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Existence and Uniqueness of Solution of Fractional Differential Equation Using Non-local Operator

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Abstract

Objectives: To investigate the existence and uniqueness of a solution to the non-local Cauchy problem of order $0 < \alpha < 1$ using the Atangana Baleanu (AB) fractional derivative operator. **Methods:** Using Schauder's fixed point theorem and the Arzela-Ascoli theorem, the study proved the existence of a solution to the given problem. Further, it obtains results for the uniqueness of the solution. **Findings:** This study proves the existence of a solution to the non-local Cauchy problem under the given conditions. Results are also provided for the uniqueness of the solution. **Novelty :** A novel approach to fractional differential equations is represented by applying Atangana-Baleanu fractional derivative operator to the non-local Cauchy problem. Schauder's fixed point theorem and Arzela-Ascoli's theorem, are used to show existence and uniqueness. Further, one detailed example has been solved.

Keywords: Existence; Uniqueness; Non-local operator; Schauder fixed point theorem; Arzela-Ascoli theorem; Cauchy problem

1 Introduction

The generalization of classical calculus that deals with operations of integration and differentiation of an arbitrary order is called Fractional Calculus (FC). The fractional calculus applications are successful due to new fractional-order models which are often more accurate than integer-order ones. During the past decades, fractional calculus had a remarkable development as shown by many researchers in [1,2]. In literature, different definitions of fractional derivative and integral exist. Determining the existence and uniqueness of a differential equation's solutions is crucial in analyzing any differential equation, including FDEs. These characteristics are essential because they guarantee that the model that the FDE describes is well-posed, which means that it has a unique solution that depends constantly on the initial conditions. Many mathematicians study the existence and uniqueness of solutions of fractional differential equations by applying different operators. Several results for the existence and uniqueness of generalized solutions for the problem of nonlinear quantitative response equations have been expanded by delay [3] using the Leray Schauder fixed point theorem and the Banach contraction principle. A. Shaikh, et.al. in [4] investigate the Dynamical behaviour of

HIV/AIDS model using Fractional Derivative with Mittag-Leffler Kernel. A. Shaikh et al.[5] proposed a mathematical model of COVID-19 using fractional derivative. They investigated the effect of several biological parameters on the dynamics of COVID-19 transmission. Recently in 2023 Kinda Abuasbeh et al. in[6], investigated a Mathematical Modelling of COVID-19 by Using a Mild Solution with a Delay Caputo Operator. The fractional order mathematical model of diabetes and its resulting complications was studied by Srivastava et al. in[7]. They also proposed some new approximate solutions of the time-fractional Nagumo equation involving fractional integrals without singular kernel, looked into the properties of spiral-like close-to-convex functions associated with conic domains, and used the Jacobi collocation method to approximate the solution of some fractional-order Riccati differential equations with variable coefficients, respectively. The Existence and Uniqueness of Positive Solutions for the Fractional Differential Equation Involving the $\rho(\tau)$-Laplacian Operator and Nonlocal Integral Condition have been established by Borisut, P. and Supak, P.[8]. Recently M. Manjula et al.[9] have studied the existence, uniqueness, and approximation of the nonlocal fractional differential equation of Sobolev type with impulses. On the other hand, fractional differential equations with boundary conditions have been investigated in a wide range of references for instance, see[10,11]

In[12], existence and uniqueness results for the solutions of the following differential equation of order $0 < \alpha < 1$, is obtained by Z. Mokhtary, M.B Ghaemi, and S. Salahshour.

$$^{CF}D^{\alpha}x(t)= f\,(t,\, x(t)),\; t\in[0,\,1]\;,$$

$$x\,(0) = \int_0^1 g\,(s)\,x\,(s)\,ds,$$

Where $^{CF}D^{\alpha}$ is the Caputo-Fabrizio derivative operator of order $0 < \alpha < 1,\;\; g\in L^1\left([0,1]\,,R_+\right),\; g\,(t)\in[0,1)$and f is E-valued function.

Many more researchers in[13–18] investigated the existence and uniqueness of solutions for various types of fractional differential equation using local and non-local derivative operators.

Motivated and persuaded by the work specified above, we study the existence and uniqueness of solutions to the nonlocal Cauchy problem for the following fractional differential equations using the Atangana Baleanu derivative operator in Banach space E:

$$^{AB}D^{w}y(t)= F\,(t,\, y(t)),\; 0\leq t\leq 1, \tag{1}$$

$$y\,(0) = \int_0^1 h\,(p)\,y\,(p)\,dp,$$

Where $^{AB}D^{w}$ is the Atangana Baleanu derivative operator of order $0 < w < 1,\; h\in L^1\left([0,1]\,,R_+\right),\; h\,(t)\in[0,1)$, and F is E-valued function.

2 Methodology

This section introduces several definitions, notations, and fractional calculus results that will be used to get the desired result.

Define E as Banach space. Let $C([0,\,1],\,E)$ as Banach space of all of the continuous functions where $y:[0,1]\to E$, and $||y||_c = Sup_{t\in[0,1]}\,||y\,(t)||$,are defined on $C\left([0,\,1],\,E\right)$. $L^1([0,\,1],\,E)$ is set as the Banach space of measurable function $y\,:\,[0,\,1]\to\,E$, as integral and is equipped with

$\parallel y\parallel_{L^1} = \int_0^1 \parallel y(p)\parallel dp.$

Definition 2.1. (Equicontinuous)

$F\subset C(X)$ is equicontinuous if for every $\epsilon > 0$, $\exists\,\delta > 0$ (which depends only on ϵ) such that for $x,\,y\in X$, $d\,(x-y) < \delta \Rightarrow |f\,(x) - f\,(y)| < \epsilon$, $\forall f\in F$, where d is metric on X.

Definition 2.2. (Atangana Baleanu Fractional Derivative Operator)

If $f\,(x)\in C^2[a,\,b]$ and $a < x < b$ then Atangana Baleanu Fractional Derivative is denoted by $^{AB}D^{\alpha}f\,(x)$ and is defined as follows

$$^{AB}D^{\alpha}y\,(t) = \frac{M(\alpha)}{1-\alpha}\int_0^x E_{\alpha}\left(\frac{-\alpha\,(x-t)}{1-\alpha}\right)f'\,(t)\,dt.$$

Definition 2.3. (Atangana Baleanu Fractional Integral Operator)

If $f\,(x)\,\in\,C^2[a,\,b]$ and $a\,<\,x\,<\,b$ then Atangana Baleanu Fractional integral is denoted by ${}^{AB}I^{\alpha}f\,(x)$ and is defined as follows

$$ {}^{AB}I^{\alpha}f\ (x) = \frac{1-\alpha}{M\,(\alpha)}f\,(x) + \frac{\alpha}{M\,(\alpha)\,\Gamma\alpha}\int_0^x f\,(s)\,(x-s)^{\alpha-1}ds. $$

Theorem 2.1. (Arzela Ascoli Theorem)

Let $(X,\,d)$ be a compact space. A subset F of $C(X)$ is relatively compact if and only if F is uniformly bounded and equicontinuous.

Theorem 2.2. (Schauder's Fixed Point Theorem)

Let $(E,\,d)$ be a complete metric space, let U be a closed convex subset of E, and Let $A:\,U\,\rightarrow\,U$ be a mapping such that the set $\{Au:\;u\,\in\,U\,\}$ is relatively compact in E. Then A has at least one fixed point.

Lemma 2.1. The solution of the fractional differential Equation (1) with boundary condition y(0) = c is given below

$$ y\,(t) = y\,(0) + \,(1\,-\,w)\,F\,\left(t,\,y\,(t)\right) + \,w\int_0^t (t-p)^{w-1}F\,(p,y\,(p))\,dp,\; w\in(0,1)\,. $$

Lemma 2.2. [15]

$$ \frac{G(\tau)}{\Gamma w} < e,\; \frac{\int_0^t (t-p)^{w-1}\,dp}{\Gamma w} < e, \tag{2} $$

where $G\,(\tau) = \int_\tau^1 h(p)(p-\tau)^{w-1}dp,\; p,\tau\in[0,1]\,.$

Proof: Let us consider

$$ \frac{G(\tau)}{\Gamma w} = \frac{\int_\tau^1 h(p)(p-\tau)^{w-1}dp}{\int_0^\infty p^{w-1}\,e^{-p}dp} $$

$$ \leq \frac{\int_\tau^1 (p-\tau)^{w-1}dp}{\int_0^\infty p^{w-1}\,e^{-p}dp} $$

$$ = \frac{\int_0^{1-\tau} p^{w-1}dp}{\int_0^\infty p^{w-1}\,e^{-p}dp} $$

$$ \leq e\frac{\int_0^{1-\tau} p^{w-1}e^{-p}dp}{\int_0^\infty p^{w-1}\,e^{-p}dp} $$

$$ < e $$

and

$$ \frac{\int_0^t (t-p)^{w-1}dp}{\Gamma w} = \frac{\int_0^t p^{w-1}dp}{\int_0^\infty p^{w-1}\,e^{-p}dp} \leq e\frac{\int_0^t p^{w-1}e^{-p}dp}{\int_0^\infty p^{w-1}\,e^{-p}dp} < e $$

3 Results and Discussion

Assume the following conditions hold:

(A_1) Let $F \in C([0,\,1],\,E)$ such that there is a constant $M > 0, P_F(t) \leq M \, for \, t \in [0,\,1]$ and every $y \in E$, there is $P_F(t) \in L^1([0,1], R_+)$ such that $\| F(t,\,y) \| \leq P_F(t) \| y \|$.

(A_2) for each $t \in [0,\,1]$ and $R > 0, F(t, B_R) = \{F(t,y) | y \in B_R\}$is relatively compact in E where $B_R = \{y \in C([0,\,1],\,E), \| y \|_C \leq R\}$, $\lambda_1 = \frac{2-\mu_1}{1-\mu_1} M(e+1) < 1$ and $\mu_1 = \int_0^1 h(p)\,dp$.

Lemma 3.1. Assume that the condition A_1hold then the problem 1 (Equation (1)) is equivalent to equation

$$y(t) = \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,F(p,y(p))\,dp + \frac{w}{1-\mu_1}\int_0^1 G(\eta)\,F(\eta,y(\eta))\,d\eta + (1-w)\,F(t,y(t))$$

$$+w\int_0^t (t-p)^{w-1} F(p,y(p))\,dp.$$

Proof. Using lemma (2.1) we have

$$y(t) = y(0) + (1-w)\,F(t,y(t)) + w\int_0^t (t-p)^{w-1} F(p,y(p))\,dp \tag{3}$$

Then we find

$$y(0) = \int_0^1 h(p)\,y(p)dp$$

$$y(0) = \int_0^1 h(p)\left(y(0) + (1-w)\,F(p,y(p)) + w\int_0^p (p-\eta)^{w-1} F(\eta,y(\eta))\,d\eta\right)dp$$

$$= \int_0^1 h(p)\,y(0)\,dp$$
$$+(1-w)\int_0^1 h(p)\,F(p,y(p))\,dp + w\int_0^1 h(p)\int_0^p (p-\eta)^{w-1} F(\eta,y(\eta))\,d\eta dp$$

$$y(0)\left(1-\int_0^1 h(p)\,dp\right)$$
$$= (1-w)\int_0^1 h(p)\,F(p,y(p))\,dp + w\int_0^1 h(p)\left(\int_0^p (p-\eta)^{w-1} F(\eta,y(\eta))\,d\eta\right)dp$$

Here $0 \leq \eta \leq p \leq 1$ by change of order of integration we get $\eta \leq p \leq 1,\; 0 \leq \eta \leq 1.$

$$\therefore \int_0^1 h(p)\left(\int_0^p (p-\eta)^{w-1} F(\eta,y(\eta))\,d\eta\right)dp = \int_0^1 F(\eta,y(\eta))\left(\int_\eta^1 h(p)(p-\eta)^{w-1}dp\right)d\eta$$

Let

$$G(\eta) = \int_\eta^1 h(p)(p-\eta)^{w-1}dp$$

$$y(0)\left(1-\int_0^1 h(p)\,dp\right) = (1-w)\int_0^1 h(p)\,F(p,y(p))\,dp + w\int_0^1 G(\eta)\,F(\eta,y(\eta))d\eta$$

$$y(0) = \frac{(1-w)}{(1-\mu_1)}\int_0^1 h(p)\,F(p,y(p))\,dp + \frac{w}{(1-\mu_1)}\int_0^1 G(\eta)\,F(\eta,y(\eta))d\eta$$

From Equation (3) we get

$$y(t) = \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,F(p,y(p))\,dp + \frac{w}{1-\mu_1}\int_0^1 G(\eta)\,F(\eta,y(\eta))\,d\eta + (1-w)\,F(t,y(t))$$

$$+w\int_0^t (t-p)^{w-1} F(p,y(p))\,dp$$

Conversely, consider the solution of the problem 1 (Equation (1)) is y(t), then using the definition of Atangana Baleanu fractional derivative, for each $t \in [0,\ 1]$, we get

$${}^{AB}D^w y(t) = {}^{AB}D^w \begin{pmatrix} \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,F(p,y(p))\,dp + \frac{w}{1-\mu_1}\int_0^1 G(\eta)\,F(\eta,y(\eta))\,d\eta \\ +(1-w)\,F(t,y(t)) + w\int_0^t (t-p)^{w-1} F(p,y(p))\,dp \end{pmatrix}$$

$$= {}^{AB}D^w \left((1-w)\,F(t,y(t)) + w\int_0^t (t-p)^{w-1} F(p,y(p))\,dp\right)$$

$$= {}^{AB}D^w \left({}^{AB}I^w F(t,y(t))\right)$$

$${}^{AB}D^w y(t) = F(t,y(t))$$

This completes the proof.

Theorem 3.1. Assume that both A_1 and A_2 are true and function satisfied Lipschitz condition, then problem 1 (Equation (1)) has at least one solution.

Proof. Consider the operator

$A : C([0,\ 1],\ E) \to C([0,\ 1],\ E)$,

given below

$$Ay(t) = \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,F(p,y(p))\,dp + \frac{w}{1-\mu_1}\int_0^1 G(\eta)\,F(\eta,y(\eta))\,d\eta +$$

$$(1-w)\,F(t,y(t)) + w\int_0^t (t-p)^{w-1} F(p,y(p))\,dp,$$

The fixed point of operator A is a solution of problem 1 (Equation (1)).

Now consider $B_R = \{y \in C([0,\ 1],\ E),\ \| y \|_C \le R\}$.

Clearly, B_R is convex, closed, and bounded.

The above theorem is proved in the several steps shown below:

Step 1: Initially, we show the continuity of operator A.

Take

$y_n, \bar{y} \in C([0,\ 1],\ E),\ \| y_n - \bar{y} \|_C \to 0$, Then we have $r = sup_i \| y_i \|_C < \infty,\ \| y \|_C \le r$,

For each $t \in [0,\ 1]$

$$||Ay_n - A\overline{y}|| = \left\| \begin{matrix} \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,F(p,y_n(p))\,dp + \frac{w}{1-\mu_1}\int_0^1 G(\eta)\,F(\eta,y_n(\eta))\,d\eta \\ +(1-w)\,F(t,y_n(t)) + w\int_0^t (t-p)^{w-1} F(p,y_n(p))\,dp - \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,F(p,\overline{y}(p))\,dp \\ -\frac{w}{1-\mu_1}\int_0^1 G(\eta)\,F(\eta,\overline{y}(\eta))\,d\eta - (1-w)\,F(t,\overline{y}(t)) - w\int_0^1 (t-p)^{w-1} F(p,\overline{y}(p))\,dp \end{matrix} \right\|$$

$$\|Ay_n - A\overline{y}\| \le \frac{1-w}{1-\mu_1}\int_0^1 h(p)\,\|F(p,y_n(p)) - F(p,\overline{y}(p))\|\,dp$$

$+\frac{w}{1-\mu_1}\int_0^1 G(\eta)\|F(\eta,y_n(\eta))-F(\eta,\overline{y}(\eta))\|d\eta$
$+(1-w)\|F(t,y_n(t))-F(t,\overline{y}(t))\|+$
$w\int_0^t (t-p)^{w-1}\|F(p,y_n(p))-F(p,\overline{y}(p))\|dp$

By Equation (2) we get

$$\|Ay_n-A\overline{y}\|\leq \frac{1-w}{1-\mu_1}\int_0^1 h(p)\|F(p,y_n(p))-F(p,\overline{y}(p))\|dp$$
$$+\frac{ew\Gamma w}{1-\mu_1}\int_0^1 \|F(\eta,y_n(\eta))-F(\eta,\overline{y}(\eta))\|d\eta$$
$$+(1-w)\|F(t,y_n(t))-F(t,\overline{y}(t))\|+w\int_0^1 (t-p)^{w-1}\|F(p,y_n(p))-F(p,\overline{y}(p))\|dp$$

As $n\to\infty,\ F(t,y_n(t))\to F(t,\overline{y}(t)),\ \|Ay_n-A\overline{y}\|\to 0\ as\ n\to\infty$.

Hence A is a continuous operator.

$\|F(t,y_n(t))-F(t,\overline{y}(t))\|\leq P_F(t)(\|y_n\|+\|y\|)\leq 2Mr$

Step 2: Claim $A(B_R)$ is equicontinuous.

Let $t_1,\ t_2\in[0,1],t_1<t_2$, and $y\in B_R$, then

$$\|Ay(t_2)-Ay(t_1)\|=\left\|\begin{matrix}\frac{1-w}{1-\mu_1}\int_0^1 h(p)F(p,y(p))dp+\frac{w}{1-\mu_1}\int_0^1 G(\eta)F(\eta,y(\eta))d\eta\\ +(1-w)F(t_2,y(t_2))\\ +w\int_0^{t_2}(t_2-p)^{w-1}F(p,y(p))dp-\frac{1-w}{1-\mu_1}\int_0^1 h(p)F(p,y(p))dp\\ -\frac{w}{1-\mu_1}\int_0^1 G(\eta)F(\eta,y(\eta))d\eta-(1-w)F(t_1,y(t_1))\\ -w\int_0^{t_1}(t_1-p)^{w-1}F(p,y(p))dp\end{matrix}\right\|$$

$\leq(1-w)\|F(t_2,y(t_2))-F(t_1,y(t_1))\|$
$+w\left[\int_0^{t_2}(t_2-p)^{w-1}-\int_0^{t_2}(t_1-p)^{w-1}\right]\|F(p,y(p))\|dp$
$\leq(1-w)\|Ft_2,y(t_2)-F(t_1,y(t_1))\|$
$+w\left[\int_0^{t_1}(t_2-p)^{w-1}+\int_{t_1}^{t_2}(t_2-p)^{w-1}-\int_0^{t_1}(t_1-p)^{w-1}\right]\|F(p,y(p))\|dp$
$\leq(1-w)\|F(t_2,y(t_2))-F(t_1,y(t_1))\|+$
$w\int_0^{t_1}\left[(t_2-p)^{w-1}-(t_1-p)^{w-1}\right]\|F(p,y(p))\|dp+$
$w\int_{t_1}^{t_2}(t_2-p)^{w-1}\|F(p,y(p))\|dp$
$\because\|F(p,y(p))\|\leq MR$
$\leq(1-w)\|F(t_2,y(t_2))-F(t_1,y(t_1))\|+wMR\int_0^{t_1}\left[(t_2-p)^{w-1}-(t_1-p)^{w-1}\right]dp$
$+wMR\int_{t_1}^{t_2}(t_2-p)^{w-1}dp$
$\leq(1-w)\|F(t_2,y(t_2))-F(t_1,y(t_1))\|+MR[t_2^w-t_1^w]$

As $t_1\to t_2$ R.H.S of the above inequality is zero. Hence $A(B_R)$ is equicontinuous.

Step 3:

$$\|Ay(t)\|=\left\|\begin{matrix}\frac{1-w}{1-\mu_1}\int_0^1 h(p)F(p,y(p))dp\\ +\frac{w}{1-\mu_1}\int_0^1 G(\eta)F(\eta,y(\eta))d\eta+(1-w)F(t,y(t))\\ +w\int_0^t(t-p)^{w-1}F(p,y(p))dp\end{matrix}\right\|$$

$\leq\frac{1-w}{1-\mu_1}\int_0^1 h(p)\|F(p,y(p))\|dp+\frac{w}{1-\mu_1}\int_0^1 G(\eta)\|F(\eta,y(\eta))\|d\eta$
$+(1-w)\|F(t,y(t))\|+w\int_0^t(t-p)^{w-1}\|F(p,y(p))\|dp$
$\because\|F(t,y(t))\|\leq P_F(t)\|y\|_c\leq M\|y\|_c$
$\leq\frac{2-\mu_1}{1-\mu_1}M(1+e)\|y\|_c$

Choose $\lambda_1=\frac{2-\mu_1}{1-\mu_1}M(1+e)<1$

$\|Ay(t)\|\leq\|y\|_c\leq R$

Hence uniformly bounded condition is proven. Therefore it is relatively compact. By Schauder's fixed point theorem, there exist a fixed point of operator A in B_R.

Theorem 3.2. Under the assumptions of the above theorem and$\|F(t,y)-F(t,z)\|\leq L\|y-z\|$,,where $y,z\in B_R,\ 0<L<\frac{1-\mu_1}{(2-\mu_1)(e+1)}$ then $y(t)$ is the solution of problem (1). Moreover, it is unique in B_R.

Proof. We have to prove that, there exists a unique solution to IVP (1).

Consider $y_1(t),y_2(t)\in B_R$ be the two solutions of IVP (1), then we have

$$\|y_1(t)-y_2(t)\| = \left\| \begin{array}{c} \frac{1-w}{1-\mu_1}\int_0^1 h(p)F(p,y_1(p))\,dp + \frac{w}{1-\mu_1}\int_0^1 G(\eta)F(\eta,y_1(\eta))\,d\eta \\ +(1+w)F(t,y_1(t)) + w\int_0^t (t-p)^{w-1}F(p,y_1(p))\,dp \\ -\frac{1-w}{1-\mu_1}\int_0^1 h(p)F(p,y_2(p))\,dp - \frac{w}{1-\mu_1}\int_0^1 G(\eta)F(\eta,y_2(\eta))\,d\eta - 1(1-w)F(t,y_2(t)) \\ -w\int_0^t (t-p)^{w-1}F(p,y_2(p))\,dp \end{array} \right\|$$

$\leq \frac{1-w}{1-\mu_1}\int_0^1 h(p)\|F(p,y_1(p)) - F(p,y_2(p))\|\,dp$

$+\frac{w}{1-\mu_1}\int_0^1 G(\eta)\|F(\eta,y_1(\eta)) - F(\eta,y_2(\eta))\|\,d\eta + (1-w)\|F(t,y_1(t)) - F(t,y_2(t))\|$

$+w\int_0^t (t-p)^{w-1}\|F(p,y_1(p)) - F(p,y_2(p))\|\,dp$

$\leq \left[\frac{1-w}{1-\mu_1}L + \frac{ew\Gamma wL}{1-\mu_1} + (1-w)L + wL\right]\|y_1(p)-y_2(p)\|_c$

$\leq \left[\frac{L(2-\mu_1)(e+1)}{1-\mu_1}\right]\|y_1(p)-y_2(p)\|_c$

Hence we get

$\|y_1-y_2\| \leq \left[\frac{L(2-\mu_1)(e+1)}{1-\mu_1}\right]\|y_1-y_2\|_c$

$\because \|y_1-y_2\| = 0$

Hence proved.

Illustration

Example 4.1. Consider the following differential equation of order $0 < w < 1$ on $E = [0,\,1]$:

$$^{AB}D^{w}y(t) = \frac{1+t}{100}y(t),\ t\in[0,1],\ y(0) = \int_0^1 \frac{1}{2}y(p)\,dp, \tag{5}$$

Then the solution to the equation above in [0, 1] is unique.

Solution 4.1. Let

$$F(t,y) = \frac{1+t}{100}y(t),\ h(p) = \frac{1}{2}$$

Clearly $F \in C([0,\,1],\,E),P_F(t) \leq \frac{1}{50} = M$

$P_F \in L([0,1],R^+,\|F(t,y)\| \leq P_F\|y\|)$

Hence assumption (A_1) holds. Now we show that the assumption (A_2) also holds.

Take $\lambda_1 = \frac{2-\mu_1}{1-\mu_1}M(e+1),\ \mu_1 = \int_0^1 h(p)\,dp$

So $\lambda_1 \leq \frac{3}{50}(e+1) < 1$.

Hence, assumption (A_2) holds. It follows from Theorem 3.1, that there exists at least one solution to the problem 5 (Equation (5)).

For uniqueness, we will apply theorem 3.2.

In fact,

$$\|F(t,y)-F(t,z)\| = \left\|\frac{1+t}{100}y - \frac{1+t}{100}z\right\| \leq \frac{1}{50}\|y-z\|$$

So,

$$\lambda_2 = \frac{1-\mu_1}{(2-\mu_1)(e+1)} = \frac{1}{3(e+1)}$$

Hence:

$$L = \frac{1}{50} < \frac{1}{3(e+1)}$$

Using theorem 3.2, the Equation (5) has a unique solution in the interval [0, 1].

4 Conclusion

First, this study has introduced several key terms and theorems that will help to understand the rest of the research. Under certain assumptions, this study provides new results to determine the existence and uniqueness of the solution of the fractional differential Equation (1) by applying the Atangana Baleanu fractional derivative operator. This study has applied two well-known theorems: the Schauder fixed point theorem and the Arzela-Ascoli theorem to obtain the existence of a solution to Equation (1). Further, it proved the obtained solution is unique. Additionally, it has provided one example to show the applicability of the obtained result.

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