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RESEARCH ARTICLE

A New General Integral Transform of Modified Laguerre's Polynomial $L_{\mathbf{a,b,c,n}}(t)$

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Abstract

Objectives: The core objective of this paper is to introduce the explicit Modified Laguerre polynomial $L_{a,b,c,n}(t)$ and obtain a new general integral transform of Modified Laguerre's Polynomial, by using this general transform, some transforms of Laguerre's Polynomial are identified as special cases. **Method:** We derive transforms of the polynomials $L_{a,b,c,n}(t)$, $L_n(t)$ by using this new integral transform approach and found the transforms like Laplace, α-Laplace, Sawi, Elzaki, Sumudu, Natural, Aboodh, Pourreza, Mohand, G_transform, Kamal transforms and their relationship for the same. **Findings:** A new general integral transform of the Modified Laguerre's Polynomial $L_{a,b,c,n}(t)$ has been obtained, and a special case for the polynomial $L_n(t)$ has been derived by using this transform. **Novelty:** This is a novel general integral transform for this polynomial and by applying this transform to the Modified Laguerre's polynomial as special transform cases like Laplace, α-Laplace, Sawi, Elzaki, Sumudu, Natural, Aboodh, Pourreza, Mohand, G_, Kamal, Sumudu, Elzaki, Natural, and Pourreza are obtained.

Keywords: Modified Laguerre polynomial; Laguerre polynomial; A New Integral Transform; Laplace Transform; Sumudu Transform

1 Introduction

To solve some real-world problems, the use of integral transforms is required. Differential and integral equations can be simplified into an algebraic equation, which is simple to solve by selecting the appropriate integral transformations. In the last two decades, numerous integral transforms within the Laplace transform class have been introduced including G_transform, Sawi, Kamal, Aboodh, Sumudu, Elzaki, Natural, and Pourreza transforms [1–5].

A variety of special functions are required to solve numerous theoretical and mathematical physics problems. A special function's generating function relations, integral representations, explicit formula, recurrence relations, and other distinctive qualities are identified, during analysis [6–11]. In this paper, we primarily write the different transforms of the well-known classical Laguerre polynomials [6,7,12–15] in the field of theory of special functions by using the New Integral transform [5].

The explicit form of Modified Laguerre polynomial $L_{a,b,c,n}(t)$ was introduced [16,17] as

$$L_{a,b,c,n}(t) = \frac{b^n (c)_n}{n!} \, {}_1F_1\left[-n; c; -; \frac{at}{b}\right] \tag{1.1}$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{at}{b}\right)^k}{(c)_k \, k!}$$

But

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n! \left(\frac{at}{b}\right)^k}{(c)_k (n-k)! \, k!} \tag{1.2}$$

The definition of a new integral transform[2] is as follows: Let $f(t)$ be an integrable function defined for $t \ge 0$, $p(s) \neq 0$ and, $q(s)$ are positive real functions, the general integral transform $F(s)$ of $f(t)$ is defined by the formula

$$T\{f(t); s\} = F(s) = p(s) \int_0^{\infty} f(t) e^{-q(s)t} \, dt \tag{1.3}$$

provided, that the integral exists for some $q(s)$. The techniques and methods of writing generating functions and the results are discussed in [1,2,6,7,12–15,17]. This motivated us to find the relation between Modified Laguerre polynomial $L_{a,b,c,n}(t)$ and the new integral transform.

2 Methodology

2.1 New General Integral Transform of Modified Laguerre's Polynomial $L_{\mathbf{a,b,c,n}}(\mathbf{t})$

$$T\{L_{a,b,c,n}(t)\} = p(s) \int_0^{\infty} L_{a,b,c,n}(t) \, e^{-q(s)t} \, dt$$

$$= p(s) \int_0^{\infty} \left\{ \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{at}{b}\right)^k}{(c)_k \, k!} \right\} e^{-q(s)t} \, dt$$

$$T\{L_{a,b,c,n}(t)\} = T\left\{ \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{at}{b}\right)^k}{(c)_k \, k!} \right\}, \tag{2.1}$$

$$= T\left\{ \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{a}{b}\right)^k t^k}{(c)_k \, k!} \right\},$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{a}{b}\right)^k}{(c)_k \, k!} T\{t^k\},$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{a}{b}\right)^k}{(c)_k \; k!} \frac{\Gamma(k+1) \;\; p(s)}{(q(s))^{k+1}},$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{a}{b}\right)^k}{(c)_k \; k!} \left\{ \frac{k! \;\; p(s)}{(q(s))^{k+1}} \right\},$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{a}{b}\right)^k}{(c)_k} \left\{ \frac{p(s)}{(q(s))^{k+1}} \right\},$$

$$= \frac{b^n (c)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \; p(s)}{(c)_k \;\; q(s)} \left\{ \left(\frac{a}{b\, q(s)} \right)^k \right\}. \tag{2.2}$$

If we put $c = 1$ in Equation (2.2), then we have the following form of the transform

$$T\{L_{a,b,1,n}(t)\} = \frac{b^n (1)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \;\; p(s)}{(1)_k \;\; q(s)} \left\{ \left(\frac{a}{b\, q(s)} \right)^k \right\},$$

$$T\{L_{a,b,1,n}(t)\} = \frac{b^n \; n!}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k \;\; p(s)}{k! \;\; q(s)} \left\{ \left(\frac{a}{b\, q(s)} \right)^k \right\},$$

$$T\{L_{a,b,1,n}(t)\} = \frac{b^n \;\; p(s)}{q(s)} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left\{ \left(\frac{a}{b\, q(s)} \right)^k \right\},$$

$$T\{L_{a,b,1,n}(t)\} = \frac{b^n \;\; p(s)}{q(s)} \left(1 - \frac{a}{b\, q(s)} \right)^n. \tag{2.3}$$

2.2 The following transforms of the Modified Laguerre's Polynomial are obtained after using the relationship between the New Transform and other transforms:

2.2.1: For $p(s) = 1$ *and* $q(s) = s$ in this new transform, we get the Laplace transform
Then from Equation (2.3) we get,

$$L\{L_{a,b,1,n}(t)\} = \int_0^{\infty} L_{a,b,c,n}(t) \; e^{-st} \, dt = \frac{b^n}{s} \left(1 - \frac{a}{bs} \right)^n.$$

2.2.2: For $p(s) = 1$ *and* $q(s) = s^{\frac{1}{\alpha}}$ in this new transform, we get the α-Laplace transform
Then from Equation (2.3) we get,

$$L_{\alpha}\{L_{a,b,1,n}(t)\} = \int_0^{\infty} L_{a,b,c,n}(t) \; e^{-s^{\frac{1}{\alpha}} t} \, dt = \frac{b^n}{s^{\frac{1}{\alpha}}} \left(1 - \frac{a}{b\, s^{\frac{1}{\alpha}}} \right)^n.$$

2.2.3: For $p(s) = \frac{1}{s}$ *and* $q(s) = \frac{1}{s}$ in this new transform, we get the Sumudu transform
Then from Equation (2.3) we get,

$$S\{L_{a,b,1,n}(t)\} = \frac{1}{s} \int_0^{\infty} L_{a,b,c,n}(t) \; e^{-\frac{1}{s} t} \, dt = b^n \left(1 - \frac{as}{b} \right)^n.$$

2.2.4: For $p(s) = \frac{1}{s}\ and\ q(s) = 1$ in this new transform, we get the Aboodh transform
Then from Equation (2.3) we get,

$$A\left\{L_{a,b,1,n}(t)\right\} = \frac{1}{s}\int_0^\infty L_{a,b,c,n}(t)\ e^{-t}\,dt = \frac{b^n}{s}\ \left(1-\frac{a}{b}\right)^n$$

2.2.5: For $p(s) = s\ and\ q(s) = s^2$ in this new transform, we get the Pourreza transform
Then from Equation (2.3) we get,

$$HJ\left\{L_{a,b,1,n}(t)\right\} = s\int_0^\infty L_{a,b,c,n}(t)\ e^{-s^2 t}\,dt = \frac{b^n}{s}\ \left(1-\frac{a}{bs^2}\right)^n$$

2.2.6: For $p(s) = s\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Elzaki transform
Then from Equation (2.3) we get,

$$E\left\{L_{a,b,1,n}(t)\right\} = s\int_0^\infty L_{a,b,c,n}(t)\ e^{-\frac{1}{s}\,t}\,dt = b^n s^2\ \left(1-\frac{as}{b}\right)^n$$

2.2.7: For $p(s) = u\ and\ q(s) = \frac{s}{u}$ in this new transform, we get the Natural transform
Then from Equation (2.3) we get,

$$N\left\{L_{a,b,1,n}(t)\right\} = u\int_0^\infty L_{a,b,c,n}(ut)\ e^{-\frac{s}{u}t}\,dt = \frac{b^n\,u^2}{s}\ \left(1-\frac{au}{bs}\right)^n$$

2.2.8: For $p(s) = s^2\ and\ q(s) = s$ in this new transform, we get the Mohand transform
Then from Equation (2.3) we get,

$$M\left\{L_{a,b,1,n}(t)\right\} = s^2\int_0^\infty L_{a,b,c,n}(t)\ e^{-st}\,dt = b^n s\ \left(1-\frac{a}{bs}\right)^n$$

2.2.9: For $p(s) = \frac{1}{s^2}\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Sawi transform
Then from Equation (2.3) we get,

$$Sa\left\{L_{a,b,1,n}(t)\right\} = \frac{1}{s^2}\int_0^\infty L_{a,b,c,n}(t)\ e^{-\frac{1}{s}t}\,dt = \frac{b^n}{s}\ \left(1-\frac{as}{b}\right)^n$$

2.2.10: For $p(s) = 1\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Kamal transform
Then from Equation (2.3) we get,

$$K\left\{L_{a,b,1,n}(t)\right\} = \int_0^\infty L_{a,b,c,n}(t)\ e^{-\frac{1}{s}t}\,dt = b^n s\ \left(1-\frac{as}{b}\right)^n$$

2.2.11: For $p(s) = s^\alpha\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the G_transform
Then from Equation (2.3) we get,

$$G\left\{L_{a,b,1,n}(t)\right\} = s^\alpha\int_0^\infty L_{a,b,c,n}(t)\ e^{-\frac{1}{s}t}\,dt = b^n s^{\alpha+1}\ \left(1-\frac{as}{b}\right)^n$$

2.3 Particular case:

If $a = b = 1$, then the Equation (2.3) reduces to

$$T\left\{L_n(t)\right\} = \frac{p(s)}{q(s)}\ \left(1-\frac{1}{q(s)}\right)^n, \tag{2.4}$$

$$T\left\{L_n(t)\right\} = p(s)\int_0^\infty L_n(t)\ e^{-q(s)t}\,dt = \frac{p(s)}{q(s)}\ \left(1-\frac{1}{q(s)}\right)^n.$$

2.4 Using the relation between the New Transform and other transforms,we get the following transforms of Laguerre's Polynomial

2.4.1: For $p(s) = 1\ and\ q(s) = s$ in this new transform, we get the Laplace transform
Then from Equation (2.4) we get,

$$L\{L_n(t)\} = \int_0^\infty L_n(t)\ e^{-st}\,dt = \frac{1}{s}\left(1-\frac{1}{s}\right)^n.$$

2.4.2: For $p(s) = 1\ and\ q(s) = s^{\frac{1}{\alpha}}$ in this new transform, we get the α-Laplace transform. Then from Equation (2.4) we get,

$$L_\alpha\{L_n(t)\} = \int_0^\infty L_n(t)\ e^{-s^{\frac{1}{\alpha}}t}\,dt = \frac{1}{s^{\frac{1}{\alpha}}}\left(1-\frac{1}{s^{\frac{1}{\alpha}}}\right)^n.$$

2.4.3: For $p(s) = \frac{1}{s}\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Sumudu transform
Then from Equation (2.4) we get,

$$S\{L_n(t)\} = \frac{1}{s}\int_0^\infty L_n(t)\ e^{-\frac{1}{s}t}\,dt = (1-s)^n.$$

2.4.4: For $p(s) = \frac{1}{s}\ and\ q(s) = 1$ in this new transform, we get the Aboodh transform
Then from Equation (2.4) we get,

$$A\{L_n(t)\} = \frac{1}{s}\int_0^\infty L_n(t)\ e^{-t}\,dt = 0$$

2.4.5: For $p(s) = s\ and\ q(s) = s^2$ in this new transform, we get the Pourreza transform
Then from Equation (2.4) we get,

$$HJ\{L_n(t)\} = s\int_0^\infty L_n(t)\ e^{-s^2t}\,dt = \frac{1}{s}\left(1-\frac{1}{s^2}\right)^n$$

2.4.6: For $p(s) = s\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Elzaki transform
Then from Equation (2.4) we get,

$$E\{L_n(t)\} = s\int_0^\infty L_n(t)\ e^{-\frac{1}{s}\ t}\,dt = s^2\ (1-s)^n$$

2.4.7: For $p(s) = u\ and\ q(s) = \frac{s}{u}$ in this new transform, we get the Natural transform
Then from Equation (2.4) we get,

$$N\{L_n(t)\} = u\int_0^\infty L_n(ut)\ e^{-\frac{s}{u}t}\,dt = \frac{u^2}{s}\left(1-\frac{u}{s}\right)^n$$

2.4.8: For $p(s) = s^2\ and\ q(s) = s$ in this new transform, we get the Mohand transform
Then from Equation (2.4) we get,

$$M\{L_n(t)\} = s^2\int_0^\infty L_n(t)\ e^{-st}\,dt = s\left(1-\frac{1}{s}\right)^n$$

2.4.9: For $p(s) = \frac{1}{s^2}\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Sawi transform
Then from Equation (2.4) we get,

$$Sa\{L_n(t)\} = \frac{1}{s^2}\int_0^\infty L_n(t)\ e^{-\frac{1}{s}t}\,dt = \frac{1}{s}\ (1-s)^n$$

2.4.10: For $p(s) = 1\ and\ q(s) = \frac{1}{s}$ in this new transform, we get the Kamal transform

Then from Equation (2.4) we get,

$$K\{L_n(t)\} = \int_0^\infty L_n(t)\, e^{-\frac{1}{s}t}\, dt = s\,(1-s)^n$$

2.4.11: For $p(s) = s^\alpha$ and $q(s) = \frac{1}{s}$ in this new transform, we get the G_transform

Then from Equation (2.4) we get,

$$G\{L_n(t)\} = s^\alpha \int_0^\infty L_n(t)\, e^{-\frac{1}{s}t}\, dt = s^{\alpha+1}\,(1-s)^n$$

3 Results and Discussion

This study has obtained a new general integral transform of Modified Laguerre's Polynomial. Using this obtained transform and its relation with the transforms like, Laplace, α-Laplace, Sawi, Elzaki, Sumudu, Natural, Aboodh, Pourreza, Mohand, G_transform, and Kamal transforms, and derived the above transforms of the polynomial $L_{a,b,c,n}(t)$. Then, for the Laguerre polynomial $L_n(t)$, the new general integral transform is used to achieve all of the above transformations. These findings are innovative and easy to obtain.

4 Conclusion

This study has used a general integral transform and its relation with transforms like, Laplace, α-Laplace, Sawi, Elzaki, Sumudu, Natural, Aboodh, Pourreza, Mohand, G_ and Kamal to compute transforms of the Modified Laguerre polynomial $L_{a,b,c,n}(t)$ and Laguerre polynomial $L_n(t)$. The new integral transform covers the existing Laplace, α-Laplace, Sawi, Elzaki, Sumudu, Natural, Aboodh, Pourreza, Mohand, G_transform, and Kamal transforms for various values of $p(s)$ and $q(s)$. The new integral transform provides a simple way of writing the different transforms of special functions.

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