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RESEARCH ARTICLE

Cardinality and Isomorphic Properties of Hajós Graphs and Hajós Fuzzy Graphs

K Radha[1]*, A Jasmine Kingsly[2]

1 Associate Professor, PG and Research Department of Mathematics, Thanthai Periyar Government Arts and Science College (Autonomous), Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India
2 Research Scholar, PG and Research Department of Mathematics,, Thanthai Periyar Government Arts and Science College (Autonomous), Affiliated to Bharathidasan University, Tirchy, Tamil Nadu, India

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* **Corresponding author**.

radhagac@yahoo.com

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Abstract

Objectives: Fuzzy graphs allow uncertainty in the ideas characterizing vertices and edges to be included when modelling real-world scenarios into graph models. This study aims to introduce Hajós fuzzy graph, cardinality of Hajós graph and Hajós fuzzy graph and to explore many properties. **Methods :** Here, the Hajós construction is applied to two fuzzy graphs, and the Hajós fuzzy graph is defined by assigning membership values to all the edges and vertices of the newly formed fuzzy graph. Fuzzy graph conditions are verified for the assigned membership values. Examples are used to illustrate the concept. **Findings:** The Hajós fuzzy graph's size and order are established. Isomorphic property is discussed. The number of Hajós (fuzzy) graphs that exist for some combinations of two (fuzzy) graphs is calculated. **Novelty:** Based on Hajós construction, the Hajós fuzzy graph is defined. Based on the fact that different choices of vertices and edges produce different Hajós (fuzzy) graph, its cardinality is defined. The notion of cardinality is a novel concept in Hajos graph which helps to find the total number of Hajós (fuzzy) graphs. The cardinality of the Hajós (fuzzy) graph which arises from two (fuzzy) graphs that can be a cycle, path, regular graph, complete graph and complete bipartite graph are derived. **Mathematics subject classification.** (2010): 05C72, 05C76

Keywords: Hajós Graph; Hajós fuzzy graph; Order; Size; Isomorphism; Cardinality of Hajós fuzzy graph

1 Introduction

Graph theory is a part of mathematics that possesses numerous applications in the real world. As it comes to uncertain data, it is essential to apply fuzzy theory to graphs, which results in fuzzy graph theory. Fuzzy graph theory is an extension of graph theory, which plays a crucial role in real-world circumstances like medical diagnosis, social network analysis, and natural language processing. Rosenfeld initially introduced fuzzy graph theory in 1975.

The Hajós construction, named after György Hajós (1961), is an operation on graphs that can be used to construct any critical graph or any graph whose chromatic number is at least a certain threshold. Yamuna M and Karthika K have proved that, if G_1 and G_2 are Euler graphs, then the Hajós graph is also Euler in 'A note on Hajós stable graphs'[1]. They have obtained conditions for Hajós graphs to be Hamiltonian. Also, they have studied the just excellent and very excellent properties of Hajós stable graphs. Benjamin Braun and Julianne Vega have shown that for a large class of Hajós merges and for any identification of a pair of vertices having distance at least five from each other, the resulting graph has an S1-wedge summand in its neighborhood complex in 'Hajós-Type Constructions and Neighborhood Complexes'[2]. They also introduce two graph construction algorithms using Hajós-type operations. Juan Carlos García-Altamirano, Mika Olsen, Jorge Cervantes-Ojeda solved the problem of determining the Hajós operations to construct the symmetric 5-cycle from the complete symmetric digraph of order 3 using only Hajós constructions in 'How to construct the symmetric cycle of length 5 using Hajós construction with an adapted Rank Genetic Algorithm'[3]. They generalize the construction of the symmetric cycle of length 5 and determine the computational complexity Hajós construction in 'Computacional complexity of Hajós constructions of symmetric odd cycles'[4]. G. Gutin, A.V. Kostochka and B. Toft provided bounds of Nordhaus–Gaddum type for the Hajós number[5]. Though Hajós construction has been applied in many areas many properties have been discussed, the total number of Hajós graphs has not yet been discussed.

Different choices of vertices and edges to get Hajós graphs from two given graphs, by applying Hajós construction, produce different Hajós graphs. When combining two graphs using Hajós construction, especially graph models of road networks or railway networks, knowing the total number of Hajós graphs and their properties will be helpful in choosing the best possible combination. Till now, no attempt has been made to determine the number of possible Hajós graphs that can be obtained from two given graphs. Therefore, the cardinality of Hajós graphs is introduced. It is obtained for various combinations of graphs. It is observed that the cardinality is preserved under isomorphism.

Hajós construction and cardinality are extended to fuzzy graphs. Fuzzy graphs enable to incorporate uncertainty in the concepts that characterize vertices and edges into graph models to represent real-world circumstances. For example, while modelling the road connection between cities into a graph, the number of people in the cities who may use the road and the quality of the road can be formulated as the membership values of the vertices and edges. Therefore, it will be useful to introduce and develop the features of Hajós fuzzy graphs for various kinds of applications in real life.

The crisp graph is denoted by $G^*(V, E)$, and the fuzzy graph on G^* is denoted by $G:(\sigma, \mu)$.

The basic things that we have used in developing our new concepts are as follows:

Let G_1 and G_2 be any two graphs, (u_1, v_1) be an edge of G_1 and (u_2, v_2) be an edge of G_2. Then the Hajós construction produce a different graph H that combines the two graphs by

i. Merging vertices u_1 and u_2 into a single vertex u_{12}

ii. Eliminating the two edges (u_1, v_1) and (u_2, v_2)

iii. Adding a new edge (v_1, v_2)[6]

The order of H is $O(G_1) + O(G_2) - 1$ and the size of H is $S(G_1) + S(G_2) - 1$.

A fuzzy graph G is defined as an ordered pair G: (σ, μ) on graph G^* : (V, E) where V is the set of vertices, a vertex is also called a node, and E is the set of edges which is a subset of $V \times V$, the membership functions are $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$.

The degree of a vertex $u \in V$ is defined by $d_G(u) = \sum_{u \neq v} \mu(uv) = \sum_{uv \in E} \mu(uv)$.[7]

If $d_G(v) = k$, $\forall v \in V$, that is if each vertex has same degree k, then G is said to be a regular *fuzzy graph of degree k* or *a k-regular fuzzy graph.*

A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$ is denoted by Km,n.[8] A complete graph Kn is a simple graph such that every pair of vertices is joined by an edge[9].

A walk is a series of pairwise incident edges in a graph; a trail is a walk with no repeated edges; a path is a trail with no repeated vertices. A u-v walk/trail/path is considered closed whenever u = v. Closed paths are known as cycles[10].

An isomorphism of fuzzy graphs $f : G \to G'$ is a bijective map $f : V \to V'$ such that $\sigma(u) = \sigma'(f(u)), \forall u \in V, \mu(uv) = \mu'(f(u) f(v)), u, v \in V$[11]

2 Methodology

Here, the Hajós construction is applied on fuzzy graphs. Hajós construction is one of the effective ways that are used to combine two networks. The Hajós fuzzy graph is a fuzzy graph obtained by applying the binary operation, namely Hajós construction, on two fuzzy graphs. The properties of Hajós fuzzy graph and the cardinality of Hajós fuzzy graph are obtained.

Definition 2.1: Let $G_1 : (\sigma_1, \mu_1)$ and G_2:$\left(\sigma_2, \mu_2\right)$ be two fuzzy graphs on $G_1* : (V_1, E_1)$ and $G_2* : (V_2, E_2)$respectively. Let u_1v_1 be any edge in G_1 and u_2v_2 be any edge in G_2. The Hajós fuzzy graph of G_1 and G_2 with respect to the edge-vertices $u_1v_1 - u_1$ and $u_2v_2 - u_2$, denoted by $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)] : (\sigma, \mu)$ on $H*(V, E)$ where $V = (V_1 - \{u_1\}) \cup (V_2 - \{u_2\}) \cup \{u_1.u_2\}$ and $E = (E_1 - \{u_1v_1\}) \cup (E_2 - \{u_2v_2\}) \cup \{v_1.v_2\}$ is constructed as follows:

(i) Merge the vertices u_1 in G_1 and u_2 in G_2 into a new single vertex $u_1 \cdot u_2$ with membership value $\sigma(u_1 \cdot u_2) = \sigma_1(u_1) \vee \sigma_2(u_2)$

(ii) Remove the edges $u_1v_1$in G_1 and $u_2v_2$in G_2.

(iii) Insert a new edge v_1v_2 in H with membership value $\mu(v_1v_2) = \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)$.

The edges $vw \in E_i - \{u_iv_i\}$ are of two types:

$vw \in E_i,\ v \neq u_i,\ w \neq u_i; v \in V_i,\ w = u_1 \cdot u_2,\ i = 1, 2$. Assign the membership values for these edges as

$$\mu(vw) = \begin{cases} \mu_1(vw), & if\ vw \in E_1, \quad v \neq u_1, w \neq u_1 \\ \mu_2(vw), & if\ vw \in E_2, \quad v \neq u_2, w \neq u_2 \\ \mu_1(vu_1), & if\ v \in V_1, \quad w = u_1.u_2 \\ \mu_2(vu_2), & if\ v \in V_2, \quad w = u_1.u_2 \end{cases}$$

(iv) For all the vertices $v \neq u_i$ i=1, 2, assign their membership value as

$$\sigma(v) = \begin{cases} \sigma_1(v), & if\ v \in V_1 \\ \sigma_2(v), & if\ v \in V_2 \end{cases}$$

Let us verify that σ and μ satisfy the conditions of fuzzy graph.

Let vw be any edge of H. Then $vw \in E_i - \{u_iv_i\}$ or $vw = v_1v_2$.

If $vw \in E_1, v \neq u_1, w \neq u_1, \mu(vw) = \mu_1(vw) \leq \sigma_1(v) \wedge \sigma_1(w) = \sigma(v) \wedge \sigma(w)$.

If $vw \in E_2, v \neq u_2, w \neq u_2, \mu(vw) = \mu_2(vw) \leq \sigma_2(v) \wedge \sigma_2(w) = \sigma(v) \wedge \sigma(w)$.

For any edge $v(u_1 \cdot u_2)$ with $v \in V_1$, $\mu(v(u_1 \cdot u_2)) = \mu_1(vu_1) \leq \sigma_1(v) \wedge \sigma_1(u_1) \leq \sigma_1(v) \wedge (\sigma_1(u_1) \vee \sigma_1(u_2)) = \sigma(v) \wedge \sigma((u_1 \cdot u_2))$.

Similarly for any edge $v(u_1 \cdot u_2)$ with $v \in V_2$, $\mu(v(u_1 \cdot u_2)) \leq \sigma(v) \wedge \sigma((u_1 \cdot u_2))$.

For the new edge v_1v_2 in H,

$\mu(v_1v_2) = \mu(u_1v_1) \wedge \mu(u_2v_2) \leq \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2) \leq \sigma_1(v_1) \wedge \sigma_2(v_2)$

$= \sigma(v_1) \wedge \sigma(v_2)$.

Therefore for all the edges vw in H, $\mu(vw) \leq \sigma(v) \wedge \sigma(w)$

Hence $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)] : (\sigma, \mu)$ is a fuzzy graph, called the Hajós fuzzy graph on G*(V, E).

Example 2.2: The definition of Hajós graph is explained in the following figure. Consider the two fuzzy graphs in G_1 and G_2 in Figure 1. Let us construct $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)] : (\sigma, \mu)$ with $\sigma(u_2 \cdot v_1) = \sigma_1(u_2) \vee \sigma_2(v_1) = 0.4$and $\mu(u_3u_4) = \mu_1(u_2u_3) \vee \mu_2(v_4v_1) = 0.2$. The Hajós fuzzy graph $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)] : (\sigma, \mu)$ is given in Figure 1.

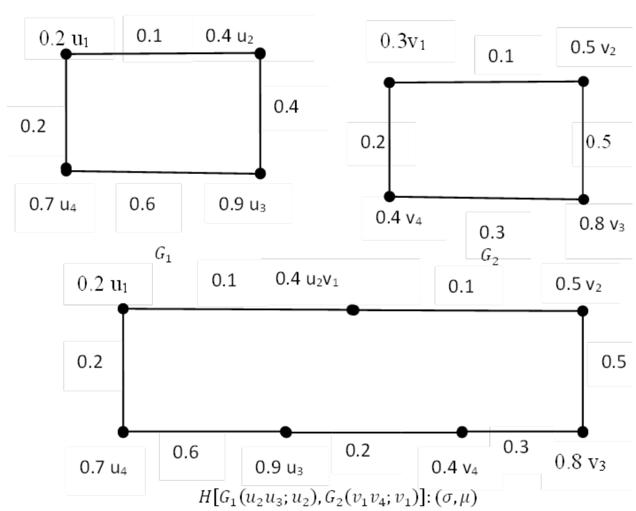

Fig 1. Hajós fuzzy graph of G_1 and G_2

3 Results and Discussions

The Hajós construction of graphs has been widely used in colouring of graphs. Till now, isomorphic properties of Hajós graphs and the number of Hajós graphs that can exist from two given graphs have not been analysed in detail. Apart from the properties of Hajós graphs that already exist including Hajós number which gives the minimum number of steps required to construct a k-chromatic graph from K_k, this paper contributes to the study of Hajós graph by discussing its isomorphic properties and by introducing its cardinality. Hajós construction on fuzzy graphs will be helpful in various real-life applications. It is helpful to combine two networks in an effective way, and its applications can be found in scheduling problems, image processing, and computer networks. Hajós fuzzy graph is introduced by applying the Hajós construction to fuzzy graphs, and its properties are discussed.

3.1: Order and Size of Hajós Fuzzy Graph

Depending on the membership values of the vertices and edges chosen, the formulae for the order and size of the Hajós fuzzy graph are obtained. They will be helpful in determining those values from the given graphs without actually constructing the Hajós fuzzy graph.

Theorem 3.1.1:

If $G_j : (\sigma_j, \mu_j)$ is a fuzzy graph on $G_j{}^{*}$: $G_j * (V_j, E_j)$,, j = 1, 2, then the order of the Hajós Fuzzy Graph $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)] : (\sigma, \mu)$ is $O(H) = O(G_1) + O(G_2) - \sigma_1(u_1) \wedge \sigma_2(u_2)$.

Proof:

The order of the Hajós fuzzy graph H is

$$O(H) = \sum_{u\in V(H)} \sigma(u) = \sum_{u\in V_1-\{u_1\}} \sigma(u) + \sum_{u\in V_2-\{u_2\}} \sigma(u) + \sigma(u_1 \cdot u_2)$$

$$= \sum_{u\in V_1} \sigma(u) - \sigma_1(u_1) + \sum_{u\in V_2} \sigma(u) - \sigma_2(u_2) + \sigma_1(u_1) \vee \sigma_2(u_2)$$

$$= \sum_{u\in V_1} \sigma(u) + \sum_{u\in V_2} \sigma(u) + \sigma_1(u_1) \vee \sigma_2(u_2) - \sigma_1(u_1) - \sigma_2(u_2)$$

$$Since\ \sigma_1(u_1) + \sigma_2(u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2) + \sigma_1(u_1) \vee \sigma_2(u_2)$$

$$\sigma_1(u_1) + \sigma_2(u_2) - \sigma_1(u_1) \vee \sigma_2(u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$$

$$Hence\ O(H) = \sum_{u\in V_1} \sigma_1(u) + \sum_{u\in V_2} \sigma_2(u) - \sigma_1(u_1) \wedge \sigma_2(u_2)$$
$$= O(G_1) + O(G_2) - \sigma_1(u_1) \wedge \sigma_2(u_2).$$

Theorem 3.1.2:

If $G_j : (\sigma_j, \mu_j)$ is a fuzzy graph on $G_j * : (V_j, E_j)$, j = 1, 2, then the size of the Hajós Fuzzy Graph $H[G_1(w_1z_1;w_1), G_2(w_2z_2;z_2)] : (\sigma, \mu)$ is $S(H) = S(G_1) + S(G_2) - \mu_1(w_1z_1) \vee \mu_2(w_2z_2)$

Proof:

The size of the Hajós Fuzzy Graph $H[G_1(w_1z_1;w_1), G_2(w_2z_2;w_2)] : (\sigma, \mu)$ is$S(H) = \sum_{e\in E(H)} \mu(e) = \sum_{e\in E_1-(w_1z_1)} \mu(e) + \sum_{e\in E_2-(w_2z_2)} \mu(e) + \mu(z_1z_2)$

$= \sum_{e\in E_1} \mu(e) - \mu_1(w_1z_1) + \sum_{e\in E_2} \mu(e) - \mu_2(w_2z_2) + \mu_1(w_1z_1) \wedge \mu_2(w_2z_2)$

Since $\mu_1(w_1z_1) + \mu_2(w_2z_2) = \mu_1(w_1z_1) \wedge \mu_2(w_2z_2) + \mu_1(w_1z_1) \vee \mu_2(w_2z_2)$

$\mu_1(w_1z_1) + \mu_2(w_2z_2) - \mu_1(w_1z_1) \wedge \mu_2(w_2z_2) = \mu_1(w_1z_1) \vee \mu_2(w_2z_2)$.

Hence $S(H) = S(G_1) + S(G_2) - \mu_1(w_1z_1) \vee \mu_2(w_2z_2)$

3.2. Isomorphism on Hajós graphs and Hajós Fuzzy Graphs

Theorem 3.2.1: Let G_i* and $G_i^{'}*$be isomorphic graphs and let $h_i : G_i* \to G_i*$ be an isomorphism, i = 1, 2. Then the Hajós graphs $H[G_1^*(u_iu_j, u_i); G_2^*(v_lv_k, v_l)]$ and $H'[G_1^*{}'(h_1(u_i)h_1(u_j), h_1(u_i)); G_2^*{}'(h_2(v_l)h_2(v_k), h_2(v_l))]$ are isomorphic.

Proof:

Adjacency among vertices in $G_i*$and $G_i'*$are preserved under isomorphism h_i. In other words, uv is an edge in G_i*if and only if $h_i(u)h_i(v)$ is an edge in $G_i'*$.

Define $\phi : H \to H'$ by

$\phi(u_i \cdot v_l) = h_1(u_i) \cdot h_2(v_l)$,$\phi(v) = h_1(v)$, if $v \neq u_i, v \in V_2$ and $\phi(v) = h_2(v)$, if $v \neq v_l$,, $v \in V_2$

Then ϕ is a bijection from V(H) to V(H$'$).

To prove adjacency among vertices under ϕ, consider any edge vw in H.

Case 1:$vw \in E_1, v \neq u_i, w \neq u_i$.

Then $\phi(v) = h_1(v)$ and $\phi(w) = h_1(w)$

Since vw is an edge in G_1*, $h_1(v)h_1(w)$ is an edge in $G_1'*$. Therefore, by Hajós construction, $h_1(v)h_1(w)$ is an edge in H'. Hence $\phi(v)\phi(w)$ is an edge in H$'$.

Case 2: $vw \in E_2, v \neq v_l, w \neq v_l$.

As in case 1, $\phi(v)\phi(w)$ is an edge in H$'$.

Case 3: $v \in V_1, w = u_i \cdot v_l$.

Here vu_i is an edge in G_1*. Therefore $h_1(v)h_1(u_i)$ is an edge in $G_1'*$.

Hence $h_1(v)$ and $h_1(u_i) * h_2(v_l)$ are adjacent in H$'$. That is, $\phi(v)\phi(u_i \cdot v_l)$ is an edge in H$'$.

Case 4: $v \in V_2, w = u_i \cdot v_l$.

As in case 3, $\phi(v)\phi(u_i \cdot v_l)$ is an edge in H$'$.

Case 5: $v = u_j, w = v_k$.

In the construction of H$'$, the edges $h_1(u_i)h_1(u_j)$ in $G_1'*$ and $h_2(v_l)h_2(v_k)$ in $G_2'*$are removed and an edge between the vertices $h_1(u_j)$ and $h_2(v_k)$ is newly added. So $h_1(u_j)$ and $h_2(v_k)$ are adjacent in H$'$. Hence $\phi(u_j)$ and $\phi(v_k)$ are adjacent in H$'$.

Theorem 3.2.2:

Let G_i and G_i' be isomorphic fuzzy graphs and let $h_i : G_i \to G_i'$ be an isomorphism, (i = 1, 2). Then the Hajós fuzzy graphs $H[G_1(u_iu_j, u_i); G_2(v_lv_k, v_l)]$ and $H'[G_1'(h_1(u_i)h_1(u_j), h_1(u_i)); G_2'(h_2(v_l)h_2(v_k), h_2(v_l))]$ are isomorphic.

Proof:

Define $\phi : H \to H'$ by

$\phi(u_1 \cdot v_1) = h_1(u_i) \cdot h_2(v_1)$,, $\phi(v) = h_1(v)$, if v $\neq u_i, v \in V_1$ and $\phi(v) = h_2(v)$, if v $\neq v_l, v \in V_2$

From theorem 3.2.1, the Hajós graphs $H[G_1^*(u_iu_j, u_i); G_2^*(v_lv_k, v_l)]$ and $H'[G_1^*{}'(h_1(u_i)h_1(u_j), h_1(u_i)]$;

$G_2^*{}'[h_2(v_l)h_2(v_k), h_2(v_l))]$are isomorphic under ϕ. So we have to prove that the membership values of the vertices and edges in H are preserved under ϕ in H'.

Since h_i is an isomorphism from fuzzy graph G_i to $G_i', i = 1, 2$, the membership values of the vertices and edges in G_i are preserved under h_i in G_i', $i = 1, 2$.

Now $\sigma_H(u_i \cdot v_l) = \sigma_1(u_i) \vee \sigma_2(v_l) = \sigma_1'(h_1(u_i)) \vee \sigma_2'(h_2(v_l)) = \sigma_H'(h_1(u_i) \cdot h_2(v_l)) = \sigma_{H'}(\phi(u_i \cdot v_l))$.

If v is any other vertex of H, then either v is in G_1 or v is in G_2.

If v is in G_1, then $\sigma_H(v) = \sigma_1(v) = \sigma_1'(h_1(v)) = \sigma_{H'}(h_1(v)) = \sigma_{H'}(\phi(v))$.

If v is in G_2, then $\sigma_H(v) = \sigma_2(v) = \sigma_2'(h_2(v)) = \sigma_{H'}(h_2(v)) = \sigma_{H'}(\phi(v))$.

Hence the membership values of the vertices are preserved under ϕ.

Now consider any edge vw in H.

Case 1:$vw \in E_1, v \neq u_i, w \neq u_i$.

$\mu_H(vw) = \mu_1(vw) = \mu_1'(h_1(vw)) = \mu_{H'}(h_1(vw)) = \mu_{H'}(\phi(vw))$.

Case 2: $vw \in E_2, v \neq v_l, w \neq v_l$.

$\mu_H(vw) = \mu_2(vw) = \mu_2'(h_2(vw)) = \mu_{H'}(h_2(vw)) = \mu_{H'}(\phi(vw))$.

Case 3: $v \in V_1, w = u_i \cdot v_l$.

$\mu_H(vw) = \mu_1(vu_i) = \mu_1'(h_1(vu_i)) = \mu_{H'}(h_1(vu_i)) = \mu_{H'}(\phi(vw))$.

Case 4: $v \in V_2, w = u_i \cdot v_l$.

$\mu_H(vw) = \mu_2(vv_l) = \mu_2'(h_2(vv_l)) = \mu_{H'}(h_2(vv_l)) = \mu_{H'}(\phi(vw))$.

Case 5: $v = u_j, w = v_k$.

$\mu_H(u_j v_k) = \mu_1(u_i u_j) \wedge \mu_2(v_l v_k) = \mu_1'(h_1(u_i u_j)) \wedge \mu_2'(h_2(v_l v_k)) = \mu_{H'}(\phi(u_j v_k))$.
Hence the membership values of the edges are preserved under ϕ.
Hence Hajós fuzzy graphs H and H' are isomorphic.

3.3. Cardinality of Hajós Fuzzy Graphs

The following examples illustrate that for different choices of edges and vertices, different Hajós graphs are obtained. Then by considering, Hajós graphs for every choices of vertices and edges as unique, cardinality of Hajós graph and Hajós fuzzy graph are defined. Then the cardinality of Hajós (fuzzy) graphs obtained from various combinations of (fuzzy) graphs are derived.

Example 3.3.1: Here, Figure 2 shows two Hajós graphs, got by identifying u_1 to v_1 and u_2 to v_2 respectively, with the same choice of the edges u_1u_2 in G_1 and v_1v_2 in G_2. Here two Hajós graphs are isomorphic. But the merged vertex $u_1 \cdot v_1$ have degree 1 in H_1 and $u_2 \cdot v_2$ have degree 2 in H_2.

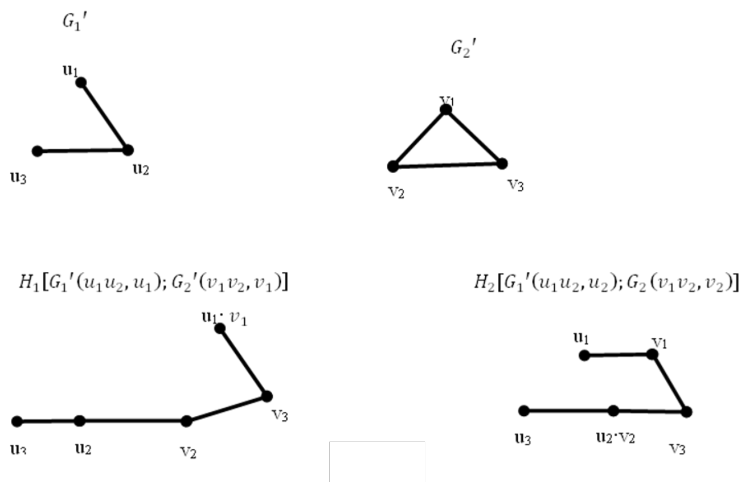

Fig 2. H_1 *and* H_2 **are different Hajós graphs of** $G_1{}'$ *and* $G_2{}'$

Example 3.3.2: The Hajós graph obtained by merging u_1 and v_1 is different from the Hajós graph obtained by merging u_1 and v_2. Though both Hajós graphs are disconnected, they are not isomorphic. So, in general, there is no relationship between the Hajós graphs obtained by identifying different choices of the pairs of vertices.

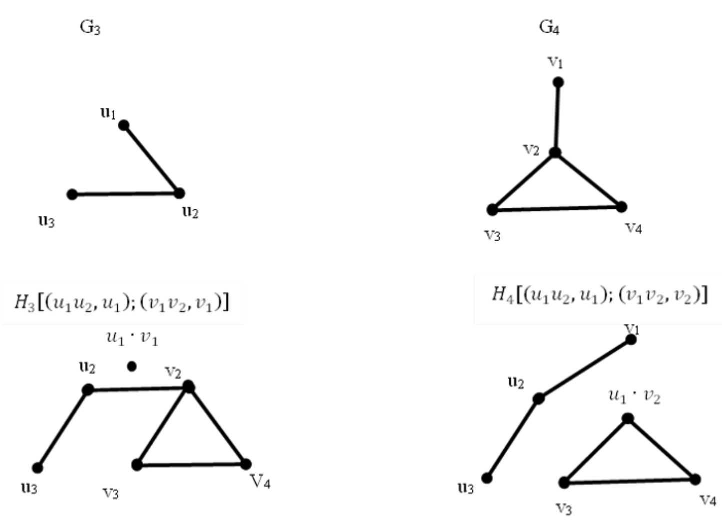

Fig 3. H_3 *and* H_4 **are two different Hajós graphs of** G_3 *and* G_4

Definition 3.3.3: The total number of Hajós graphs that can be constructed from two graphs G_1^* and G_2^* is defined as the cardinality of Hajós graph from G_1^* and G_2^*. It is denoted by $C\left(H\left[G_1^*;G_2^*\right]\right)$ or by $|H[G_1^*;\,G_2^*]|$, where $H\left[G_1^*;G_2^*\right]$ denotes the set of all Hajós graphs from G_1^* and G_2^*.

Definition 3.3.4: The total number of Hajós fuzzy graphs that can be obtained from two fuzzy graphs G_1 and G_2 is defined as the cardinality of Hajós fuzzy graph from G_1 and G_2. It is denoted by $C\left(H\left[G_1;G_2\right]\right)$or by $|H[G_1;\,G_2]|$, where $H\left[G_1;G_2\right]$denotes the set of all Hajós fuzzy graphs from G_1 and G_2.

Theorem 3.3.5:

(i) If $G_i^*$and $G_i^{'}*$are isomorphic graphs, i = 1, 2, then $C\left(H\left[G_1^*;G_2^*\right]\right)=C\left(H\left[G_1'*;G_2'*\right]\right)$.

(ii) If G_i and G_i' are isomorphic fuzzy graphs, i = 1, 2, then $C\left(H\left[G_1;G_2\right]\right)=C\left(H\left[G_1';G_2'\right]\right)$

Theorem 3.3.6:

The number of Hajós graphs that can be constructed from the underlying crisp graphs $G_1^*$and $G_2^*$is the same as the number of Hajós fuzzy graphs that can be constructed from the fuzzy graphs G_1 and G_2.

Theorem 3.3.7:

If $G_j*:\left(V_j,E_j\right)j=1,\ 2$are two cycles with $|V_1|=\ n_{\ 1}$ and $|V_2|=\ n_{\ 2}$, then the number of Hajós graphs that can be obtained from $G_j*, j=1,2$ is $4\,n_1n_2$.

Proof:

Let G_1^*be the cycle $u_1u_2u_3\cdots u_{n_1}u_1$ on $n_{\ 1}$ vertices and G_2^*be the cycle $v_1v_2v_3\cdots v_{n_2}v_1$ on $n_{\ 2}$ vertices.

To obtain the Hajós graph for each choice of the vertex u_i in G_1^*, there are two choices, $u_{i-1}u_i$ and $u_i\,u_{i+1}$ for the edges in G_1^*. Let $u_i$be identified with the vertex $v_j$in G_2^*. There are two choices for the edges, $v_{j-1}v_j$ and v_jv_{j+1}, in G_2^*. Also G_2^* has $n_{\ 2}$ vertices. Therefore, for each vertex u_i of G_1^*, the number of Hajós graphs that can be obtained is $2{\times}2{\times}n_2 = 4n_2$.

Since G_1^* has $n_{\ 1}$ vertices, the total number of Hajós graph is $n_1 \times 4n_2 = 4n_1n_2$.

Theorem 3.3.8:

If$G_j:\left(\sigma_j,\,\mu_j\right)$,, j = 1, 2 are two fuzzy graphs on the cycles $G_j*:\left(V_j,E_j\right)$, j = 1, 2 respectively such that $|V_1|=\ n_{\ 1}$ and $|V_2|=\ n_{\ 2}$, then the number of Hajós fuzzy graph that can be obtained from G_j*, j = 1, 2 is $4n_1n_2$.

Theorem 3.3.9:

If $G_j^*:\left(\sigma_j,\,\mu_j\right)$,, j = 1, 2 are two complete graphs with $|V_1|=\ n_{\ 1}$ and $|V_2|=\ n_{\ 2,}$ then the number of Hajós graphs that can be obtained from $G_j^*j=1,2$is $n_1n_2(n_1-1)(n_2-1)$.

Proof:

Let G_j*, j = 1, 2 be two complete graphs on the n_1 vertices $u_1,u_2,\,u_3,\cdots,\,u_{n_1}$ and on n_2 vertices $v_1,v_2,v_3,\cdots,v_{n_2}$ respectively.

To obtain a Hajós graph, for each choice of the vertex u_i in G_1^*, there are $n-1$ edges, $u_iu_{\ 1},u_iu_{\ 2},\,u_iu_3,\cdots,\,u_iu_{\ n_1}$ to choose.

When $u_i$is identified with the vertex v_j of G_2^*, there are $n_{\ 2}-1$ possible choices for the edges, namely, $v_jv_1,\,v_jv_2,\,v_jv_3,\cdots,\,v_{n_2}$. Also there exist $n_{\ 2}$ vertices in G_2^*. Therefore for a fixed vertex u_i of G_1^*, the number of Hajós graphs is $(n_1-1)(n_2-1)n_{\ 2}$. Since $|V_1|=n_{\ 1,}$ the total number of Hajós graphs is $n_1\times(n_1-1)(n_2-1)n_2{=}n_1n_2(n_1-1)(n_2-1)$.

Theorem 3.3.10:

Let $G_j:\left(\sigma_j,\,\mu_j\right)$, j = 1, 2 be two fuzzy graphs on the complete graphs $G_j*:\left(V_j,E_j\right)$ j = 1, 2 respectively such that $|V_1|=n_1$and $|V_2|=\ n_{\ 2}$. Then the number of Hajós fuzzy graph that can be obtained from $G_j,j\ =\ 1,\ 2$ is $n_1n_2(n_1-1)(n_2-1)$.

Theorem 3.3.11:

If $G_j*:\left(V_j,\,E_j\right)$, is a k_j-regular graph, $j\ =1,\ 2$ with $|V_1|=\ n_{\ 1}$ and $|V_2|=\ n_{\ 2}$ respectively, then the number of Hajós graphs that can be obtained from $G_j*,\ j\ =\ 1,\ 2$ is $k_{\ 1}k_{\ 2}n_{\ 1}n_{\ 2}$.

Proof:

Let G_j*be a k_j -regular graph on $n_{\ j}$vertices $j\ =1,\ 2$ with $u_1,u_2,\,u_3,\cdots,\,u_{n_1}$and $v_1,v_2,v_3,\cdots,v_{n_2}$ respectively.

To obtain a Hajós graph for each choice of the vertex u_j in G_1^* one edge can be chosen from any of the $k_{\ 1}$ edges incident at u_j.

When u_j is identified with the vertex v_i in G_2^*, an edge can be chosen from any of the $k_{\ 2}$ edges incident at $v_i,i\ =\ 1,\ 2,\ \cdots,\ n_{\ 2}$.

Therefore for a fixed vertex u_i of G_1^*, the number of Hajós graphs is $k_{\ 1}k_{\ 2}n_{\ 2}$

Since G_1^*has $n_{\ 1}$ vertices, the total number of Hajós graphs in $n_{\ 1}\times k_{\ 1}k_{\ 2}n_{\ 2}= k_{\ 1}k_{\ 2}n_{\ 1}n_{\ 2}$.

Theorem 3.3.12:

If $G_j:(\sigma_j,\,\mu_j)$, is a fuzzy graph on k_j-regular graph, G_j*: $(V_j,\,E_j)$, j =1, 2 with $|V_1| = n_{\ 1}$ and $|V_2| = n_{2,}$ then the number of Hajós fuzzy graph that can be obtained from G_1 and G_2 is $k_{\ 1}k_{\ 2}n_{\ 1}n_{\ 2}$

Theorem 3.3.13:

If G_j∗:(V_j, E_j), is a cycle for j = 1 and a complete graph for j = 2, with $|V_j| = n_j$, j = 1, 2, then the number of Hajós graphs that can be obtained from G_j∗, j =1, 2 is $2n_1 n_2(n_2-1)$.

Proof:

Let G_j∗be a cycle with $u_1u_2u_3\cdots u_{n_1}u_1$and G_2^*be a complete graph with $v_1, v_2, v_3, \cdots, v_{n_2}$.

To obtain the Hajós Graph for each choice of vertex u_i in G_1^*there are two choices $u_{i-1}u_i$ and u_iu_{i+1} of edges incident at u_i.

When u_i is identified with vertex v_j in G_2^*, there are n_2 – 1 choices of edges incident at $v_j$namely, $v_jv_1, \cdots v_jv_{j-1},\ v_jv_{j+1}, \cdots v_jv_{n_2}$. Also the number of vertices in $G_2^*$is n_2. Therefore for a fixed vertex u_i in $G_1^*$the number of Hajós graphs is $2n_2(n_2-1)$.

Since $|V_1| = n_1$, the total number of Hajós graphs is $n_1 \times 2n_2(n_2-1) = 2n_1n_2(n_2-1)$.

Theorem 3.3.14:

If $G_j : (\sigma_j, \mu_j)$, is a fuzzy graph on G_j∗: (V_j, E_j), where G_1∗:(V$_1$, E$_1$) is a cycle and G_2^*: (V_2, E_2) is a complete graph, with $|V_j| = n_j$, $j = 1, 2$, then the number of Hajós fuzzy graphs that can be obtained from G_j, $j = 1, 2$ is $2n_1n_2(n_2-1)$.

Theorem 3.3.15:

If G_j∗: (V_j, E_j) is a cycle for j = 1 and a k-regular graph for j = 2 with $|V_1| = n_1$ and $|V_2| = n_2$, then the number of Hajós graphs that can be obtained from G_j∗, $j = 1, 2$ is $2kn_1n_2$.

Proof:

Let G_1^*be a cycle $u_1u_2u_3\cdots u_{n_1}u_1$ and G_2^*be a k-regular graph on $n_2$vertices $v_1, v_2, v_3, \cdots, v_{n_2}$.

To obtain the Hajós graph, for each choice of u_i in G_1^*there are two choices, $u_{i-1}u_i$ and u_iu_{i+1} for the edges in G_1^*.

When u_i in G_1^* is identified with the vertex v_j in G_2^*, an edge can be chosen from any of the k edges incident at v_j. Therefore for a fixed vertex u_i in $G_1^*$the number of Hajós graphs is $2kn_2$.

Since $|V_1| = n_1$, the total number of Hajós graphs is $n_1 \times 2kn_2 = 2kn_1n_2$.

Theorem 3.3.16:

If $G_j : (\sigma_j, \mu_j)$, is a fuzzy graph on G_j∗:(V_j, E_j), with $|V_j| = n_j$ for $j = 1,\ 2$, where $G_1^*$is a cycle and $G_2^*$is a k-regular graph, then the number of Hajós fuzzy graph that can be obtained from G_j,$j = 1, 2$ is $2kn_1n_2$.

Theorem 3.3.17:

If G_2^*:(V$_1$, E$_1$) is a cycle with $|V_1| = n_1$ and G_2^*: (V_2, E_2), is a complete bipartite graph K_{mn}∗, then the number of Hajós graphs that can be obtained from G_j^*, $j = 1, 2$ is $4mnn_1$.

Proof:

Let G_1^* be a cycle $u_1u_2u_3\cdots u_{n_1}u_1$.

Let G_2^* be a complete bipartite graph with bipartition $\{v_{21}, v_{22}, v_{23}, \cdots, v_{2m}\}$ and $\{w_{21}, w_{22}, w_{23}, \cdots, w_{2n}\}$.

To obtain the Hajós Graph for each choice of vertex u_i in G_1^*, there are two choices for the edges, $u_{i-1}u_i$ and u_iu_{i+1}, in G_1^*.

Case 1: u_i is identified with the vertex v_{2j} in G_2^* where $v_{2j} \in \{v_{21}, v_{22}, v_{23}, \cdots, v_{2m}\}$

When u_i is identified with the vertex v_{2j} in G_2^*, there are n choices for edges namely, $v_{2j}w_{21}, v_{2j}w_{22}, v_{2j}w_{23}, \cdots, v_{2j}w_{2n} j = 1, 2, \cdots, m$. Therefore, since $|V_{21}| = m$, the number of Hajós graphs that results in this case is mn.

Case 2: $u_i$is identified with the vertex w_{2j} in G_2^*, where $w_{2j} \in \{w_{21}, w_{22}, w_{23}, \cdots, w_{2n}\}$.

When $u_i$is identified with the vertex w_{2j} in G_2^*, there are m choices of edges namely, $w_{2j}v_{21}, w_{2j}v_{22}, w_{2j}v_{23}, \cdots, w_{2j}v_{2m}$, $j = 1, 2, \cdots, n$. Therefore, since $|V_{22}| = n$, the number of Hajós graphs that results in this case is mn.

From the above two cases, the number of Hajós graphs obtained by identifying $u_i$with the vertex v_{2j} or $w_{2j}$is $2mn$.

Therefore for a fixed vertex u_i in G_1^*, as two edges are incident at u_i, the number of corresponding Hajós graphs is $2 \times 2mn = 4mn$.

Since G_j^* has n_1 vertices, the total number of Hajós graphs is $n_1 \times 4mn = 4mnn_1$.

Theorem 3.3.18:

If $G_j : (\sigma_j, \mu_j)$, is a fuzzy graph on G_j^*:(V_j, E_j), with | $|V_j|$= n_j for $j = 1,\ 2$, where $G_1^*$is a cycle and G_2^* is a complete bipartite graph K_{mn}∗,then the number of Hajós fuzzy graph that can be obtained from G$_j$, j = 1, 2 is $4mnn_1$.

Theorem 3.3.19:

If G_1^*: $(V_1, E_1)$is a cycle with $|V_1| = n_1$ and G_2^*: (V_2, E_2), is a path P_{n_2}, then the number of Hajós graphs that can be obtained from G_j^*, $i = 1, 2$ is $4n_1(n_2-1)$.

Proof:

Let G_1^*be the cycle $u_1u_2u_3\cdots u_{n_1}u_1$and G_2^* be the path P_{n_2}on n$_2$ vertices$v_1v_2v_3\cdots v_{n_2}$.

To obtain the Hajós graph for each choice of vertex u$_i$ in G_1^*, there are two choices $u_{i-1}u_i$ and u_iu_{i+1} for the edges in G_1^*.

When u_i is identified with the vertex v_j in G_2^*, the possible cases for v_i in G_2^* are as follows:

Case 1: u_i is identified with the end vertex v_j in G_2^*.

Since v_j in G_2^* is any one of the end vertices of the path P_{n_2}, there is only one edge which is incident at v_j. Since there are two end vertices, 2 Hajós graphs are obtained.

Case 2:u_i is identified with the internal vertex v_j in G_2^*

If v_j in G_2^* is any one of the vertices, there are two choices, $v_{j-1}v_j$ and v_jv_{j+1}, for the edges in G_1^*. Since there are (n_2-2) vertices other than the end vertices, there are $2(n_2-2)$ choices to consider. So the number of Hajós graphs in this case is $2(n_2-2)$.

From the above two cases, for a fixed vertex u_i in G_1^*, the number of Hajós graphs is $2 \times [2 + 2(n_2-2)] = 4(n_2-1)$. Since G_1^* has n_1 vertices, the total number of Hajós graphs is $4n_1(n_2-1)$.

Theorem 3.3.20:

If $G_j : (\sigma_j, \mu_j)$, is a fuzzy graph on G_j^*:(V_j, E_j), with | $|V_j| = n_j$ for $j = 1, 2$, where $G_1^*$is a cycle and $G_2^*$is a path, then the number of Hajós fuzzy graph that can be obtained from G_j, $j = 1, 2$ is $4n_1(n_2-1)$.

Theorem 3.3.21:

If G_1^*: (V_1, E_1) is a complete graph with $|V_1| = n_1$ and G_2^*: (V_2, E_2) is a path P_{n_2}, then the number of Hajós graphs that can be obtained from G_j^*, $j= 1, 2$ is $2n_1(n_1-1)(n_2-1)$.

Proof:

Let G_2^*be a complete graph $u_1u_2u_3\cdots u_{n_1}$ and G_2^*be a path P_{n_2} on $n_2$vertices $v_1, v_2, v_3, \cdots, v_{n_2}$.

To obtain a Hajós graph, for each choice of the vertex u_i in G_1^*, there are $n-1$ edges, $u_iu_1, \cdots u_iu_{i-1}, u_iu_{i+1}, \cdots, u_iu_{n_1}$, to choose.

Suppose that u_i is identified with the vertex v_j in G_2^*.

Case 1: v_j in $G_2^*$is any one of the end vertices of the path P_{n_2}.

Then there is only one choice for edge: if v_j is v_1, it is v_1v_2 and if v_j is v_{n_1}, it is v_{n_1-1}, v_{n_1}. Since there are two end vertices there are 2 choices. Therefore the number of Hajós graph is 2.

Case 2: v_j in G_2^* is any one of the vertices other than the end vertices.

There are two choices $v_{j-1}v_j$ and v_jv_{j+1} for the edges in G_2^*. Since there are (n_2-2) vertices other than the end vertices, there exist $2(n_2-2)$ choices for edges. Therefore the number of Hajós graph is $2(n_2-2)$.

From the above two cases, for a fixed vertex u_i in G_1^*, the number of Hajós graphs is $(n_1-1) \times [2+2(n_2-2)] = 2(n_2-1)(n_1-1)$

Since G_1^*has n_1 vertices, the total number of Hajós graphs is $2n_1(n_1-1)(n_2-1)$.

Theorem 3.3.22:

If $G_j : (\sigma_j, \mu_j)$, is a fuzzy graph on G_j^*:(V_j, E_j), with $||V_j| = n_j$ for $j = 1, 2$, where G_1^* is a complete graph and G_2^* is a path, then the number of Hajós fuzzy graph that can be obtained from G_j, j = 1, 2 is $2n_1(n_1-1)(n_2-1)$.

4 Conclusion

Using Hajós construction on graphs, a contribution towards binary operations on fuzzy graphs is made. Also some contributions towards the study on properties of Hajós (fuzzy) graphs have been made in this paper. Isomorphic properties on Hajós (fuzzy) graphs are derived. Examples are provided to demonstrate how different vertex and edge selections result in different Hajós graphs. Based on this fact, the notion of cardinality is introduced. The cardinalities of Hajós (fuzzy) graphs, which are obtained from two (fuzzy) graphs, which include cycle, path, regular graph, complete graph and complete bipartite graph, are derived. The cardinality of Hajós (fuzzy) graphs for other combinations of (fuzzy) graphs can be developed in the future.

References

1) Yamuna M, Karthika K. A note on hajos stable graphs. *IOP Conference Series Materials Science and Engineering*. 2017;263(4):42118–42118. Available from: https://doi.org/10.1088/1757-899X/263/4/042118.
2) Braun B, Vega J. Hajós-Type Constructions and Neighborhood Complexes. *SIAM Journal on Discrete Mathematics*. 2020;34(4). Available from: https://doi.org/10.1137/19M1243476.
3) Altamirano JCG, Olsen M, Ojeda JC. How to construct the symmetric cycle of length 5 using Hajós construction with an adapted Rank Genetic Algorithm. *Discrete Mathematics and Theoretical Computer Science*. 2023;25:1–3. Available from: https://dmtcs.episciences.org/10997/pdf.
4) Ojeda JC, Altamirano JCG, Olsen M. Computacional complexity of Hajós constructions of symmetric odd cycles. *Procedia Computer Science*. 2023;223. Available from: https://doi.org/10.1016/j.procs.2023.08.220.
5) Gutin G, Kostochka AV, Toft B. On the Hajós number of graphs. *Discrete Mathematics*. 2000;213. Available from: https://doi.org/10.1016/S0012-365X(99)00175-2.
6) Jensen JB, Bellitto T, Schweser T, Stiebitz M. Hajós and Ore constructions for digraphs. *The Electronic Journal of Combinatorics*. 2020;27(1). Available from: https://www.combinatorics.org/ojs/index.php/eljc/article/download/v27i1p63/pdf.

7) Elumalai A. Recent Trends of Fuzzy Graphs. *Malaya Journal Of Mathematik*;2020:1618–1621. Available from: https://doi.org/10.26637/MJM0S20/0431.
8) Esakkiammal E, Devi PC. Triple Even Star Decomposition of Complete Bipartite Graphs. *International Journal on Recent and Innovation Trends in Computing and Communication*. 2023;11(11):479–483. Available from: https://dx.doi.org/10.17762/ijritcc.v11i11.9917.
9) Surega S, Sangeetha MR. A Study on Graph Theory of Path Graphs. *International Journal on Future Revolution in Computer Science & Communication Engineering*. 2019;p. 16–18. Available from: https://core.ac.uk/download/539895792.pdf.
10) Lewis R, Corcoran P, Gagarin A. Methods for determining cycles of a specific length in undirected graphs with edge weights. *Journal of Combinatorial Optimization*. 2023;46:29–29. Available from: https://doi.org/10.1007/s10878-023-01091-w.
11) Samanta S, Sarkar B. Isomorphism on generalized fuzzy graphs and image visualizations. *Soft Computing*. 2020;24:14401–14409. Available from: https://doi.org/10.1007/s00500-020-05260-5.