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RESEARCH ARTICLE

A Theorem on k-Expansive Mapping and its Fixed Point in G_{JS} – Metric Space

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Abstract

Objectives: To find a fixed point of k-expansive mapping on G_{JS}-metric space by establishing a theorem on it and to prove the uniqueness of such a fixed point. **Methods:** A theorem for the k-expansive map on G_{JS}-metric space is established by generalizing an existing contractive condition theorem. Generalization is done by replacing the contractive condition with an expansive type of condition. **Findings:** Certain conditions that have to be satisfied by a k-expansive mapping for the existence and uniqueness of fixed points in G_{JS}-metric space were found in this article. An example is also given to illustrate the existence of a fixed point. **Novelty:** G_{JS}-metric space is a new metric space that has been established recently. Furthermore, the fixed point theorem of k-expansive mapping on G_{JS} - metric space has not yet been proved by anyone, and hence it is proved in this article.

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1 Introduction

In non-linear analysis, fixed point theory is showing immense progress in the research field due to its enormous applications in various fields (see, for example[1] [2]). Many generalizations of the Banach contraction principle[3,4] were brought into the picture, and the process is always evergreen.

It is a well-known fact that if the distance between the elements is greater than the distance between their images of a map, then the mapping is known as contraction mapping. Further, generalizations of the contraction mapping theorem were brought into the limelight by many researchers (for example, see in[5–8]).

Moreover, replacing the contractive condition with an expansive type of condition and studying its fixed points is an interesting topic of research. There are several fixed point theorems for expansive mappings on various metric spaces, like JS-metric space and G_b – metric space[9] [10] and there are many fixed point theorems on generalized expansive mappings (for example, see in[11,12]).

It is a well-known fact that generalization of a metric space can be done by replacing or weakening one of the conditions of a metric space. By using this method, D - metric space and G - metric space were introduced. Further, G_{JS} - metric space is introduced by generalizing G - metric space through JS - metric space by Srilatha et al.[13].

The main aim of this article is to study k – expansive mapping in G_{JS} - metric space and find its fixed points.

2 Methodology

In this article, a systematic procedure is followed in order to establish a new result in G_{JS} - metric space. There have been many expansive mapping theorems in various metric spaces till now, but there has been no such work in G_{JS} - metric space. Hence, here in this article, some basic definitions that are required for the main result is given initially. Then a theorem on k – expansive mapping is given for finding fixed points in G_{JS} - metric space. Moreover, the main theorem is supported with an example.

Definition 2.1[13] Assume that E is a non-void set. Then the mapping $G_{JS}: E^3 \to [0,\infty] G_{JS}$metric on E if it satisfies the following conditions:

$(G_{JS}1) G_{JS}(X,\psi,\xi) = 0$if and only if $X = \psi = \xi$

$(G_{JS}2) 0 < G_{JS}(\chi,\chi,\psi)$for all $\chi,\psi \in E$ with $\chi \neq \psi$.

$(G_{JS}3) G_{JS}(\chi,\chi,\psi) < G_{JS}(\chi,\psi,\xi)$ for all $\chi,\psi,\xi \in E$, with $\psi \neq \xi$.

$(G_{JS}4) G_{JS}(\chi,\psi,\xi) = G_{JS}(\sigma(X,\psi,\xi))$ for all $X,\psi,\xi,\in E$,where $\sigma(X,\psi,\varepsilon)$ is a permutation of the set $\{X,\psi,\xi\}$ and

$(G_{JS}5)$ there is a constant c > 0 such that for $(\chi,\psi,\xi) \in E^3$ and $\langle X_n \rangle \in G(G_{JS},E,\chi)$,,

$G_{JS}(\chi,\psi,\xi) \leq \text{c}\lim_{n\to\infty}\sup\ G_{JS}(\chi_n,\psi,\xi)$,

where $G(G_{JS},E,x) = \{\langle X_n\rangle \subset E : \lim_{n\to\infty} G_{JS}(X_n,X,X) = 0\}$. Then the pair (E,G_{JS}) is called a G_{JS}- metric space.

Definition 2.2[13] Let (E,G_{JS})be a G_{JS}-metric space. A sequence $\langle X_n\rangle$ in E is said to be

G_{JS}- convergent to $\chi \in E$ if $\langle X_n\rangle \in G(G_{JS},E,X)$

Definition 2.3[13] Let (E,G_{JS})be a G_{JS}- metric space. A sequence $\langle X_n\rangle \subset E$is said to be

G_{JS}- Cauchy if $\lim\limits_{n,m\to\infty} G_{JS}(\chi_n,\chi_n,\chi_m) = 0$.

Definition 2.4[13] A G_{JS}- metric space is said to be G_{JS}- complete if every G_{JS}- Cauchy sequence in E is G_{JS}- convergent in E.

Definition 2.5 Let (E,G_{JS}) be a G_{JS}-metric space and $\mu: E \to E$, be a self- mapping. Then μ is called G_{JS} – continuous at $a \in E$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\chi \in E, G_{JS}(\mu a,\mu a,\mu\chi) < \varepsilon$ whenever $G_{JS}(a,a,\chi) < \delta$.

Definition 2.6 A G_{JS}-metric space $(E,G_{JS})$is claimed to be symmetric if $G_{JS}(X,\psi,\psi) = G_{JS}(\psi,X,X)$, for all $\chi,\psi \in E$.

Definition 2.7[9] Let (E,G_{JS}) be a G_{JS}-metric space and μ: E $\to$ E be an onto mapping. For k > 1, μ is said to be a k-expansive mapping if $G_{JS}(\mu\chi,\mu\chi,\mu\psi) \geq kG_{JS}(\chi,\chi,\psi)$for every $(\chi,\chi,\psi) \in E^3$.

Definition 2.8[14,15] Let μ: E $\to$ E be a mapping defined on a non-empty set E. If $\mu(\chi) = \chi$ for every $\chi \in E$, then $\chi$is called fixed point of μ.

Remark 2.8 In(E,G_{JS}), we define

$\Delta(G_{JS},\mu,\chi_0) = sup\ \{G_{JS}(\mu^i\chi_0,\mu^j\chi_0,\mu^j\chi_0) : i,j \in N\}$.

Proposition 2.9 If $(E,G_{JS})$is a G_{JS}-metric space and $\langle X_n\rangle$ converges to χ in E, then it has a unique limit point.

Proof. Assume that $\langle X_n\rangle$ converges to both χ and ψ.

For any constant c > 0, for every $(\psi,\chi,\chi) \in E^3$ and $\langle X_n\rangle \in G(G_{JS},E,\psi)$ we get

$G_{JS}(\psi,\chi,\chi) \leq c \lim\limits_{n\to\infty} sup\ G_{JS}(\chi_n,\chi,\chi) = 0$,

implying that$\chi = \psi$.

Therefore, the limit point of $\langle X_n\rangle$ is unique.

3 Results and Discussion

Priya Shahi, Jatinderdeep Kaur and Bhatia established a common fixed point theorem for expansive mappings by using the concept of weak compatibility in G-metric spaces whereas in this article, this theorem is generalized to find the fixed point of a k- expansive mapping without using the concept of weak compatibility in G_{JS}- metric space.

Theorem 3.1 In a symmetric, complete G_{JS}-metric space (E,G_{JS}), if $\mu: E \to E$is a bijective and k - expansive mapping holding the following condition,

$G_{JS}(\mu\chi,\mu^2\chi,\mu^2\chi) \geq kG_{JS}(\chi,\mu\chi,\mu\chi)$, for each $\chi \in E,\ K > 1$.

Then $\frac{1}{k^n}\Delta(G_{JS},\mu,\chi_0) < \infty$ as$n \to \infty$ for $\chi_0 \in E$.

If μ is G_{JS} - continuous on E, then μ has unique fixed point.

Proof: Since $\mu : E \to E$ is a bijective mapping, μ has inverse on its range. Therefore, for an arbitrary $\chi_0, \chi_1 \in E$ we can see that $\chi_0 = \mu^{-1}(\chi_1)$.

Similarly, we can find $\chi_2 \in E$so that $\chi_1 = \mu^{-1}(\chi_2)$.

Proceeding in a similar manner, there is $\langle X_n \rangle \in E$ with $\mu\chi_n = \chi_{n+1}$.

Also, we can observe that

$$\mu^2\chi_0 = \mu(\mu\chi_0) = \mu\chi_1 = \chi_2$$

$$\mu^3\chi_0 = \mu(\mu^2\chi_0) = \mu\chi_2 = \chi_3$$

$$\mu^n\chi_0 = \chi_n$$

If $\chi_n = \chi_{1+n}$, then evidently, $\mu(\chi_n) = \chi_n$ for every $n \in N$.

Now, for each $n \in N$ assume that $\chi_n \neq \chi_{1+n}$.

Since μ is a k-expansion, by the given condition for $\chi_0 \in E$,

$$G_{\text{JS}}(\mu\chi_0, \mu^2\chi_0, \mu^2\chi_0) \geq kG_{\text{JS}}(\chi_0, \mu\chi_0, \mu\chi_0)$$

$$G_{\text{JS}}(\mu^2\chi_0, \mu^3\chi_0, \mu^3\chi_0) \geq kG_{\text{JS}}(\mu\chi_0, \mu^2\chi_0, \mu^2\chi_0)$$

$$G_{\text{JS}}(\chi_2, \chi_3, \chi_3) \geq k^2G_{\text{JS}}(\chi_0, \mu\chi_0, \mu\chi_0) = k^2G_{\text{JS}}(\chi_0, \chi_1, \chi_1)$$

Proceeding in a similar way, for any $n \in N$, we get

$$G_{\text{JS}}(\chi_n, \chi_{1+n}, \chi_{1+n}) \geq k^nG_{\text{JS}}(\chi_0, \chi_1, \chi_1).$$

$$G_{\text{JS}}(\mu^n\chi_0, \mu^{n+1}\chi_0, \mu^{n+1}\chi_0) \geq kG_{\text{JS}}(\mu^{n-1}\chi_0, \mu^n\chi_0, \mu^n\chi_0) \geq k^nG_{\text{JS}}(\chi_0, \chi_1, \chi_1).$$

$$Or\ G_{\text{JS}}(\mu^{n-1}\chi_0, \mu^n\chi_0, \mu^n\chi_0) \leq \frac{1}{k}G_{\text{JS}}(\mu^n\chi_0, \mu^{n+1}\chi_0, \mu^{n+1}\chi_0).$$

Taking supremum on both sides and using the definition of Δ from Remark 2.8, we get

$$\Delta(G_{\text{JS}}, \mu, \mu^n, \chi_0) \leq \frac{1}{k}\Delta(G_{\text{JS}}, \mu, \mu^{n-1}, \chi_0) \leq \frac{1}{k^n}\Delta(G_{\text{JS}}, \mu, \chi_0)$$

In general, for n,$m \in N$, we get

$$G_{\text{JS}}(\mu^n\chi_0, \mu^{n+m}\chi_0, \mu^{n+m}\chi_0) \leq \Delta(G_{\text{JS}}, \mu^n, \mu^{n+m}, \chi_0) \leq \Delta(G_{\text{JS}}, \mu, \mu^n, \chi_0) \leq \frac{1}{k^n}\Delta(G_{\text{JS}}, \mu, \chi_0)$$

As n $\to \infty$ and k > 1, using $\frac{1}{k^n}\Delta(G_{\text{JS}}, \mu, \chi_0) < \infty$, we get $\lim\limits_{n,m \to \infty} G_{\text{JS}}(\chi_n, \chi_n, \chi_{n+m}) = 0$.

Therefore $\langle X_n \rangle$ is G_{JS} - Cauchy.

Moreover,

$$\lim_{n \to \infty} \chi_n = \xi,\ where,\ \xi \in E. \tag{3.1.1}$$

This implies $\lim_{n\to\infty} \chi_{2n} = \lim_{n\to\infty} \chi_{2n+1} = \xi$.

$\xi = \lim_{n\to\infty} \chi_{2n+1} = \lim_{n\to\infty} \mu\chi_{2n} = \mu\xi$, from the G_{JS}-continuity of μ.

Therefore $\mu\xi = \xi$.

If $\xi^{'} \neq \xi$ with $\mu\xi^{'} = \xi^{'}$, then the sequence $\langle \mu^n X_0 \rangle$converges to both ξ and $\xi^{'}$.

From (Equation (3.1.1)), we get

$\lim_{n\to\infty} \mu^n \chi_0 = \lim_{n\to\infty} \chi_n = \xi$and $\lim_{n\to\infty} \mu^n \chi_0 = \lim_{n\to\infty} \chi_n = \xi^{'}$.

From proposition 2.9 we can conclude that the sequence $\langle \mu^n X_0 \rangle = \langle X_n \rangle$ have unique limit point. Therefore, μ has unique fixed point.

Example 3.2 Let $E = [0, \infty)$ and $G_{JS} : E^3 \to [0,\infty)$ is defined by,

$G_{JS}(\chi,\psi,\xi) = |\xi-\chi| + |\xi-\psi| + |\psi-\chi|$,

is a G_{JS}- metric on E. Let $\mu : E \to E$be defined by $\mu(\chi) = 5X$ for every $\chi \in E$.

It is clear that μ is a G_{JS} – continuous and bijective map.

$$G_{JS}(\mu\chi,\mu\chi,\mu\psi) = G_{JS}(5_X,5_X,5\psi) = 25_X - 5_\psi = 10X - \psi \tag{3.2.1}$$

$$G_{JS}(\chi,\chi,\psi) = 2\psi - \chi \tag{3.2.2}$$

From (Equation (3.2.1)) and (Equation (3.2.2)), it is evident that

$G_{JS}(\mu\chi,\mu\chi,\mu\psi) \geq kG_{JS}(\chi,\chi,\psi)$ for any $k > 1,\ X \in E$.

Therefore, μ is a k-expansive mapping and $\frac{1}{k^n}\Delta(G_{JS},\mu,\chi_0) < \infty$ as $n \to \infty$.

$$G_{JS}(\mu\chi,\mu^2\chi,\mu^2\chi) = 40_X \tag{3.2.3}$$

$$G_{JS}(\chi,\mu\chi,\mu\chi) = 8\chi \tag{3.2.4}$$

From (Equation (3.2.3)) and (Equation (3.2.4)), we can say that$G_{JS}(\mu\chi,\mu^2\chi,\mu^2\chi) \geq kG_{JS}(\chi,\mu\chi,\mu\chi)$, for every $\chi \in E,\ K > 1$

Thus, every condition of Theorem 3.1 is satisfied.

Moreover, the one and only fixed point of $\mu$is $\chi = 0$.

4 Conclusion

In this article, a fixed point theorem for k – expansive mapping on G_{JS}- metric space is established. The uniqueness of a fixed point is proved and verified with a suitable example in G_{JS}- metric space.

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