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New and Modified Homotopy Perturbation Methods for Addressing Burger's Non-linear Equation

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Abstract

Objectives: The aim of this study is to find a novel solution procedure for solving a fluid dynamical problem, especially to solve two-dimensional coupled Burger's non-linear equation. **Methods:** New and Modified homotopy perturbation techniques are used to solve the two-dimensional coupled Burger's equation. The methods intend to make homotopy perturbation method a more robust and trustworthy tool for fluid dynamics researchers by addressing convergence concerns, improving solution accuracy and allowing it to handle a broader range of problems. **Findings:** The solution for two-dimensional coupled Burger's non-linear equation is obtained by constructing homotopies. Numerical results thus obtained are compared with the exact solution. The obtained results and the exact solutions to the stated problem are closely related. **Novelty:** The New and Modified Homotopy Perturbation Method proposed in this paper are effective mathematical tools for getting an exact solution to the coupled Burgers' equations. It is also a potential approach for solving various linear and nonlinear partial differential equations.

Keywords: Perturbation; Homotopy; New homotopy perturbation; Modified homotopy perturbation; Twodimension; Linear; Coupled burgers' equation

1 Introduction

The Homotopy Perturbation Method (HPM), introduced by Ji-Huan He [1], has become a popular tool for obtaining approximate solutions to nonlinear differential equations. Its key strengths lie in its versatility and parameter independence. HPM may have limitations in handling strongly nonlinear problems or those with rapidly varying solutions. The convergence rate can also depend on the choice of the embedding parameter. Unlike some perturbation methods, HPM does not require a small parameter to be present in the equation, which broadens its applicability to real-world scenarios where such a parameter might not exists [2]. However, ongoing research strives to improve HPM's effectiveness. New and modified versions address potential limitations and enhance its capabilities.

Numerous enhancements are intended to accelerat the convergence of the solution series acquired using HPM. This leads to more accurate results with fewer iterations, making the method more efficient [3]. HPM can be combined with other techniques like variational methods to create a hybrid approach that leverages the strengths of both for even more robust solutions. Many researchers have focused on solving nonlinearpartial differential equations using various methods. Kumar and his co-authors presented the generalized aspects of least square homotopy perturbation to treat the system of nonlinear fractional partial differential equations [4]. Several methods exist, including the Laplace modified homotopy [5], Li-He's modified homotopy [6], optimal and modified homotopy perturbation [7] and they provide approximate solutions for possible fastest convergence. The convergence of these methods can be sensitive to the choice of the embedding parameter and the auxiliary parameter. Finding optimal values can be challenging, especially for complex problems. The approximate solutions obtained from these methods are often represented as infinite series. Truncating these series can introduce errors, especially for highly nonlinear problems. However, their effectiveness and limitations depend on the specific problem characteristics and the appropriate choice of parameters.

Recently, Awatif[8] apply HPM to provide light on the use of this method and the generic fractional derivative to solve the generalized Burgers equation, making a contribution in the field of nonlinear differential equation. Salima et.al. [9] used HPM in conjunction with the Mohandtransform (MHPM) to solve Burger's equations. Mohand transform is a useful technique for solving linear differential equations, but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms.

Here we used new and modified homotopy methods to solve two-dimensional Burger's equation and the results show that these methods are more effective, convenient to use and of high accuracy. The new and modified homotopy parameters involved in this method helps to improve the convergence rate and accuracy of the solution. In this method, the solution is considered as an infinite series expansion where it converges rapidly to the exact solution. The current approach lessens the computational burden and all computations may be performed with straightforward manipulations. If the closed form of the solutions is needed, it can be obtained using the solutions presented in this paper.

2 Methodology

2.1. Homotopy perturbation method

In topology two continuous functions from one topological space to another is called "homo-topic". Formally, a homotopy between two continuous function f and g from a topological space Y is defined to be a continuous function.

$$H:\ X \times [0,1] \rightarrow Y$$

such that $H(x,0) = f(x)$

$$H(x,1) = g(x) \forall x \in X$$

The homotopy perturbation method does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter.

Now, to illustrate the basic idea of the homotopy perturbation method, we consider the following non-linear equation:

$$A(u) = f(x) \ \ x \in \Omega \tag{1}$$

where A is any operator

f is a known function of x with boundary condition $B\left(U, \frac{\partial u}{\partial n}\right), x \in \Gamma$.

The operator A can generallly be divided into two parts L and N, where L is a linear operator,

N is a non-linear operator. B is a boundary operator and Γ is boundary of domain Ω

Rewrite Equation (1) as

$$L(u) + N(u) - f(x) = 0$$

By the homotopy technique, we construct a homotopy

$$v(r,p) : \Omega \times [0,1] \rightarrow R$$

which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1], r \in \Omega$$

[OR]

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0$$
$$H(v,0) = L(u) - L(u_0) = 0$$
$$H(v,1) = A(v) - f(r) = 0$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

2.2. New Homotopy Perturbation Technique for solving two-dimensional Burger's equation

General form of system of Burgers' equations can be considered as the following forms

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \eta u \frac{\partial u}{\partial x} + \alpha \frac{\partial}{\partial x}(uv) = f(x,t), \\ \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + \eta v \frac{\partial v}{\partial x} + \beta \frac{\partial}{\partial x}(uv) = g(x,t), \end{aligned}\right\} \tag{2}$$

with initial conditions

$$u(x,0) = \varphi_1(x)$$
$$v(x,0) = \varphi_2(x)$$

$$u(x,0) = \varphi_1(x)$$
$$v(x,0) = \varphi_2(x)$$

where

$\alpha,\ \beta,\ \eta$ are constants, and

$f,\ g$ are inhomogeneous terms.

The Burgers' model [4] of turbulence is a very important fluid dynamic model, the study of this model and the theory of shock waves have been considered by many authors, both to obtain a conceptual understanding of a class of physical flows and for testing various numerical methods.

2.2.1. Algorithm

For solving system 2 (Equation (2)) by NHPM we construct the following homotopies:

$$\left.\begin{aligned} \frac{\partial U}{\partial t} = u_0 - p\left[u_0 - \frac{\partial^2 U}{\partial x^2} + \eta U \frac{\partial U}{\partial x} + \alpha \frac{\partial}{\partial x}(UV) - f(x,t)\right], \\ \frac{\partial V}{\partial t} = v_0 - p\left[v_0 - \frac{\partial^2 V}{\partial x^2} + \eta V \frac{\partial V}{\partial x} + \beta \frac{\partial}{\partial x}(UV) - g(x,t)\right] \end{aligned}\right\} \tag{3}$$

Applying the inverse operator, $L^{-1} = \int_{t_0}^{t} (.)dt$ to both sides of Equation (3), we obtain

$$\left.\begin{aligned} U(x,t) = U(x,t_0) + \int_{t_0}^{t} u_0 dt - p\int_{t_0}^{t}\left[u_0 - \frac{\partial^2 U}{\partial x^2} + \eta U \frac{\partial U}{\partial x} + \alpha \frac{\partial}{\partial x}(UV) - f(x,t)\right], \\ V(x,t) = V(x,t_0) + \int_{t_0}^{t} v_0 dt - p\int_{t_0}^{t}\left[u_0 - \frac{\partial^2 V}{\partial x^2} + \eta V \frac{\partial V}{\partial x} + \alpha \frac{\partial}{\partial x}(UV) - g(x,t)\right], \end{aligned}\right\} \tag{4}$$

where

$$U(x,t_0) = u(x,t_0),$$
$$V(x,t_0) = v(x,t_0).$$

Let's present the solution of the system 4 (Equation (4)) as the following

$$\left.\begin{aligned} U = U_0 + pU_1 + p^2 U_2 + \cdots \\ V = V_0 + pV_1 + p^2 V_2 + \cdots \end{aligned}\right\} \tag{5}$$

where $U_i, V_i, i = 1, ..., n,$ are functions that should be determined.Suppose that the initial approximation of Equation (2) is in the following form

$$\left.\begin{array}{l} u_0(x,t) = \sum_{j=0}^{\infty} a_j(x) P_j(t), \\ v_0(x,t) = \sum_{j=0}^{\infty} b_j(x) P_j(t), \end{array}\right\} \tag{6}$$

where $a_{ij}(x_1, x_2, ..., x_{n-1}), i = 1, ..., n, j = 0, ..., n,$ are unknown coefficients and $P_0(t), P_1(t), P_2(t), \cdots$ are specific functions. Substituting Equations (5) and (6) into Equation (4) and equating the coefficients of p with the same powers lead to

$$p^0 : \begin{cases} U_0(x,t) = \varphi_1(x) + \sum_{j=0}^{\infty} a_j \int_{t_0}^{t} P_j(t)\, dt \\ V_0(x,t) = \varphi_2(x) + \sum_{j=0}^{\infty} b_j \int_{t_0}^{t} P_j(t)\, dt \end{cases}$$

$$p^1 : \begin{cases} U_1(x,t) = -\sum_{j=0}^{\infty} a_j \int_{t_0}^{t} P_j(t)\, dt - \int_{t_0}^{t} [-\frac{\partial^2 U_0}{\partial x^2} + \eta U_0 \frac{\partial U_0}{\partial x} + \alpha \frac{\partial}{\partial x}(U_0 V_0) - f(x,t)] dt \\ V_1(x,t) = -\sum_{j=0}^{\infty} b_j \int_{t_0}^{t} P_j(t)\, dt - \int_{t_0}^{t} [-\frac{\partial^2 V_0}{\partial x^2} + \eta V_0 \frac{\partial V_0}{\partial x} + \beta \frac{\partial}{\partial x}(U_0 V_0) - g(x,t)] dt \end{cases}$$

$$\vdots$$

$$p^{j+1} : \begin{cases} U_{j+1}(x,t) = -\int_{t_0}^{t} (-\frac{\partial^2 U_j}{\partial^2} + \eta \sum_{k=0}^{j} U_j \frac{\partial U_{j-k}}{\partial x} + \alpha \sum_{k=0}^{j} V_{j-k} \frac{\partial U_j}{\partial x} + \alpha \sum_{k=0}^{j} U_{j-k} \frac{\partial V_j}{\partial x}) dt \\ V_{j+1}(x,t) = -\int_{t_0}^{t} (-\frac{\partial^2 V_j}{\partial^2} + \eta \sum_{k=0}^{j} V_j \frac{\partial V_{j-k}}{\partial x} + \beta \sum_{k=0}^{j} V_{j-k} \frac{\partial U_j}{\partial x} + \beta \sum_{k=0}^{j} U_{j-k} \frac{\partial V_j}{\partial x}) dt \end{cases} \tag{7}$$

And so on…

Now if we solve these equations in such a way that $U_{i,1}(x_1, x_2, \cdots, x_{n-1}, t) = 0$, then Equation (7) results in $U_{i,1}(x_1, x_2, \cdots, x_n, t) = U_{i,2}(x_1, x_2, \cdots, x_n, t) = \cdots = 0$. Therefore the exact solution may be obtained as

$$\begin{aligned} u_i(x_1, x_2, \cdots, x_{n-1}, t) &= U_{i,0}(x_1, x_2, \cdots, x_{n-1}, t) \\ &= f_i(x_1, x_2, \cdots, x_{n-1}) + \sum_{j=0}^{\infty} a_{ij} \int_{t_0}^{t} P_j(t), \end{aligned}$$

$$i = 1, 2, ..., n \tag{8}$$

It is worthwhile to mention that if $g(x_1, x_2, \cdots, x_{n-1}, t)$, and $u_{i,0}(x_1, x_2, \cdots, x_{n-1}, t)$, are analytic around $t = t_0$, then their Taylor series can be defined as

$$\begin{array}{c} u_0(x,t) = \sum_{j=0}^{\infty} a_j(x)(t-t_0)^n, \\ v_0(x,t) = \sum_{j=0}^{\infty} b_j(x)(t-t_0)^n, \\ f(x,t) = \sum_{j=0}^{\infty} a_j^*(x)(t-t_0)^n, \\ g(x,t) = \sum_{j=0}^{\infty} b_j^*(x)(t-t_0)^n, \end{array}$$

can be used in Equation (7) where $a_j(x), j = 0, \cdots, n, b_j(x), j = 0, \cdots, n,$ are unknown coefficients which must be computed, and $a_j(x), j = 0, \cdots, n, b_j^*(x), j = 0, \cdots, n.$ are known ones.

2.2.2. Numerical result

Consider the system of coupled Burgers' equations.

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-2u\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}(uv)=0\\ \frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}-2u\frac{\partial v}{\partial x}+\frac{\partial}{\partial x}(uv)=0\end{array}\right\} \tag{9}$$

with initial conditions

$$u(x,0)=cosx,$$
$$v(x,0)=cosx$$

The exact solutions are

$$u(x,t)=e^{-t}cosx$$
$$v(x,t)=e^{-t}cosx$$

To solve Equation (9), by NHPM, we construct the following homotopies

$$\left.\begin{array}{l}\frac{\partial U}{\partial t}=u_0-p\left[u_0-\frac{\partial^2 u}{\partial x^2}-2u\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}(uv)\right]\\ \frac{\partial V}{\partial t}=v_0-p\left[v_0-\frac{\partial^2 v}{\partial x^2}-2u\frac{\partial v}{\partial x}+\frac{\partial}{\partial x}(uv)\right]\end{array}\right\} \tag{10}$$

Applying the inverse operator, $L^{-1}=\int_0^t(.)dt$ to both sides of these equations, we obtain

$$\left.\begin{array}{l}U(x,t)=U(x,0)+\int_0^t u_0(x,t)\,dt-p\int_0^t\left[u_0(x,t)-\frac{\partial^2 U}{\partial x^2}-2U\frac{\partial U}{\partial x}+\frac{\partial}{\partial x}(UV)\right]dt\\ V(x,t)=V(x,0)+\int_0^t v_0(x,t)\,dt-p\int_0^t\left[v_0(x,t)-\frac{\partial^2 V}{\partial x^2}-2V\frac{\partial V}{\partial x}+\frac{\partial}{\partial x}(UV)\right]dt\end{array}\right\} \tag{11}$$

where

$$U(x,t_0)=u(x,t_0)$$
$$V(x,t_0)=v(x,t_0)$$

Let's present the solution of the system 11 (Equation (11)) as the following

$$\left.\begin{array}{l}U=U_0+pU_1+p^2U_2+\cdots\\ V=V_0+pV_1+p^2V_2+\cdots\end{array}\right\} \tag{12}$$

Substituting Equation (12) into Equation (11), we get

$$U_0(x,t)+pU_1(x,t)+p^2U_2(x,t)+\cdots$$

$$\begin{array}{l}=u(x,0)+t\int_0^t u_0(x,t)\,dt-p\int_0^t\{u_0(x,t)\\ -\frac{\partial^2}{\partial x^2}(U_0+pU_1+\cdots]-2[U_0+pU_1+p^2U_2+\cdots]\\ \frac{\partial}{\partial x}[U_0+pU_1+p^2U_2+\cdots]+\\ \frac{\partial}{\partial x}[(U_0+pU_1+p^2U_2+\cdots)(V_0+pV_1+p^2V_2+\cdots)]\}dt\end{array}$$

$$V_0(x,t)+pV_1(x,t)+p^2V_2(x,t)+\cdots$$

$$\begin{array}{l}=v(x,0)+t\int_0^t v_0(x,t)\,dt-p\int_0^t\{v_0(x,t)\\ -\frac{\partial^2}{\partial x^2}[V_0+pV_1+\cdots]-2[V_0+pV_1+p^2V_2+\cdots]\\ \frac{\partial}{\partial x}[V_0+pV_1+p^2V_2+\cdots]+\\ \frac{\partial}{\partial x}[(U_0+pU_1+p^2U_2+\cdots)(V_0+pV_1+p^2V_2+\cdots)]\}dt\end{array}$$

Now, Collecting the same powers of P, and equating each coefficient of p to zero, results in

$$p_0 : \begin{cases} U_0(x,t) = u(x,0) + \int_0^t u_0(x,t)\,dt \\ V_0(x,t) = v(x,0) + \int_0^t v_0(x,t)\,dt \end{cases}$$

$$p_1 : \begin{cases} U_1(x,t) = -\int_0^t \left[u_0(x,t) - \frac{\partial^2 U_0}{\partial x^2} - 2U_0 \frac{\partial U_0}{\partial x} + \frac{\partial}{\partial x}(U_0 V_0)\right] dt \\ V_1(x,t) = -\int_0^t \left[v_0(x,t) - \frac{\partial^2 V_0}{\partial x^2} - 2V_0 \frac{\partial V_0}{\partial x} + \frac{\partial}{\partial x}(U_0 V_0)\right] dt \end{cases}$$

And so on…

Assuming

$$\begin{cases} u_0(x,t) = \sum_{n=0}^{\infty} a_n(x) P_n(t), P_n(t) = t^n, U(x,0) = u(x,0) \\ v_0(x,t) = \sum_{n=0}^{\infty} b_n(x) P_n(t), P_n(t) = t^n, V(x,0) = v(x,0) \end{cases}$$

Substituting the values in the above equations we get,

$$\begin{aligned} U_0(x,t) &= cosx + a_0(x)t + a_1(x)\tfrac{t^2}{2} + a_2(x)\tfrac{t^3}{3} + \cdots \\ V_0(x,t) &= cosx + b_0(x)t + b_1(x)\tfrac{t^2}{2} + b_2(x)\tfrac{t^3}{3} + \cdots \end{aligned}$$

$$\begin{aligned} &U_1(x,t) = t(-a_0(x) - \cos x) + t^2\,[-\tfrac{1}{2}\,a_1(x) + \tfrac{1}{2}\,a_{0_{xx}}(x) \\ &+ \cos x a_{0x}(x) - a_0(x)\sin x + \tfrac{1}{2}\,a_0(x)\sin x \\ &- \tfrac{1}{2} b_{0x}(x)\cos x + \tfrac{1}{2}\,b_0(x)\sin x - \tfrac{1}{2} a_{0x}(x)\cos x] \\ &+ t^3\,[-\tfrac{1}{3}\,a_2(x) + \tfrac{1}{6}\,a_{1xx}(x) - \tfrac{1}{3}\,a_1(x)\sin x \\ &+ \tfrac{2}{3}\,a_0(x)\,a_{0x}(x) + \tfrac{1}{3}\,\cos x a_{1x}(x) + \tfrac{1}{6}\,a_1(x)\sin x - \tfrac{1}{3}\,b_{0x}(x)\,a_0(x) \\ &- \tfrac{1}{6}\cos x b_{1x}(x) + \tfrac{1}{6}\,\sin x b_1(x) - \tfrac{1}{3}\,b_0(x)\,a_{0x}(x) - \tfrac{1}{6}\cos x a_{1x}(x)] + \ldots \end{aligned}$$

$$\begin{aligned} &V_1(x,t) = t(-b_0(x) - sinx) + t^2[-\tfrac{1}{2} b_1(x) + \tfrac{1}{2} b_{0xx}(x) \\ &+ cosx b_{0x}(x) - b_0(x)\,sinx + \tfrac{1}{2} b_0(x)\,sinx \\ &- \tfrac{1}{2} b_{0x}(x)\,cosx + \tfrac{1}{2} a_0(x)\,sinx - \tfrac{1}{2} a_{0x}(x)\,cosx] \\ &+ t^3[-\tfrac{1}{3} b_2(x) + \tfrac{1}{6} b_{1xx}(x) - \tfrac{1}{3} b_1(x)\,sinx \\ &+ \tfrac{2}{3} b_0(x)\,b_{0x}(x) + \tfrac{1}{3} cosx b_{1x}(x) + \tfrac{1}{6} b_1(x)\,sinx - \tfrac{1}{3} a_{0x}(x)\,b_0(x) \\ &- \tfrac{1}{6} cosx a_{1x}(x) + \tfrac{1}{6} sinx a_1(x) - \tfrac{1}{3} a_0(x)\,b_{0x}(x) - \tfrac{1}{6} cosx b_{1x}(x)] + \cdots \end{aligned}$$

By Vanishing $U_1(x,t), V_1(x,t)$ coefficients, $a_n(x), b_n(x)\,(n = 1, 2, \cdots)$ are determined as

$$\begin{aligned} a_0(x) &= -cosx \\ b_0(x) &= -cosx \\ a_1(x) &= cosx \\ b_1(x) &= cosx \\ a_2(x) &= -\tfrac{cosx}{2} \\ b_2(x) &= -\tfrac{1}{2} cosx \end{aligned}$$

Therefore we gain the solution

$$\begin{gathered} u(x,t) = U_0(x,t) \\ = cosx + a_0(x)t + \tfrac{1}{2} a_1(x)t^2 + \tfrac{1}{3} a_2(x) + \ldots \\ = cosx - cosxt + \tfrac{1}{2} cosxt^2 - \tfrac{1}{2} cosx\tfrac{1}{3} t^3 + \cdots \\ = cosxexp(-t) \end{gathered}$$

$$\begin{gathered} v(x,t) = V_0(x,t) \\ = cosx + b_0(x)t + \tfrac{1}{2} b_1(x)t^2 + \tfrac{1}{3} b_2(x) + \cdots \\ = cosx - cosxt + \tfrac{1}{2} cosxt^2 - \tfrac{1}{2} cosx\tfrac{1}{3} t^3 + \cdots \\ = cosxexp(-t) \end{gathered}$$

2.3. Modified Homotopy Perturbation Technique for solving two-dimensional Burger's equation

The two-dimensional Burgers' equation [10] is a fundamental mathematical model from fluid mechanics which has the same convective and diffusion terms as the Navier-Stokes equation, and is widely used in various areas as a simple model for understanding of various physical flows and problems, such as modeling of dynamics the phenomena of turbulence and flow through a shock wave traveling in a viscous fluid and traffic flow.

Numerical solution of Burgers' equation is a logical first step towards developing methods for the computation of complex flows. Burgers' equation is also a useful tool for examining the robustness of numerical discretization schemes. It has become customary to test new numerical approaches in computational fluid dynamics by implementing novel numerical approaches to the Burgers' equation.

2.3.1. Results Consider the two dimensional nonlinear Burgers' differential equation

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]\end{array}\right\} \tag{13}$$

with initial conditions

$$\left.\begin{array}{l}u(x,y,0)=x+y\\ v(x,y,0)=x-y\end{array}\right\} \tag{14}$$

Using Modified algorithm of Homotopy Perturbation Method, the homotopies for Equation (13) are

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}=p\left[\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-D_t u\right]\\ \frac{\partial v}{\partial t}=p\left[\frac{\partial v}{\partial t}-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-D_t v\right]\end{array}\right\} \tag{15}$$

As above, the basic assumption is that the solutions of the Equation (13) can be written as a power series in p.

$$\left.\begin{array}{l}u=u_0+pu_1+p^2u_2+p^3u_3+\cdots\\ v=v_0+pv_1+p^2v_2+p^3v_3+\cdots\end{array}\right\} \tag{16}$$

On substituting Equation (16) and the initial conditions Equation (14) into Equation (14), we get $\frac{\partial}{\partial t}\left(u_0+pu_1+p^2u_2+p^3u_3+\cdots\right)$

$$\begin{array}{c}=p[\frac{\partial}{\partial t}\left(u_0+pu_1+p^2u_2+\ldots\right)-\left(u_0+pu_1+p^2u_2+p^3u_3+\ldots\right)\\ \frac{\partial}{\partial x}\left(u_0+pu_1+p^2u_2+p^3u_3\right)-\left(v_0+pv_1+p^2v_2+p^3v_3+\ldots\right)\\ \frac{\partial}{\partial y}\left(u_0+pu_1+p^2u_2+p^3u_3+\ldots\right)+\frac{1}{Re}(\frac{\partial^2}{\partial x^2}\left(u_0+pu_1+p^2u_2+\ldots\right)\\ +\frac{\partial^2}{\partial y^2}\left(u_0+pu_1+p^2u_2+\ldots\right))-D_t\left(u_0+pu_1+p^2u_2+\ldots\right)]\end{array}$$

$$\frac{\partial}{\partial t}\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)$$

$$\begin{array}{c}=p[\frac{\partial}{\partial t}\left(v_0+pv_1+p^2v_2+\cdots\right)-\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)\frac{\partial}{\partial x}\left(v_0+pv_1+p^2v_2+p^3v_3\right)\\ -\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)\frac{\partial}{\partial y}\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)\\ +\frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}\left(v_0+pv_1+p^2v_2+\cdots\right)+\partial^2\partial y^2\left(v_0+pv_1+p^2v_2+\cdots\right)\right)\\ -D_t\left(v_0+pv_1+p^2v_2+\cdots\right)]\end{array}$$

And

$$\left[u_0+pu_1+p^2u_2+\cdots\right](x,y,0)=x+y$$

$$\left[v_0+pv_1+p^2v_2+\cdots\right](x,y,0)=x-y$$

Equating the terms with identical powers of p, we can obtain the following set of linear partial differential equations.

$$\frac{\partial u_0}{\partial t}=0,\ u_0(x,y,0)=x+y$$
$$\frac{\partial v_0}{\partial t}=0,\ v_0(x,y,0)=x-y$$
$$\Rightarrow \frac{\partial u_1}{\partial t}=\frac{\partial u_0}{\partial t}-u_0\frac{\partial u_0}{\partial x}-v_0\frac{\partial u_0}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right]-D_t u_0,\ u_1(x,y,0)=0$$
$$\Rightarrow \frac{\partial v_1}{\partial t}=\frac{\partial v_0}{\partial t}-u_0\frac{\partial v_0}{\partial x}-v_0\frac{\partial v_0}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}\right]-D_t u_0,\ v_1(x,y,0)=0$$

$$\Rightarrow \frac{\partial u_2}{\partial t}=\frac{\partial u_1}{\partial t}-u_0\frac{\partial u_1}{\partial x}-u_1\frac{\partial u_0}{\partial x}-v_0\frac{\partial u_1}{\partial y}-v_1\partial u_0\partial y+$$
$$\frac{1}{Re}\left[\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right]-D_t u_1,\ u_2(x,y,0)=0$$
$$\Rightarrow \frac{\partial v_2}{\partial t}=\frac{\partial v_1}{\partial t}-u_0\frac{\partial v_1}{\partial x}-u_1\frac{\partial v_0}{\partial x}-v_0\frac{\partial v_1}{\partial y}-v_1\partial v_0\partial y+$$
$$\frac{1}{Re}\left[\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}\right]-D_t v_1,\ v_2(x,y,0)=0$$
$$\Rightarrow \frac{\partial u_3}{\partial t}=\frac{\partial u_2}{\partial t}-u_0\frac{\partial u_2}{\partial x}-u_1\frac{\partial u_1}{\partial x}-u_2\frac{\partial u_0}{\partial x}-v_0\frac{\partial u_2}{\partial y}-v_1\frac{\partial u_1}{\partial y}-v_2\frac{\partial u_0}{\partial y}$$
$$+\frac{1}{Re}\left[\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right]-D_t u_2,\ u_3(x,y,0)=0$$
$$\Rightarrow \frac{\partial v_3}{\partial t}=\frac{\partial v_2}{\partial t}-u_0\frac{\partial v_2}{\partial x}-u_1\frac{\partial v_1}{\partial x}-u_2\frac{\partial v_0}{\partial x}-v_0\frac{\partial v_2}{\partial y}-v_1\frac{\partial v_1}{\partial y}-v_2\frac{\partial v_0}{\partial y}$$
$$+\frac{1}{Re}\left[\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}\right]-D_t v_2,\ v_3(x,y,0)=0$$

And so on...

Consequently, the first few components of the homotopy perturbation solution for Equation (13) are derived by integrating with respect to t we get,

$$u_0=(x+y)$$

Similarly,

$$v_0=x-y$$

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial t}(x+y)-(x+y)\frac{\partial}{\partial x}(x+y)-(x-y)\frac{\partial}{\partial y}(x+y)\\ &=-2x\end{aligned}$$

On integrating with respect to t we get,

$$u_1 = -2xt+C_3$$

On substituting the initial condition $u_1(x,y,0)=0$,

$$\begin{aligned}u_1 &= -2xt\\ \frac{\partial v_1}{\partial t} &= -2y\end{aligned}$$

On integrating with respect to t and substituting the initial condition $v_1(x,y,0)=0$ we get,

$$\begin{aligned}\frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial t}(-2xt)-(x+y)\frac{\partial}{\partial x}(-2xt)-(-2xt)\frac{\partial}{\partial y}(x+y)\\ &=4xt+4yt\end{aligned}$$

Similarly,

$$\begin{aligned} u_2 &= 2xt^2 + 2yt^2 \\ \frac{\partial v_2}{\partial t} &= 4xt - 4yt \\ v_2 &= 2xt^2 - 2yt^2 + C_6 \\ \frac{\partial u_3}{\partial t} &= -12xt^2 \\ u_3 &= -4xt^2 + C_7 \\ \frac{\partial v_3}{\partial t} &= -12yt^2 \\ v_3 &= -4yt^2 + C_8 \\ \frac{\partial u_4}{\partial t} &= 16xt^3 + 16yt^3 \\ u_4 &= 4xt^4 + 4yt^4 + C_9 \\ \frac{\partial v_4}{\partial t} &= 16xt^3 - 16yt^3 \\ v_4 &= 4xt^4 - 4yt^4 \end{aligned}$$

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The approximate solution is

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)$$
$$= u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + u_4(x,y,t) + \cdots$$
$$= \frac{x - 2xt + y}{1 - 2t^2}$$

$$v(x,y,t) = \sum_{i=0}^{\infty} v_i(x,y,t)$$
$$= v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + v_3(x,y,t) + v_4(x,y,t) + \cdots$$
$$= \frac{x - 2yt - y}{1 - 2t^2}$$

which is the exact solution of the two-dimensional Burgers' equation.

3 Results and Discussion

In this section we present numerical solution of two dimensional coupled Burgers' equations obtained by New Homotopy Perturbation Method and two dimensional non-linear Burgers' equation obtained by Modified Homotopy Perturbation Method. The solutions thus obtained are compared with the exact solution.

In the following Figures 1, 2 and 3, we validate the solution for different values of t for the system of the coupled Burgers' equation by New Homotopy Perturbation Method and find that there is no difference between the exact and approximate solution.

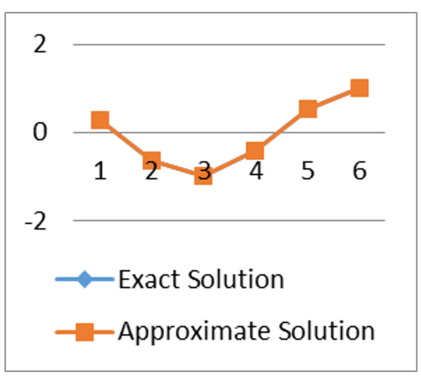

Fig 1. Comparison of approximate solution with exact solution for different values of t, t=0

In the following Figures 4, 5, 6 and 7, we validate the solution for different values of t for the system of two-dimensional non-linear Burgers' equation by Modified Homotopy Perturbation Method and find that there is no difference between the exact and approximate solution.

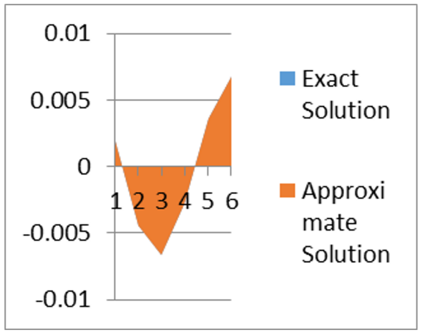

Fig 2. Comparison of approximate solution with exact solution for different values of t, t=5

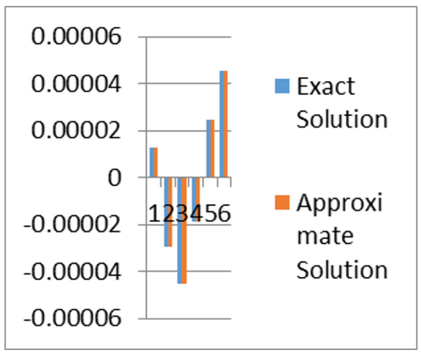

Fig 3. Comparison of approximate solution with exact solution for different values of t, t=10

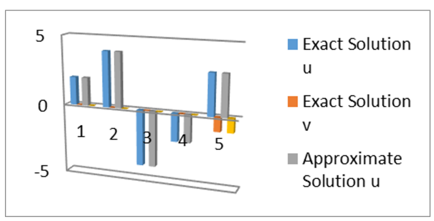

Fig 4. Comparison of approximate solution with exact solution for different values of t, t=0

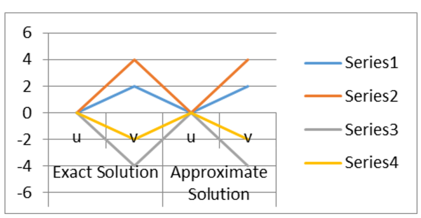

Fig 5. Comparison of approximate solution with exact solution for different values of t, t=1

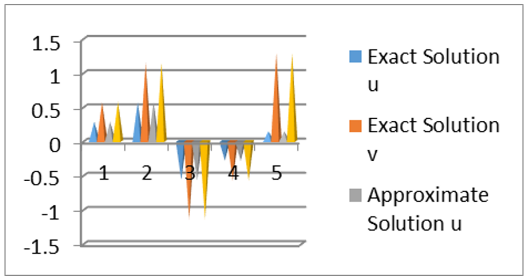

Fig 6. Comparison of approximate solution with exact solution for different values of t, t=2

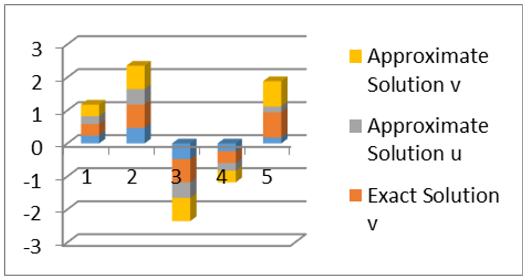

Fig 7. Comparison of approximate solution with exact solution for different values of t, t=3

4 Conclusion

In order to compare the solution to the exact solution of the two-dimensional coupled Burger's non-linear equation and to the same in a fluid dynamical method with various beginning circumstances, we investigated the New and Modified Homotopy Perturbation Method in this study. The comparison analysis shows that for the simulated problem, the approximate solutions found by NHPM and MHPM are the same. There is a close relationship between the exact solutions to the given problem and the results obtained. The convergence of the solutions and the integer-order outcomes of u and v are verified by the graphical depiction. The technique can be extended to solve real life problems by introducing new parameters and researchers can explore such aspects in near future.

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