^{'}\right)=f\left(p_{1}\right)\rightarrow4$

Thus, $rc_{2}\left[\mu\left(K_{1,2,2}\right)\right]\leq\ 7$

Hence, it is proved that $rc_{2}\left[\mu\left(K_{1,2,2}\right)\right]=\ 7$

Subcase 2: For $n\ \geq3$

From the lemma 3.1, we have

$rc_{2}\left[\mu\left(K_{1,n,n}\right)\right]\geq|D_{2}|-2+2\sum_{i=0}^{1}L_{i}\left(1-i\right)+1$

Therefore, $rc_{2}\left[\mu\left(K_{1,n,n}\right)\right]\geq2n+2$

Now, let us prove that $rc_{2}\left[\mu\left(K_{1,n,n}\right)\right]\leq2n+2$

Consider a function f_{2}, such that $f_{2}:V\left[\mu\left(K_{1,n,n}\right)\right]\rightarrow\{0,1,2,\ldots,2n+2\}$

Allocate the following $2n\ +\ 2$ - colors to $\mu\left(K_{1,n,n}\right)$ as radio 2 - chromatic:

For $1\leq i\leq n$, $f_{2}\left(p_{i}\right)\rightarrow2i+2$, $f_{2}\left(p_{i}^{'}\right)\rightarrow2i+1$

For $1\ \leq i\ \leq n\ -\ 1$, $f_{2}\left(q_{i+1}\right)\rightarrow2i+1$, $f_{2}\left(q_{i+1}^{'}\right)\rightarrow2i+2$

$f_{2}\left(q_{1}\right)\rightarrow2n+2$, $f_{2}\left(p_{1}\right)\rightarrow2n+1$

$f_{2}\left(z\right)\rightarrow0$, $f_{2}\left(t^{'}\right)\rightarrow2$, $f_{2}\left(t\right)\rightarrow1$

Thus, we have $rc_{2}\left[\mu\left(K_{1,n,n}\right)\right]\leq2n+2$

Hence, $rc_{2}\left[\mu\left(K_{1,n,n}\right)\right]=2n+2$

Case 2: When $k\ =\ 3, n\ \geq2$

From the lemma 3.1, it is clear that

$rc_{3}\left[\mu\left(K_{1,n,n}\right)\right]\geq|D_{3}|-2+2\sum_{i=0}^{1}L_{i}\left(1-i\right)+1$

Therefore, $rc_{3}\left[\mu\left(K_{1,n,n}\right)\right]\geq4n+8$

Now, let us prove that, $rc_{3}\left[\mu\left(K_{1,n,n}\right)\right]\leq4n+8$

Consider a function f_{3}, such that $f_{3}:V\left[\mu\left(K_{1,n,n}\right)\right]\rightarrow\{0,1,2,\ldots,4n+8\}$

Allocate the following $4n\ +8$ - colors to $\mu\left(K_{1,n,n}\right)$ as radio 3 - chromatic:

$f_{3}\left(t\right)\rightarrow4$, $f_{3}\left(t^{'}\right)\rightarrow6$, $f_{3}\left(z\right)\rightarrow0$, $f_{3}\left(q_{1}^{'}\right)\rightarrow2n+8$

For $1\ \leq i\ \leq n$, $f_{3}\left(p_{i}^{'}\right)\rightarrow2i+6$, $f_{3}\left(q_{i}\right)\rightarrow2$, $f_{3}\left(p_{i}\right)\rightarrow4n-2i+9$

For $1\ \leq i\ \leq n\ -\ 1$, $f_{3}\left(q_{i+1}^{'}\right)\rightarrow4n-2i+10$

Thus, we have $rc_{3}[\mu(K_{(1,n,n)})]\leq4n\ +\ 8$

Hence, $rc_{3}\left[\mu\left(K_{1,n,n}\right)\right]=4n+8$

Let us now apply the condition of radio k – coloring to the following pair of nodes:

$\left|f\left(p_{i}:1\leq i\leq n\right)-f\left(q_{i}^{'}:1\leq i\leq n\right)\right|\geq1+k-d\left(p_{i}:1\leq i\leq n,q_{i}^{'}:1\leq i\leq n\right)$

$\left|f\left(p_{i}^{'}:1\leq i\leq n\right)-f\left(p_{i}:1\leq i\leq n\right)\right|\geq1+k-d\left(p_{i}^{'}:1\leq i\leq n,p_{i}:1\leq i\leq n\right)$

$\left|f\left(t\right)-f\left(t^{'}\right)\right|\geq1+k-d\left(t,t^{'}\right)$

By the above procedure, it can be verified that the remaining pair of nodes satisfies the radio k – coloring condition.

Theorem 3.3 : For $n \geq 2$, the radio k-chromatic number for Mycielski of Triple Star Graph $K_{1,n,n,n}$ is given by

$$rc_k[\mu(K_{1,n,n,n})] = \begin{cases} 9, & if\ k=2,\ n=2 \\ 3n+2, & if\ k=2,\ n\geq 3 \\ 6n-1, & if\ k=3,\ n\geq 2 \end{cases}$$

Proof: The node and edge set of $K_{1,n,n,n}$ are denoted by

$V\left(K_{1,n,n,n}\right) = \{t\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{r_i : 1 \leq i \leq n\}$ and $E\left(K_{1,n,n,n}\right) = \{pp_i : 1 \leq i \leq n\} \cup \{p_i q_i : 1 \leq i \leq n\} \cup \{q_i r_i : 1 \leq i \leq n\}$

$\left|V\left(K_{1,n,n,n}\right)\right| = 3n+1$ and $\left|E\left(K_{1,n,n,n}\right)\right| = 3n$

By the construction of the Mycielski graph for $K_{1,n,n,n}$, then the vertex set of $K_{1,n,n,n}$ is given by $V\left[\mu\left(K_{1,n,n,n}\right)\right] = \{t\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{r_i : 1 \leq i \leq n\} \cup \{t'\} \cup \{p_i' : 1 \leq i \leq n\} \cup \{q_i' : 1 \leq i \leq n\} \cup \{r_i' : 1 \leq i \leq n\} \cup \{z\}$

$\therefore \left|V\left[\mu\left(K_{1,n,n,n}\right)\right]\right| = 6n+3$

Here, we consider the following cases based on the value of k

Case 1: When $k = 2$

Subcase 1: For $n = 2$

From the lemma 3.1, we have

$rc_2\left[\mu\left(K_{1,2,2,2}\right)\right] \geq |D_2| - 2 + 2\sum_{i=0}^{1} L_i(1-i) + 1$

Therefore, $rc_2\left[\mu\left(K_{1,2,2,2}\right)\right] \geq 9$

To prove the upper bound, $rc_2\left[\mu\left(K_{1,2,2,2}\right)\right] \leq 9$

Consider a function f_1, such that $f_1 : V\left[\mu\left(K_{1,2,2,2}\right)\right] \to \{0,1,2,\ldots.9\}$

Allocate the following 9 - colors to $\mu\left(K_{1,2,2,2}\right)$ as radio 2 - chromatic:

$f_1(z) \to 0$, $f_1\left(t'\right) \to 2$, $f_1(t) \to 1$

For $1 \leq i \leq n$, $f_1\left(p_i'\right) \to 2+i$

$f_1\left(r_1'\right) = f_1(p_2) \to 5$, $f_1\left(r_2'\right) = f_1(p_1) \to 6$

$f_1(q_1) = f_1(q_2) \to 9$, $f_1(r_1) = f_1(r_2) \to 1$

$f_1\left(q_1'\right) \to 8$, $f\left(q_2'\right) \to 7$

Thus, $rc_2\left[\mu\left(K_{1,2,2,2}\right)\right] \leq 9$

Hence, $rc_2\left[\mu\left(K_{1,2,2,2}\right)\right] = 9$

Subcase 2: For $n \geq 3$

From the lemma 3.1, it is observed that

$rc_2\left[\mu\left(K_{1,n,n,n}\right)\right] \geq |D_2| - 2 + 2\sum_{i=0}^{1} L_i(1-i) + 1$

Therefore, $rc_2\left[\mu\left(K_{1,n,n,n}\right)\right] \geq 3n+2$

Now let us prove that, $rc_2\left[\mu\left(K_{1,n,n,n}\right)\right] \leq 3n+2$

Consider a function f_2, such that $f_2 : V\left[\mu\left(K_{1,n,n,n}\right)\right] \to \{0,1,2,\ldots.3n+2\}$

Allocate the following $3n+2$ - colors to $\mu\left(K_{1,n,n,n}\right)$ as radio 2 - chromatic:

$f_2(z) \to 0$, $f_2\left(t'\right) \to 2$, $f_2(t) \to 1$, $f_2(q_1) \to 3n+1$, $f_2(p_n) \to 5$, $f_2(r_n) \to 3$

For $1 \leq i \leq n$, $f_2\left(p_i'\right) \to 3i$, $f_2\left(q_i'\right) \to 3i+1$, $f_2\left(r_i'\right) \to 3i+2$

For $1 \leq i \leq n-1$, $f_2(p_i) \to 3i+5$, $f_2(r_i) \to 3i+3$, $f_2(q_{i+1}) \to 3i+1$

Therefore, $rc_2\left[\mu\left(K_{1,n,n,n}\right)\right] \leq 3n+2$

Hence, $rc_2\left[\mu\left(K_{1,n,n,n}\right)\right] = 3n+2$

Case 2: When $k = 3$, $n \geq 2$

From the lemma 3.1, it is clear that

$rc_3\left[\mu\left(K_{1,n,n,n}\right)\right] \geq |D_3| - 2 + 2\sum_{i=0}^{1} L_i(1-i)$

Thus, $rc_3\left[\mu\left(K_{1,n,n,n}\right)\right] \geq 6n-1$

Now, let us prove that $rc_3\left[\mu\left(K_{1,n,n,n}\right)\right] \leq 6n-1$

Consider a function f_3, such that $f_3 : V\left[\mu\left(K_{1,n,n,n}\right)\right] \to \{0,1,2,\ldots..6n-1\}$

Allocate the following $6n - 1$ - colors to $\mu\left(K_{1,n,n,n}\right)$ as radio 3 - chromatic:

$f_3(z) \to 0$, $f_3\left(t^{'}\right) \to 3$, $f_3(t) \to 5$, $f_3(p_n) \to 4n+6$, $f_3\left(q_1^{'}\right) \to 2n+5$, $f_3\left(q_2^{'}\right) \to 6n-1$, $f_3(q_1) \to 2n$

For $1 \le i \le n$, $f_3(r_i) \to 2$, $f_3\left(p_i^{'}\right) \to 2i+3$, $f_3\left(r_i^{'}\right) \to 4n+2i+3$

For $1 \le i \le n-1$, $f_3(p_i) \to 4n+2i+6$, $f_3(q_{i+1}) \to 2i+4$

For $1 \le i \le n-2$, $f_3\left(q_{i+2}^{'}\right) \to 2n+2i+7$

Therefore, $rc_3\left[\mu\left(K_{1,n,n,n}\right)\right] \le 6n-1$

Hence, $rc_3\left[\mu\left(K_{1,n,n,n}\right)\right] = 6n-1$

Now, let us apply the condition of radio k – coloring to the following pair of nodes:

$|f(p_i : 1 \le i \le n) - f(q_i : 1 \le i \le n)| \ge 1+k-d(p_i : 1 \le i \le n, q_i : 1 \le i \le n)$

$|f(q_i : 1 \le i \le n) - f(r_i : 1 \le i \le n)| \ge 1+k-d(q_i : 1 \le i \le n, r_i : 1 \le i \le n)$

$\left|f(t) - f\left(t^{'}\right)\right| \ge 1+k-d\left(t, t^{'}\right)$

By a similar procedure, it can be verified that the remaining pair of nodes satisfies the condition of radio k – coloring.

Theorem 3.4 : For $n \ge 3$, the radio k-chromatic number for Mycielski of Sunlet graph S_n is given by

$$rc_k[\mu(S_n)] = \begin{cases} 11, & if\ k=2,\ n=3 \\ 3n+1, & if\ k=2,\ n \ge 4 \\ 23, & if\ k=3,\ n=3 \\ 5n+6, & if\ k=3,\ n \ge 4 \end{cases}$$

Proof: The node and edge set of S_n are denoted by

$V(S_n) = \{p_i : 1 \le i \le n\} \cup \{q_i : 1 \le i \le n\}$ and $E(S_n) = \{p_i q_i : 1 \le i \le n\} \cup \{p_i p_{i+1} : 1 \le i \le n-1\} \cup \{p_1 p_n\}$

$\therefore |V(S_n)| = 2n$ and $|E(S_n)| = 2n$

By the construction of Mycielski for S_n , then the node set of S_n is given by

$V[\mu(S_n)] = \{p_i : 1 \le i \le n\} \cup \{q_i : 1 \le i \le n\} \cup \{p_i^{'} : 1 \le i \le n\} \cup \{q_i^{'} : 1 \le i \le n\} \cup \{z\}$

Here, we consider the following cases, based on the value of k

Case 1: When $k = 2$

Subcase 1: For $n = 3$

From the lemma 3.1, we have

$rc_2[\mu(S_3)] \ge |D_2| - 2 + 2\sum_{i=0}^{1} L_i(1-i) + 1$

Therefore, $rc_2[\mu(S_3)] \ge 11$

To prove the upper bound, $rc_2[\mu(S_3)] \le 11$

Consider a function f_1, such that $f_1 : V[\mu(S_3)] \to \{0, 1, 2, \ldots .11\}$

Allocate the following 11 - colors to $\mu(S_3)$ as radio 2 chromatic:

For $1 \le i \le n$, $f_1\left(p_i^{'}\right) \to i+1$, $f_1\left(q_i^{'}\right) \to 2i+4$

$f_1(p_1) \to 1$, $f_1(p_2) \to 11$, $f_1(p_3) \to 5 f_1\left(q_3^{'}\right) = f_1(q_1) \to 10 f_1\left(q_1^{'}\right) = f_1(q_2) \to 6 f_1\left(q_2^{'}\right) = f_1(q_3) \to 8$

Therefore, $rc_2[\mu(S_3)] \le 11$

Hence, $rc_2[\mu(S_3)] = 11$

Subcase 2 : For $n \ge 4$

From the lemma 3.1, we have

$rc_2[\mu(S_n)] \ge |D_2| - 2 + 2\sum_{i=0}^{1} L_i(1-i) + 1$

Therefore $rc_2[\mu(S_n)] \ge 3n+1$

Now, let us prove that, $rc_2[\mu(S_n)] \le 3n+1$

Consider a function f_2, such that $f_2 : V[\mu(S_n)] \to \{0, 1, 2, \ldots .3n+1\}$

Allocate the following $3n + 1$ - colors to $\mu(S_n)$ as radio 2 chromatic:

$f_2(z) \to 0$, $f_2(p_1) \to 3n$, $f_2(p_2) \to 1$, $f_2(q_n) \to n+3$

For $1 \le i \le n$, $f_2\left(p_i^{'}\right) \to i+1$, $f_2\left(q_i^{'}\right) \to n+2i+1$

For $1 \le i \le n-1$, $f_2(q_i) \to n+2i+3$

For $1 \le i \le n-2$, $f_2(p_{i+2}) \to n+2i+2$

Thus, $rc_2[\mu(S_n)] \le 3n+1$

Hence, $rc_2[\mu(S_n)] = 3n+1$

Case 2: When $k = 3$

Subcase 1: For $n = 3$

From the lemma 3.1, the lower bound is
$rc_3\left[\mu\left(S_3\right)\right] \geq |D_3| - 2 + 2\sum_{i=0}^{1} L_i\left(1-i\right)$
Therefore, $rc_3\left[\mu\left(S_3\right)\right] \geq 23$
To prove the upper bound, $rc_3\left[\mu\left(S_3\right)\right] \leq 23$
Consider a function f_3, such that $f_3 : V\left[\mu\left(S_3\right)\right] \to \{0,1,2,\ldots..23\}$
Allocate the following 23 - colors to $\mu\left(S_3\right)$ as radio 3 chromatic:
$f_3\left(p_1\right) \to 13$, $f_3\left(p_2\right) \to 23$, $f_3\left(p_3\right) \to 20$
$f_3\left(p_1^{'}\right) \to 11$, $f_3\left(p_2^{'}\right) \to 16$, $f_3\left(p_3^{'}\right) \to 18$, $f_3\left(z\right) \to 0$
For $1 \leq i \leq n$, $f_3\left(q_i\right) \to i+1$, $f_3\left(q_i^{'}\right) \to 2i+3$
Thus, $f_3 : V\left[\mu\left(S_3\right)\right] \to \{0,1,2,\ldots..23\}$
Hence, $rc_3\left[\mu\left(S_3\right)\right] = 23$
Subcase 2: For $n \geq 4$
From the lemma 3.1, the lower bound is
$rc_3\left[\mu\left(S_n\right)\right] \geq |D_3| - 2 + 2\sum_{i=0}^{1} L_i\left(1-i\right)$
Therefore, $rc_3\left[\mu\left(S_n\right)\right] \geq 5n+6$
To prove the upper bound, $rc_3\left[\mu\left(S_n\right)\right] \leq 5n+6$
Consider a function f_4, such that $f_4 : V\left[\mu\left(S_n\right)\right] \to \{0,1,2,\ldots.5n+6\}$
For $1 \leq i \leq n$, $f_4\left(q_i^{'}\right) \to 2n+3i+5$, $f_4\left(p_i^{'}\right) \to 2i+5$
For all $1 \leq i \leq n$

$$f_4(q_i) = \begin{cases} C_2,\, i = 1\,mod\,2 \\ C_3,\, i = 0\,mod\,2 \end{cases}$$

For $1 \leq i \leq n-2$, $f_4\left(p_{i+2}\right) \to 2n+3i+6$
$f_4\left(p_1\right) \to 5n+3$, $f_4\left(p_2\right) \to 5n+6$, $f_4\left(z\right) \to 0$
Thus, $rc_4\left[\mu\left(S_n\right)\right] \leq 5n+6$
Hence, $rc_4\left[\mu\left(S_n\right)\right] = 5n+6$
Now, let us apply the condition of radio k – coloring to the following pair of nodes:
$\left|f\left(p_i : 1 \leq i \leq n\right) - f\left(q_i : 1 \leq i \leq n\right)\right| \geq 1+k-d\left(p_i : 1 \leq i \leq n, q_i : 1 \leq i \leq n\right)$
$\left|f\left(t\right) - f\left(t^{'}\right)\right| \geq 1+k-d\left(t,t^{'}\right)$
$\left|f\left(z\right) - f\left(t^{'}\right)\right| \geq 1+k-d\left(z,t^{'}\right)$
By a similar procedure, it can be verified that the remaining pair of nodes satisfies the condition of radio k – coloring.
Theorem 3.5 : For $n \geq 3$, the radio k-chromatic number for Mycielski of Helm graph is given by

$$rc_k[\mu(H_n)] = \begin{cases} 11, & if\; k=2,\; n=3 \\ 2n+4, & if\; k=2,\; n \geq 4 \\ 5n+8, & if\; k=3,\; n \geq 4 \end{cases}$$

Proof: The node and edge set of H_n are denoted by
$V\left(H_n\right) = \{t\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\}$ and $E\left(H_n\right) = \{pp_i : 1 \leq i \leq n\} \cup \{p_iq_i : 1 \leq i \leq n\} \cup \{p_ip_{i+1} : 1 \leq i \leq n-1\} \cup \{p_1p_n\}$
$\therefore |V\left(H_n\right)| = 2n+1$ and $|E\left(H_n\right)| = 3n$
By the construction of Mycielski for H_n , then the vertex set of H_n is given by
$V\left[\mu\left(H_n\right)\right] = \{t\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{t^{'}\} \cup \{p_i^{'} : 1 \leq i \leq n\} \cup \{q_i^{'} : 1 \leq i \leq n\} \cup \{z\}$
$\therefore V\left[\mu\left(H_n\right)\right] = 4n+3$
Here we consider the following cases, based on the value of k
Case 1: When $k = 2$
Subcase 1: For $n = 3$
From the lemma 3.1, we have
$rc_2\left[\mu\left(H_3\right)\right] \geq |D_2| - 2 + 2\sum_{i=0}^{1} L_i\left(1-i\right) + 1$
Therefore, $rc_2\left[\mu\left(H_3\right)\right] \geq 11$
To prove the upper bound, $rc_2\left[\mu\left(H_3\right)\right] \leq 11$

Consider a function f_1, such that $f_1 : V[\mu(H_3)] \rightarrow \{0,1,2,\dots 11\}$
Allocate the following 11 - colors to $\mu(H_3)$ as radio 2 chromatic:
$f_1(z) \rightarrow 0$, $f_1(t) \rightarrow 1$, $f_1\left(t^{'}\right) \rightarrow 2$, $f_1(q_3) \rightarrow 9$, $f_1(q_2) \rightarrow 11$, $f_1(q_1) \rightarrow 10$
For $1 \le i \le n$, $f_1\left(p_i^{'}\right) \rightarrow 2i+1$, $f_1(p_i) \rightarrow 2i+2$
For $1 \le i \le 3$, $f_1\left(q_i^{'}\right) \rightarrow n+5+i$
Thus, $rc_2[\mu(H_3)] \le 11$
Hence, $rc_2[\mu(H_3)] = 11$
Subcase 2: For $n \ge 4$
From the lemma 3.1, we have
$rc_2[\mu(H_n)] \ge |D_2| - 2 + 2\sum_{i=0}^{1} L_i(1-i) + 1$
Therefore, $rc_2[\mu(H_n)] \ge 2n+4$
Now, let us prove that $rc_2[\mu(H_n)] \le 2n+4$
Consider a function f_2, such that $f_2 : V[\mu(H_n)] \rightarrow \{0,1,2,\dots.2n+4\}$
Allocate the following $2n + 4$ - colors to $\mu(H_n)$ as radio 2 - chromatic:
$f_2(t) \rightarrow 2$, $f_2\left(t^{'}\right) \rightarrow 1$, $f_2(z) \rightarrow 0$, $f_2\left(q_{n-1}^{'}\right) \rightarrow 4$, $f_2\left(q_n^{'}\right) \rightarrow 6$
For $1 \le i \le n$, $f_2(p_i) \rightarrow 2i+2$, $f_2\left(p_i^{'}\right) \rightarrow 2i+1$, $f_2(q_i) \rightarrow 2n+4$
For $1 \le i \le n-2$, $f_2\left(q_i^{'}\right) \rightarrow 2i+6$
Therefore, $rc_2[\mu(H_n)] \le 2n+4$
Hence, $rc_2[\mu(H_n)] = 2n+4$
Case 2: When $k=3,\ n \ge 4$
From the lemma 3.1, we have observed that
$rc_3[\mu(H_n)] \ge |D_3| - 2 + 2\sum_{i=0}^{1} L_i(1-i)$
Therefore, $rc_3[\mu(H_n)] \ge 5n+8$
Now, let us prove that, $rc_3[\mu(H_n)] \le 5n+8$
Consider a function f_3, such that $f_3 : V[\mu(H_n)] \rightarrow \{0,1,2,\dots..5n+8\}$
Allocate the following $5n + 8$ - colors to $\mu(H_n)$ as radio 3 - chromatic:
$f_3(z) \rightarrow 0$, $f_3(t) \rightarrow 7$, $f_3\left(t^{'}\right) \rightarrow 5$, $f_3(p_1) \rightarrow 5n+5$, $f_3(p_2) \rightarrow 5n+8$
For $1 \le i \le n$, $f_3\left(p_i^{'}\right) \rightarrow 2i+7$, $f_3\left(q_i^{'}\right) \rightarrow 2n+3i+7$
For $1 \le i \le n-2$, $f_3(p_{i+2}) \rightarrow 2n+3i+8$
For all $1 \le i \le n$,

$$f_3(q_i) = \begin{cases} c_2,\ i = 0\,mod\,2 \\ c_{3,}\ i = 1\,mod\,2 \end{cases}$$

Therefore, $rc_3[\mu(H_n)] \le 5n+8$
Hence, $rc_3[\mu(H_n)] = 5n+8$
Now let us apply the condition of radio k – coloring to the following pair of nodes:
$|f(p_i : 1 \le i \le n) - f(q_i : 1 \le i \le n)| \ge 1 + k - d(p_i : 1 \le i \le n, q_i : 1 \le i \le n)$
$\left|f\left(q_i^{'} : 1 \le i \le n\right) - f(z)\right| \ge 1 + k - d\left(q_i^{'} : 1 \le i \le n, z\right)$
$\left|f(t) - f\left(t^{'}\right)\right| \ge 1 + k - d\left(t, t^{'}\right)$
By a similar procedure, it can be verified that the remaining pair of nodes satisfies the condition of radio k – coloring.
Theorem 3.6 : For $n \ge 3$, the radio k-chromatic number for Mycielski of Closed Helm graph CH_n is given by

$$rc_k[\mu(CH_n)] = \begin{cases} 4n+2, & if\ k=2,\ n \ge 3 \\ 6n+5, & if\ k=3,\ n \ge 4 \end{cases}$$

Proof: The node and edge set of CH_n are denoted by
$V(CH_n) = \{t\} \cup \{p_i : 1 \le i \le n\} \cup \{q_i : 1 \le i \le n\}$ and $E(CH_n) = \{pp_i : 1 \le i \le n\} \cup \{p_i q_i : 1 \le i \le n\} \cup \{p_i p_{i+1} : 1 \le i \le n\} \cup \{p_1 p_n\} \cup \{q_i q_{i+1} : 1 \le i \le n-1\} \cup \{q_i q_n\}$
$\therefore |V(CH_n)| = 2n+1$ and $|E(CH_n)| = 4n$
By the construction of Mycielski for CH_n , then the vertex set of CH_n is given by

$V[\mu(CH_n)] = \{t\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{t'\} \cup \{p_i' : 1 \leq i \leq n\} \cup \{q_i' : 1 \leq i \leq n\} \cup \{z\}$
$\therefore |V[\mu(CH_n)]| = 4n+3$
Here we consider the following cases, based on the value of k
Case 1: When $k = 2,\ n \geq 3$
From the lemma 3.1, we have
$rc_2[\mu(CH_n)] \geq |D_2| - 2 + 2\sum_{i=0}^{1} L_i(1-i) + 1$
Therefore, $rc_2[\mu(CH_n)] \geq 4n+2$
To prove the upper bound, $rc_2[\mu(CH_n)] \leq 4n+2$
Consider a function f_1 such that $f_1 : V[\mu(CH_n)] \to \{0,1,2,,\ldots.4n+2\}$
Allocate the following $4n + 2$ - colors to $\mu(CH_n)$ as radio 2 - chromatic:
Subcase 1: For $n = 3$
$f_1(t) \to 1$, $f_1(t') \to 2$, $f_1(z) \to 0$,
For $1 \leq i \leq n$, $f_1(p_i') \to 2i+1$, $f_1(q_i') \to 8+i$, $f_1(p_i) \to 2i+2$, $f_1(q_i) \to 3n+6-i$
Subcase 2: For $n \geq 4$
$f_1(t) \to 1$, $f_1(t') \to 2$, $f_1(z) \to 0$
For $1 \leq i \leq n$, $f_1(q_i) \to 2n+2i+2$, $f_1(p_i') \to 2i+2$
For $1 \leq i \leq n-2$, $f_1(q_{i+2}) \to 2i+1$, $f_1(p_{i+2}) \to 2n+2i+1$
For $1 \leq i \leq 2$, $f_1(p_i) \to 4n+2i-3$, $f_1(q_i) \to 2n+2i-3$
Thus, $rc_2[\mu(CH_n)] \leq 4n+2$
Hence, $rc_2[\mu(CH_n)] = 4n+2$
Case 2: When $k = 3,\ n \geq 4$
From the lemma 3.1, the lower bound is
$rc_3[\mu(CH_n)] \geq |D_3| - 2 + 2\sum_{i=0}^{1} L_i(1-i)$
Therefore, $rc_3[\mu(CH_n)] \geq 6n+5$
To prove the upper bound, $rc_3[\mu(CH_n)] \leq 6n+5$
Consider a function f_2, such that $f_2 : V[\mu(CH_n)] \to \{0,1,2,\ldots.6n+5\}$
Allocate the following $6n + 5$ - colors to $\mu(CH_n)$ as radio 3 – chromatic:
$f_2(t) \to 3$, $f_2(z) \to 0$, $f_2(t') \to 5$, $f_2(p_1) \to 6n$, $f_2(p_2) \to 6n+3$
For $1 \leq i \leq n-2$, $f_2(p_{i+2}) \to 3n+3i+3$, $f_2(q_i) \to 3i+10$
For $1 \leq i \leq n$, $f_2(p_i') \to 3i+5$, $f_2(q_i') \to 3n+3i+5$
Thus, $rc_3[\mu(CH_n)] \leq 6n+5$
Therefore, $rc_3[\mu(CH_n)] = 6n+5$
Now, let us apply the condition of radio k – coloring to the following pair of nodes:
$|f(p_i : 1 \leq i \leq n) - f(q_i : 1 \leq i \leq n)| \geq 1 + k - d(p_i : 1 \leq i \leq n, q_i : 1 \leq i \leq n)$
$|f(t) - f(t')| \geq 1 + k - d(t, t')$
$|f(t') - f(z)| \geq 1 + k - d(t', z)$
By a similar procedure, it can be verified that the remaining pair of nodes satisfies the condition of radio k – coloring.

4 Conclusion

In this paper, the radio k-chromatic number for the Mycielski graph of the Double Star graph, Triple Star graph, Helm graph, Closed Helm graph, and Sunlet graph are obtained, and the results provide an exact value to the radio k-chromatic number for those graphs.

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