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Equitable Color Class Domination Number of Honeycomb Networks Graph and Inverse Equitable Color Class Domination Number of a Graph

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Abstract

Objectives: In this study, we find a new graph domination parameter named "Inverse Equitable Color Class Domination" and discuss the equitable color class domination number of honeycomb networks graph. **Methods:** Let $G = (V, E)$ be a connected graph. Assume a group CC containing colors. Let $\tau : V(G) \rightarrow CC$ be an equitably colorable function. A dominating subset D of V is called an equitable color class dominating set if the number of dominating nodes in each color class is equal. The least possible cardinality of an equitable color class dominating set of G is the equitable color class domination number itself. It is indicated by $\gamma_{ECC}(G)$. If there exists another equitable color class dominating set $D^{'}$ in $V-D$ is called an inverse equitable color class dominating set of G corresponding to D. It is indicated by $\gamma_{ECC}^{-1}(G)$. **Findings:** In this paper we conclude the equitable color class domination number of honeycomb networks, honeycomb cup networks graphs and the inverse equitable color class domination number of some graphs. **Novelty:** This study introduces the concept of "Inverse Equitable Color Class Domination in Graphs". It obtains many bounds in terms of nodes, links, and other different parameters.

2020 Mathematics Subject Classification: 05C15, 05C69

Keywords: Dominating Set; Equitable Coloring; Color Class (CC); Equitable Color Class (ECC); Equitable Color Class Dominating Set; Inverse Equitable Color Class Domination Number

1 Introduction

The improved areas in Graph Theory are Colourings and Domination. In particular, Domination Theory in graphs deserves good attention. The domination theory was introduced in the 1950s but it was highly researched from the mid-1970s[1]. Oystein Ore introduced the terms "Dominating Set" and "Domination Number" in 1962[2]. In this paper, equitable Colourings are discussed and different Domination parameters are investigated. All graphs considered here are simple, finite, undirected, and connected. The order of the graph is defined as the number of distinct vertices in the graph and is denoted by $|V|$ and the size of the graph G is defined as the number of distinct edges of the graph and is denoted by $|E|$.

2 Methodology

Definition 2. 1: [3] Proper coloring of a graph G is when neighboring nodes do not share a similar color. The chromatic number of a graph G, described as by $x(G)$, is the least amount of colors that can be applied to it.

Definition 2. 2: [4] A subset of nodes having a similar color is termed a color class(CC).

Definition 2. 3: [3] An equitable coloring graph is one in which the cardinality of color classes differs by ≤ 1 and adjacent nodes do not share a similar color. An equitable chromatic number $\mathrm{x_E(G)}$ of G is the minimum quantity of colors that might be employed to color it equally.

Definition 2. 4: [5] A subset D of nodes in a graph $G=(V,E)$ is the dominating set if each node in $V-D$ is neighboring to a node in D. A domination number of G symbolized by $\gamma(G)$, which is the lowest cardinality that can exist for the dominating set.

Definition 2.5: [6] A dominating subset D of V is called an equitable color class dominating set if the number of dominating nodes in each color class is equal. The equitable color class domination number of G symbolized by $\gamma_{ECC}(G)$, which is the lowest cardinality that can exist for the equitable color class dominating set.

Notation 2.6 : [7] Whereas $\theta>0$ and a,b are integers, following that $a\equiv b\ (mod\ \theta)$. The prediction shows $\theta|a-b$.

Notation 2.7 : [7] Consider any real number X. Subsequently, $\lfloor X\rfloor$ depicts the largest integer $\leq X$, whereas, $\lceil X\rceil$ stands for the smallest integer $\geq X$.

3 Results and Discussion

3.1. Equitable Color Class Domination Number of Honeycomb Networks and Honeycomb Cup Networks Graph

3.1.1. Honeycomb Networks:

The use of hexagonal tessellations allows for the recursive construction of Honeycomb networks. The $HC(1)$ is a hexagon. Six hexagons are added to the $HC(1)$'s border links to create the $HC(2)$.By Induction $HC(\theta)$ is acquired from $HC(\theta-1)$ by adding a layer of hexagons around the boundary of $HC(\theta-1)$ [8]. The number of nodes and links of $HC(\theta)$ are $6\theta^2$ and $9\theta^2-3\theta$ respectively. The diameter of $HC(\theta)$ is $4\theta-1$. The $HC(2)$ and its brick representation are shown in Figure 1 below.

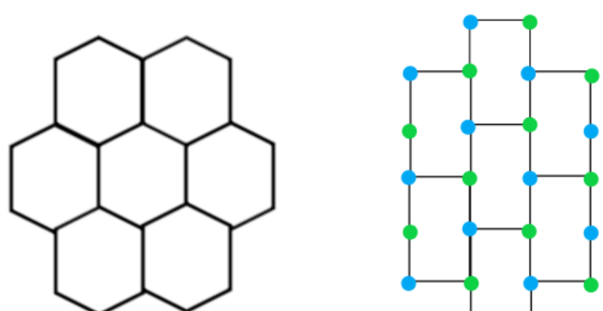

Fig 1. Honeycomb network of dimension 2 and its brick representation

In computer graphics, mobile phone base stations, image processing, and chemistry, honeycomb networks are extensively utilized for the representation of benzenoid hydrocarbons and carbon hexagons of carbon Nanotubes.

Theorem 3.1: The ECC-Domination number of Honeycomb network graph $HC(\theta)$,

$$\gamma_{ECC}(HC(\theta))=\begin{cases}2\left(\frac{(\theta-1)(\theta+2)}{2}+2\sum_{\lambda=1}^{\theta-1}\lfloor\frac{\lambda}{4}\rfloor+2\lfloor\frac{\theta}{4}\rfloor+\lceil\frac{2\theta+1}{3}\rceil\right), \theta\, is\, odd.\\ 2\left(\frac{\theta^2-\theta+2}{2}+2\sum_{\lambda=2}^{\theta}\lceil\frac{\lambda}{4}\rceil\right) \qquad\qquad\qquad\qquad , \theta\, is\, even.\end{cases}$$

Proof: let $HC(\theta)$ is the honeycomb graph with $V(HC(\theta)) = \{v_{\lambda,\mu}, u_{\lambda,\mu} : 1 \le \lambda \le \theta, 1 \le \mu \le 4\theta - (2\lambda - 1)\}$ and $E(HC(\theta)) = \{v_{\lambda,\mu}v_{\lambda,\mu+1}, u_{\lambda,\mu}u_{\lambda,\mu+1} : 1 \le \lambda \le \theta, 1 \le \mu \le 4\theta - 2\lambda\}$
$\cup\{v_{\lambda,\mu}v_{\lambda+1,\mu-1}, u_{\lambda,\mu}u_{\lambda+1,\mu-1} : 1 \le \lambda \le \theta - 1, \mu\, is\, even, 2 \le \mu \le 4\theta - 2\lambda\} \cup \{v_{1,\mu}u_{1,\mu} : \mu\, is\, odd, 1 \le \mu \le 4\theta - 1\}$. $|V(HC(\theta))| = 6\theta^2$ and $|E(HC(\theta))| = 9\theta^2 - 3\theta$. χ_E of $HC(\theta)$ is 2. Let $\tau : V(HC(\theta)) \to CC$ be an equitably colorable function. Where $CC = \{\alpha, \beta\}$. Ordain the color α to the nodes $v_{\lambda,\mu}$ where μ is odd and $u_{\lambda,\mu}$ where μ is even. Ordain the color β to the nodes $v_{\lambda,\mu}$ where μ is even and $u_{\lambda,\mu}$ where μ is odd. Now choose dominating nodes as of the following cases.

Case i: θ is odd

Choose $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ for $1 \le \lambda \le \theta - 1$ and let $1 \le \kappa \le \theta - (\lambda - 1) + 2\lfloor \frac{\lambda}{4} \rfloor$

Subcase i: If $\lambda + 3 \equiv 0\,(mod\,4)$, then $\mu = 4\kappa - 2$.

Subcase ii: If $\lambda + 2 \equiv 0\,(mod\,4)$, then $\mu = 4\kappa - 1$.

Subcase iii: If $\lambda + 1 \equiv 0\,(mod\,4)$, then $\mu = 1,\ 4\theta - (2\lambda - 1),\ 4\kappa$.

Subcase iv : If $\lambda \equiv 0\,(mod\,4)$, then $\mu = 4\kappa - 3$.

Subcase v: For $\lambda = \theta$

If $\theta + 3 \equiv 0\,(mod\,6)$ and let $1 \le \kappa \le \lceil \frac{2\theta+1}{3} \rceil - 1$, $\mu = 3\kappa - 1$ and $2\theta + 1$.

Otherwise, let $1 \le \kappa \le \lceil \frac{2\theta+1}{3} \rceil$, $\mu = 3\kappa - 1$.

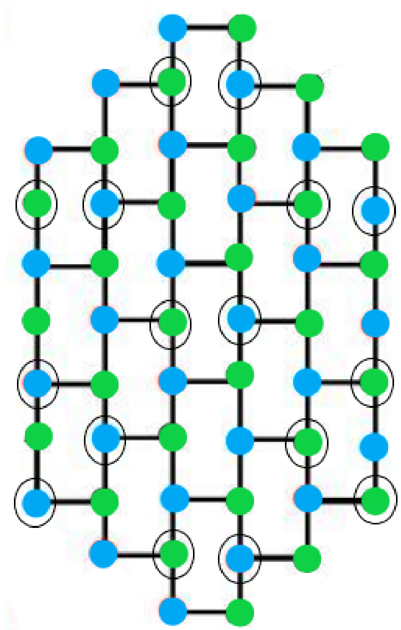

Fig 2. Brick representation of $HC(3)$

From the first four subcases number of dominating nodes in each subcase is $2(\theta - (\lambda - 1) + 2\lfloor \frac{\lambda}{4} \rfloor)$ and for each $\lambda + 1 \equiv 0\ (mod\ 4)$ need an extra 4 dominating nodes that is $4\lfloor \frac{\theta}{4} \rfloor$. So, for $1 \le \lambda \le \theta - 1$, the total number of dominating nodes is $\sum_{\lambda=1}^{\theta-1} 2(\theta - (\lambda - 1) + 2\lfloor \frac{\lambda}{4} \rfloor + 2\lfloor \frac{\theta}{4} \rfloor) = 2\left(\sum_{\lambda=1}^{\theta-1} \theta - (\lambda - 1) + 2\sum_{\lambda=1}^{\theta-1} \lfloor \frac{\lambda}{4} \rfloor + 2\lfloor \frac{\theta}{4} \rfloor\right) = 2\left(\frac{(\theta-1)(\theta+2)}{2} + 2\sum_{\lambda=1}^{\theta-1} \lfloor \frac{\lambda}{4} \rfloor + 2\lfloor \frac{\theta}{4} \rfloor\right)$. For $\lambda = \theta$, the number of dominating nodes is $2\lceil \frac{2\theta+1}{3} \rceil$. Hence $\gamma_{ECC}(HC(\theta)) = 2\left(\frac{(\theta-1)(\theta+2)}{2} + 2\sum_{\lambda=1}^{\theta-1} \lfloor \frac{\lambda}{4} \rfloor + 2\lfloor \frac{\theta}{4} \rfloor + \lceil \frac{2\theta+1}{3} \rceil\right)$.

Case ii: θ is even

Choose $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ for $2 \le \lambda \le \theta$ and let $1 \le \kappa \le \theta - \lambda + 2\lceil \frac{\lambda}{4} \rceil$

Subcase i: If $\lambda + 3 \equiv 0\,(mod\,4)$, then $\mu = 4\kappa$.

Subcase ii: If $\lambda + 2 \equiv 0\,(mod\,4)$, then $\mu = 4\kappa - 3$.

Subcase iii: If $\lambda + 1 \equiv 0\,(mod\,4)$, then $\mu = 4\kappa - 2$.

Subcase iv: If $\lambda \equiv 0\,(mod\,4)$, then $\mu = 4\kappa - 1$.

Subcase v: For $\lambda = 1$ and let $1 \le \kappa \le \theta - 1$, $\mu = 4\kappa$, $v_{1,1}$ and $u_{1,4\theta-1}$.

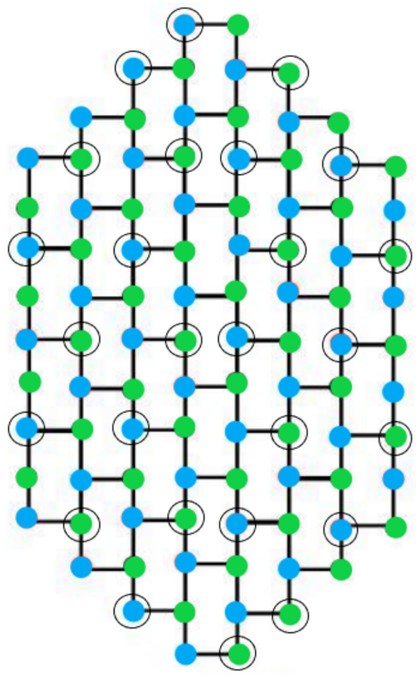

Fig 3. Brick representation of $HC(4)$

From the first four subcases number of dominating nodes in each subcase is $2\left(\theta-\lambda+2\left\lceil\frac{\lambda}{4}\right\rceil\right)$. For $2\leq\lambda\leq\theta$, the total number of dominating nodes is $\sum_{\lambda=2}^{\theta}2\left(\theta-\lambda+2\left\lceil\frac{\lambda}{4}\right\rceil\right)=2\left(\sum_{\lambda=2}^{\theta}\theta-\lambda+2\sum_{\lambda=2}^{\theta}\left\lceil\frac{\lambda}{4}\right\rceil\right)$
$=2\left(\frac{\theta^2-\theta+2}{2}+2\sum_{\lambda=2}^{\theta}\left\lceil\frac{\lambda}{4}\right\rceil\right)$. For $\lambda=1$, number of dominating nodes is $2(\theta-1)+2=2\theta$. Hence $\gamma_{ECC}(HC(\theta))=2\left(\frac{\theta^2-\theta+2}{2}+2\sum_{\lambda=2}^{\theta}\left\lceil\frac{\lambda}{4}\right\rceil\right)$.

From the above cases $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ dominates all other nodes, and each $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ are in different color classes. So, the number of dominating nodes in each color class is equal.

Example 3.2: The equitable color class domination number of $HC(3)$ and $HC(4)$.

For $HC(3)$, θ is odd.

Therefore, $\gamma_{ECC}(HC(\theta))=2\left(\frac{(\theta-1)(\theta+2)}{2}+2\sum_{\lambda=1}^{\theta-1}\left\lfloor\frac{\lambda}{4}\right\rfloor+2\left\lfloor\frac{\theta}{4}\right\rfloor+\left\lceil\frac{2\theta+1}{3}\right\rceil\right)$

$\gamma_{ECC}(HC(3))=2\left(\frac{(3-1)(3+2)}{2}+2\sum_{\lambda=1}^{3-1}\left\lfloor\frac{\lambda}{4}\right\rfloor+2\left\lfloor\frac{3}{4}\right\rfloor+\left\lceil\frac{2(3)+1}{3}\right\rceil\right)$
$=2\left(5+2\left(\left\lfloor\frac{1}{4}\right\rfloor+\left\lfloor\frac{2}{4}\right\rfloor\right)+0+3\right)$
$=2(8)=16.$

The way how the dominating nodes are chosen is shown in Figure 2. The colors blue and green denote α and β respectively. The circled nodes are ECC dominating nodes.

The equitable color class dominating set
$D=\{v_{1,2},v_{1,6},v_{1,10}\}\cup\{v_{2,3},v_{2,7}\}\cup\{v_{3,2},v_{3,5},v_{3,7}\}\cup\{u_{1,2},u_{1,6},u_{1,10}\}\cup\{u_{2,3},u_{2,7}\}\cup\{u_{3,2},u_{3,5},u_{3,7}\}$.

For $HC(4)$, θ is even.

Therefore, $\gamma_{ECC}(HC(\theta))=2\left(\frac{\theta^2-\theta+2}{2}+2\sum_{\lambda=2}^{\theta}\left\lceil\frac{\lambda}{4}\right\rceil\right)$

$\gamma_{ECC}(HC(4))=2\left(\frac{4^2-4+2}{2}+2\sum_{\lambda=2}^{4}\left\lceil\frac{\lambda}{4}\right\rceil\right)$
$=2\left(7+2\left(\left\lceil\frac{2}{4}\right\rceil+\left\lceil\frac{3}{4}\right\rceil+\left\lceil\frac{4}{4}\right\rceil\right)\right)$
$=2(13)=26.$

The way how the dominating nodes are chosen is shown in Figure 3. The colors blue and green denote α and β respectively. The circled nodes are ECC dominating nodes.

The equitable color class dominating set
$D=\{v_{1,1},v_{1,4},v_{1,8},v_{1,12}\}\cup\{v_{2,1},v_{2,5},v_{2,9},v_{2,13}\}\cup\{v_{3,2},v_{3,6},v_{3,10}\}\cup\{v_{4,3},v_{4,7}\}\cup\{u_{1,15},u_{1,4},u_{1,8},u_{1,12}\}\cup$

$\{u_{2,1},u_{2,5},u_{2,9},u_{2,13}\}\cup\{u_{3,2},u_{3,6},u_{3,10}\}\cup\{u_{4,3},u_{4,7}\}$.

3.1.2. Honeycomb Cup Networks

The construction of Honeycomb cup networks is done repeatedly with hexagonal tessellations. The $HCC(1)$ is acquired by adding two hexagons adjacent and also adding one hexagon to the boundary of two hexagons[8]. The number of nodes and links of $HCC(\theta)$ are $2(3\theta^2+4\theta+1)$ and $(9\theta^2+9\theta+1)$ respectively. The $HCC(1)$ and its brick representation are shown in Figure 4.

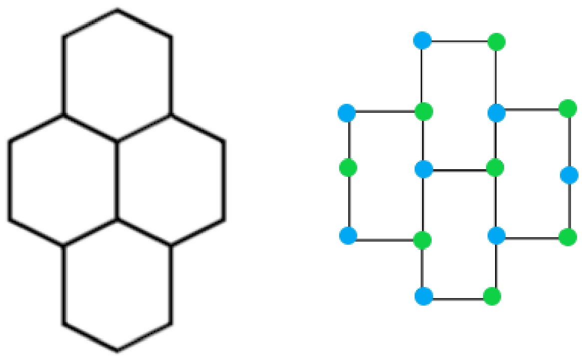

Fig 4. Honeycomb cup network of dimension 1 and its brick representation

Sizing one of the upper and lower nodes in the straight lines results in the brick structure. There are hence the same number of nodes and links in the brick representation as well.

Theorem 3.3: The ECC-Domination number of Honeycomb cup network graph $HCC(\theta)$, $\theta>2$,

$$\gamma_{ECC}(HCC(\theta))=\begin{cases} 2\left(\frac{3\theta-1}{2}+2\sum_{\lambda=1}^{\frac{\theta-1}{2}}\theta-\lambda+\left\lceil\frac{2(\theta+1)}{3}\right\rceil\right), \theta\, is\, odd. \\ 2\left(\frac{3\theta}{2}+2\sum_{\lambda=1}^{\frac{\theta-2}{2}}\theta-\lambda+\left\lceil\frac{4\theta+1}{3}\right\rceil\right), \theta\, is\, even. \end{cases}$$

Proof: let $HCC(\theta)$ is the honeycomb cup network graph with $V(HCC(\theta))=\{v_{\lambda,\mu},u_{\lambda,\mu}\ :1\leq\lambda\leq\theta+1, 1\leq\mu\leq 4\theta-(2\lambda-3)\}$ and $E(HCC(\theta))=\{v_{\lambda,\mu}v_{\lambda,\mu+1},u_{\lambda,\mu}u_{\lambda,\mu+1}\ :1\leq\lambda\leq\theta+1,\ 1\leq\mu\leq 4\theta-2(\lambda-1)\}\cup\{v_{\lambda,\mu}v_{\lambda+1,\mu-1},u_{\lambda,\mu}u_{\lambda+1,\mu-1}\ :1\leq\lambda\leq\theta,\ \mu\, is\, even,\ 2\leq\mu\leq 4\theta-2(\lambda-1)\}\cup\{v_{1,\mu}u_{1,\mu}\ :\mu\, is\, odd,\ 1\leq\mu\leq 4\theta+1\}$. $|V(HCC(\theta))|=2(3\theta^2+4\theta+1)$ and $|E(HCC(\theta))|=9\theta^2+9\theta+1$. χ_E of $HCC(\theta)$ is 2. Let $\tau\ :\ V(HCC(\theta))\to CC$ be an equitably colorable function. Where $CC=\{\alpha,\beta\}$. Ordain the color α to the nodes $v_{\lambda,\mu}$ where μ is odd and $u_{\lambda,\mu}$ where μ is even. Ordain the color β to the nodes $v_{\lambda,\mu}$ where μ is even and $u_{\lambda,\mu}$ where μ is odd. Now choose dominating nodes as of the following cases.

Case i: θ is odd

Choose $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ for $1\leq\lambda\leq\theta+1$.

Subcase i: If λ is odd, then $\mu=4\kappa-2$. Where $1\leq\kappa\leq\theta-(\frac{\lambda-1}{2})$.

Subcase ii: If λ is even, then $\mu=4\kappa-1$. Where $1\leq\kappa\leq\theta-(\frac{\lambda}{2})$.

Subcase iii: If $\theta+1\, or\, \theta+3\equiv 0\,(mod\, 6)$

If $\lambda+2\equiv 0\ (mod\ 3)$, then $\mu=4\theta-(2\lambda-2)$.

If $\lambda\equiv 0\ (mod\ 3)$, then $\mu=4\theta-(2\lambda-3)$.

Subcase iv: If $\theta+5\equiv 0\ (mod\ 6)$

If $\lambda+2\equiv 0\ (mod\ 3)$, then $\mu=4\theta-(2\lambda-2)$.

If $\lambda\equiv 0\ (mod\ 3)$, then $\mu=4\theta-(2\lambda-3)$.

If $\lambda=\theta+1$, then $\mu=4\theta-(2\lambda-2)$.

In the first subcase number of dominating nodes is $2\left(\sum_{\lambda=1}^{\theta+1}\theta-(\frac{\lambda-1}{2})\right)$ where λ is odd and in the second subcase number of dominating nodes is $2\left(\sum_{\lambda=1}^{\theta+1}\theta-(\frac{\lambda}{2})\right)$ where λ is even. The sum of the number of dominating nodes in the first two

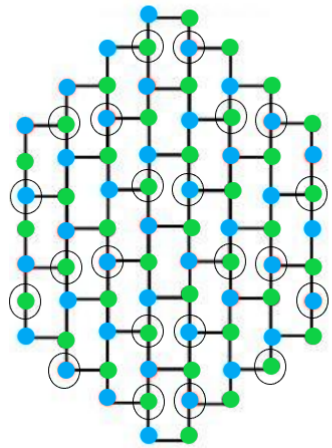

Fig 5. Brick representation of $HCC\,(\mathbf{3})$

subcases is $2\left(\frac{3\theta-1}{2}+2\sum_{\lambda=1}^{\frac{\theta-1}{2}}\theta-\lambda\right)$. From the remaining subcases, we conclude there are 2 dominating nodes for each 3λ's. So the sum of the number of dominating nodes from the remaining subcases is $2\left(\left\lceil\frac{2(\theta+1)}{3}\right\rceil\right)$. Hence $\gamma_{ECC}\left(HCC\left(\theta\right)\right)=2\left(\frac{3\theta-1}{2}+2\sum_{\lambda=1}^{\frac{\theta-1}{2}}\theta-\lambda+\left\lceil\frac{2(\theta+1)}{3}\right\rceil\right)$.

Case ii: θ is even

Choose $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ for $1\leq\lambda\leq\theta$.

Subcase i: If λ is odd, then $\mu=4\kappa-2$. Where $1\leq\kappa\leq\theta-(\frac{\lambda-1}{2})$.

Subcase ii: If λ is even, then $\mu=4\kappa-1$. Where $1\leq\kappa\leq\theta-(\frac{\lambda}{2})$.

Subcase iii: If $\lambda+2\equiv 0\,(mod\,3)$, then $\mu=4\theta-(2\lambda-2)$.

Subcase iv: If $\lambda\equiv 0\;(mod\;3)$ then $\mu=4\theta-(2\lambda-3)$.

Subcase v: For $\lambda=\theta+1$

If $\theta+4\equiv 0\,(mod\,6)$, then $\mu=3\kappa-1$. Where $1\leq\kappa\leq\left\lceil\frac{2\theta+1}{3}\right\rceil$.

If $\theta+2\equiv 0\,(mod\,6)$, then $\mu=3\kappa+1$ and 1. Where $1\leq\kappa\leq\left\lceil\frac{2\theta+1}{3}\right\rceil-1$.

If $\theta\equiv 0\;(mod\;6)$, then $\mu=3\kappa$ and 1. Where $1\leq\kappa\leq\left\lceil\frac{2\theta+1}{3}\right\rceil-1$.

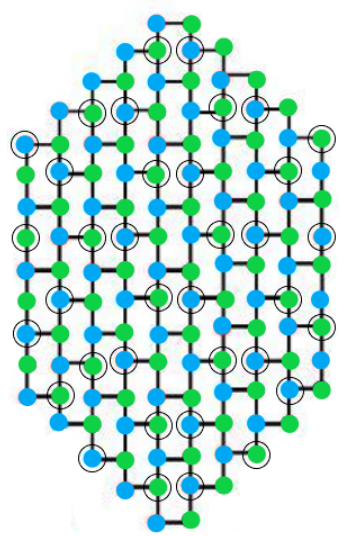

Fig 6. Brick representation of $HCC\,(\mathbf{4})$

In the first subcase number of dominating nodes is $2\left(\sum_{\lambda=1}^{\theta}\theta-\left(\frac{\lambda-1}{2}\right)\right)$ where λ is odd and in the second subcase number of dominating nodes is $2\left(\sum_{\lambda=1}^{\theta}\theta-\left(\frac{\lambda}{2}\right)\right)$ where λ is even. The sum of the number of dominating nodes in the first two subcases is $2\left(\frac{3\theta}{2}+2\sum_{\lambda=1}^{\frac{\theta-2}{2}}\theta-\lambda\right)$. From the remaining subcases, there are 2θ and $2\theta+1$ nodes that are to be dominated and these nodes are connected in a path. So the number of dominating nodes from the remaining subcases is $2\left(\left\lceil\frac{4\theta+1}{3}\right\rceil\right)$. Hence $\gamma_{ECC}(HCC(\theta))=2\left(\frac{3\theta}{2}+2\sum_{\lambda=1}^{\frac{\theta-2}{2}}\theta-\lambda+\left\lceil\frac{4\theta+1}{3}\right\rceil\right)$.

From the above cases $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ dominates all other nodes, and each $v_{\lambda,\mu}$ and $u_{\lambda,\mu}$ are in different color classes. So, the number of dominating nodes in each color class is equal.

Example 3.4: The equitable color class domination number of $HCC(3)$ and $HCC(4)$.

For $HCC(3)$, θ is odd.

Therefore, $\gamma_{ECC}(HCC(\theta))=2\left(\frac{3\theta-1}{2}+2\sum_{\lambda=1}^{\frac{\theta-1}{2}}\theta-\lambda+\left\lceil\frac{2(\theta+1)}{3}\right\rceil\right)$

$$\gamma_{ECC}(HCC(3))=2\left(\frac{3(3)-1}{2}+2\sum_{\lambda=1}^{\frac{3-1}{2}}3-\lambda+\left\lceil\frac{2(3+1)}{3}\right\rceil\right)$$
$$=2(4+2(2)+0+3)$$
$$=2(11)=22.$$

The way how the dominating nodes are chosen is shown in Figure 5. The colors blue and green denote α and β respectively. The circled nodes are ECC dominating nodes.

The equitable color class dominating set

$D=\{v_{1,2},v_{1,6},v_{1,10},v_{1,12}\}\cup\{v_{2,3},v_{2,7}\}\cup\{v_{3,2},v_{3,6},v_{3,9}\}\cup\{v_{4,3},v_{4,6}\}\cup\{u_{1,2},u_{1,6},u_{1,10},u_{1,12}\}\cup\{u_{2,3},u_{2,7}\}\cup\{u_{3,2},u_{3,6},u_{3,9}\}\cup\{u_{4,3},u_{4,6}\}$.

For $HCC(4)$, θ is even.

Therefore, $\gamma_{ECC}(HCC(\theta))=2\left(\frac{3\theta}{2}+2\sum_{\lambda=1}^{\frac{\theta-2}{2}}\theta-\lambda+\left\lceil\frac{4\theta+1}{3}\right\rceil\right)$

$$\gamma_{ECC}(HCC(4))=2\left(\frac{3(4)}{2}+2\sum_{\lambda=1}^{\frac{4-2}{2}}4-\lambda+\left\lceil\frac{4(4)+1}{3}\right\rceil\right)$$
$$=2(6+2(3)+6)$$
$$=2(18)=36.$$

The way how the dominating nodes are chosen is shown in Figure 6. The colors blue and green denote α and β respectively. The circled nodes are ECC dominating nodes.

The equitable color class dominating set

$D=\{v_{1,2},v_{1,6},v_{1,10},v_{1,14},v_{1,16}\}\cup\{v_{2,3},v_{2,7},v_{2,11}\}\cup\{v_{3,2},v_{3,6},v_{3,10},v_{3,13}\}\cup\{v_{4,3},v_{4,7},v_{4,10}\}\cup\{v_{5,1},v_{5,4},v_{5,7}\}\cup\{u_{1,2},u_{1,6},u_{1,10},u_{1,14},u_{1,16}\}\cup\{u_{2,3},u_{2,7},u_{2,11}\}\cup\{u_{3,2},u_{3,6},u_{3,10},u_{3,13}\}\cup\{u_{4,3},u_{4,7},u_{4,10}\}\cup\{u_{5,1},u_{5,4},u_{5,7}\}$.

3.2. Inverse Equitable Color Class Domination Number of Graphs

The collection of all transmitting stations in a communication network is connected to at least one station. Searching for another disjoint collection of stations is necessary if this one fails. By doing so, the inverse dominance number can be defined[9]. Let D be the equitable color class dominating set of the graph G then there exists another equitable color class dominating set D' in $V-D$ is called an inverse equitable color class dominating set of G corresponding to D. It is indicated by $\gamma_{ECC}^{-1}(G)$.

Theorem 3.5: Inverse equitable color class domination number of a path graph $\gamma_{ECC}^{-1}(P_\theta)=2\left\lceil\frac{\theta+1}{6}\right\rceil$, where $\theta>3$ and $\theta\neq 7$.

Proof: Let P_θ be the path with nodes $v_i(1\le i\le\theta)$. X_E of P_θ is 2. Let $\tau : V(P_\theta)\to CC$ be an equitably colorable function where $CC=\{\alpha,\beta\}$. Ordain the color α to the odd nodes and β to the even nodes. Choosing equitable color class dominating set as $D=v_{3i-1}\left(1\le i\le 2\left\lceil\frac{\theta}{6}\right\rceil\right)$ and choosing inverse equitable color class dominating node set as $D^{-1}=v_{3i-2}\left(1\le i\le 2\left\lceil\frac{\theta+1}{6}\right\rceil\right)$. For $\theta\le 3$ and $\theta=7$, $|D|>|V-D|$ so we cannot choose D' from $V-D$. For some cases, we change some equitable color class dominating nodes and inverse equitable color class dominating nodes for choosing a minimum inverse equitable color class dominating set.

Case i: $\theta\equiv 0\ (mod\ 6)$

In D, there is no change.

In D', $v_{3i-2}\left(i=2\left\lceil\frac{\theta+1}{6}\right\rceil\right)$ is replaced by v_θ and $v_{3i-2}\left(i=2\left\lceil\frac{\theta+1}{6}\right\rceil-1\right)$ is replaced by $v_{\theta-3}$.

Case ii: $\theta+1\equiv 0\ (mod\ 6)$

In D, there is no change.

In $D^{'}$, there is no change.

Case iii: $\theta+2\equiv 0\ (mod\ 6)$

In D, $v_{3i-1}\left(i=2\left\lceil \frac{\theta}{6}\right\rceil\right)$ is replaced by $v_{\theta-1}$.

In $D^{'}$, there is no change.

Case iv: $\theta+3\equiv 0\ (mod\ 6)$

In D, $v_{3i-1}\left(i=2\left\lceil \frac{\theta}{6}\right\rceil-1\right)$ is replaced by $v_{\theta-3}$ and $v_{3i-1}\left(i=2\left\lceil \frac{\theta}{6}\right\rceil\right)$ is replaced by v_{θ}.

In $D^{'}$, $v_{3i-2}\left(i=2\left\lceil \frac{\theta+1}{6}\right\rceil\right)$ is replaced by $v_{\theta-1}$.

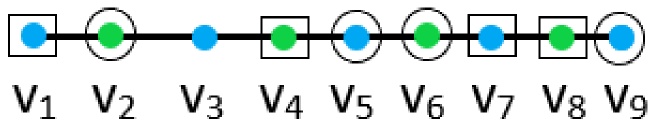

Fig 7. P_9

Case v: $\theta+4\equiv 0\ (mod\ 6)$

In D, $v_{3i-1}\left(i=2\left\lceil \frac{\theta}{6}\right\rceil\right)$ is replaced by $v_{\theta-5}$.

In $D^{'}$, $v_{3i-2}\left(i=2\left\lceil \frac{\theta+1}{6}\right\rceil\right)$ is replaced by $v_{\theta-2}$.

Case vi: $\theta+5\equiv 0\ (mod\ 6)$

In D, $v_{3i-1}\left(i=2\left\lceil \frac{\theta}{6}\right\rceil-1\right)$ is replaced by $v_{\theta-1}$ and $v_{3i-1}\left(i=2\left\lceil \frac{\theta}{6}\right\rceil\right)$ is replaced by $v_{\theta-4}$.

In $D^{'}$, $v_{3i-2}\left(i=2\left\lceil \frac{\theta+1}{6}\right\rceil\right)$ is replaced by $v_{\theta-7}$.

From all the above cases the inverse equitable color class dominating nodes in each color class are equal. Hence, $\gamma_{ECC}^{-1}(P_\theta)=2\left\lceil \frac{\theta+1}{6}\right\rceil$.

Example 3.6: The inverse equitable color class domination number of P_9.

$\gamma_{ECC}^{-1}(P_\theta)=2\left\lceil \frac{\theta+1}{6}\right\rceil$

$\gamma_{ECC}^{-1}(P_9)=2\left\lceil \frac{9+1}{6}\right\rceil$

$=2(2)=4.$

The way how the equitable dominating nodes and inverse equitable dominating nodes are chosen is shown in Figure 7. The colors blue and green denote α and β respectively. The circled and squared nodes are ECC dominating nodes and inverse ECC dominating nodes respectively.

The equitable color class dominating set $D=\{v_2,v_5,v_6,v_9\}$.

The inverse equitable color class dominating set $D^{'}=\{v_1,v_4,v_7,v_8\}$.

Theorem 3.7: Inverse equitable color class domination number of a cycle graph

$$\begin{cases} 3\left\lceil \frac{\theta}{9}\right\rceil\ if\,\theta\,is\,odd,\,\theta\geq 7\,and\,\theta\neq 11 \\ 2\left\lceil \frac{\theta}{6}\right\rceil\ if\,\theta\,is\,even \end{cases}.$$

Proof: Let C_θ be the cycle with nodes $v_i(1\leq i\leq\theta)$ and links $v_iv_{i+1}\cup v_\theta v_1(1\leq i\leq\theta-1)$.

Case i: if θ is odd.

X_E of C_θ is 3. Let $\tau\ :V(C_\theta)\rightarrow CC$ be an equitably colorable function where $CC=\{\alpha,\beta,\omega\}$. For $9i-k\leq\theta$ and $1\leq i\leq\left\lceil \frac{\theta}{9}\right\rceil$, ordain the color α to the nodes $v_{9i-k}(k=8,3,1)$, Ordain the color β to the nodes $v_{9i-k}(k=7,5,0)$ and Ordain the color ω to the nodes $v_{9i-k}(k=6,4,2)$. Choosing equitable color class dominating set as $D=v_{3i-1}\left(1\leq i\leq 3\left\lceil \frac{\theta}{9}\right\rceil\right)$ and choosing inverse equitable color class dominating node set as $D^{-1}=v_{3i-2}\left(1\leq i\leq 3\left\lceil \frac{\theta}{9}\right\rceil\right)$. For $\theta=3,\,5\,and\,11$, $|D|>|V-D|$ so we cannot choose $D^{'}$ from $V-D$. For some cases, we change some equitable color class dominating nodes and inverse equitable color class dominating nodes for choosing a minimum inverse equitable color class dominating set.

Subcase i: $\theta+1\equiv 0\ (mod\ 18)$

Change the node color of v_1 from α to ω and $v_{\theta-1}$ from ω to β.

In D, there is no change.

In $D^{'}$, $v_{3i-2}\left(i=3\left\lceil \frac{\theta}{9}\right\rceil\right)$ is replaced by $v_{\theta-2}$.

Subcase ii: $\theta+3\equiv 0\ (mod\ 18)$

Change the node colors between v_θ and $v_{\theta-2}$.

In D, $v_{3i-1}\left(i=3\left\lceil \frac{\theta}{9}\right\rceil\right)$ is replaced by $v_{\theta-2}$.

In $D^{'}$, $v_{3i-2}\left(i=3\left\lceil \frac{\theta}{9}\right\rceil\right)$ is replaced by $v_{\theta-3}$ and $v_{3i-2}\left(i=3\left\lceil \frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_θ.

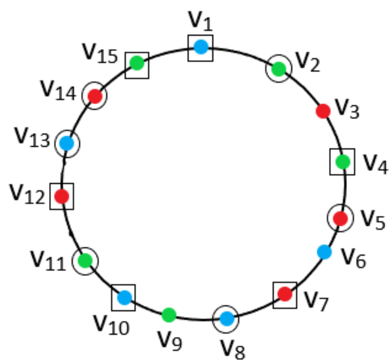

Fig 8. C_{15}

Subcase iii: $\theta+5\equiv 0\ (mod\ 18)$

In D, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil\right)$ is replaced by v_6 and $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by $v_{\theta-1}$.

In $D^{'}$, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_3.

Subcase iv: $\theta+7\equiv 0\ (mod\ 18)$

In D, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil\right)$ is replaced by v_6 and $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_{12}.

In $D^{'}$, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_3 and $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_9.

Subcase v: $\theta+9\equiv 0\ (mod\ 18)$

In D, there is no change.

In $D^{'}$, there is no change.

Subcase vi: $\theta+11\equiv 0\ (mod\ 18)$

In D, $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by $v_{\theta-1}$.

In $D^{'}$, there is no change.

Subcase vii: $\theta+13\equiv 0\ (mod\ 18)$

In D, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil\right)$ is replaced by v_6.

In $D^{'}$, $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil\right)$ is replaced by v_3.

Subcase viii: $\theta+15\equiv 0\ (mod\ 18)$

In D, $v_{3i-1}\left(i=3)\left\lceil\frac{\theta}{9}\right\rceil\right.$ is replaced by v_6 and $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_{12}.

In $D^{'}$, $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil\right)$ is replaced by v_3 and $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_9.

Subcase ix: $\theta+17\equiv 0\ (mod\ 18)$

Change the color of the node v_θ from α to ω.

In D, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil\right)$ is replaced by v_{15}, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by v_θ and $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-2\right)$ is replaced by v_9.

In $D^{'}$, $v_{3i-1}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-2\right)$ is replaced by v_3, $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-1\right)$ is replaced by $v_{\theta-1}$ and $v_{3i-2}\left(i=3\left\lceil\frac{\theta}{9}\right\rceil-2\right)$ is replaced by v_6.

From all the above subcases the inverse equitable color class dominating nodes in each color class are equal. Hence, $\gamma_{ECC}^{-1}(C_\theta)=3\left\lceil\frac{\theta}{9}\right\rceil$ when θ is odd, $\theta\geq 7$ and $\theta\neq 11$.

Case ii: if θ is even.

X_E of C_θ is 2. Let $\tau\ : V(C_\theta)\rightarrow CC$ be an equitably colorable function where $CC\ =\ \{\alpha,\beta\}$. Ordain the color α to the odd nodes and β to the even nodes. Choosing equitable color class dominating set as $D=v_{3i-1}\left(1\leq i\leq 2\left\lceil\frac{\theta}{6}\right\rceil\right)$ and choosing inverse equitable color class dominating node set as $D^{-1}=v_{3i-2}\left(1\leq i\leq 2\left\lceil\frac{\theta}{6}\right\rceil\right)$. For some cases, we change some equitable color class dominating nodes and inverse equitable color class dominating nodes for choosing a minimum inverse equitable color class dominating set.

Subcase i: $\theta\equiv 0\ (mod\ 6)$

In D, there is no change.

In $D^{'}$, there is no change.

Subcase ii: $\theta+2\equiv 0\ (mod\ 6)$

In D, $v_{3i-1}\,(i=2\lceil\frac{\theta}{6}\rceil)$ is replaced by $v_{\theta-1}$.
In $D^{'}$, there is no change.
Subcase iii: $\theta+4\equiv 0\;(mod\;6)$
In D, $v_{3i-1}\,(i=2\lceil\frac{\theta}{6}\rceil)$ is replaced by v_3.
In $D^{'}$, $v_{3i-2}\,(i=2\lceil\frac{\theta}{6}\rceil)$ is replaced by v_6.

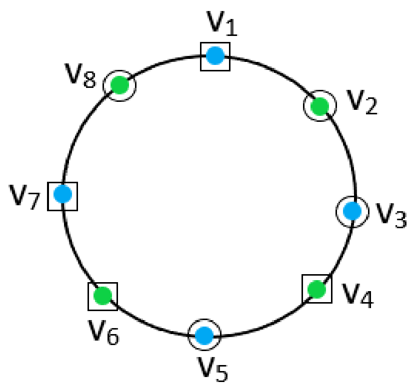

Fig 9. C_8

From all the above subcases the inverse equitable color class dominating nodes in each color class are equal. Hence, $\gamma^{-1}_{ECC}(C_\theta)=2\lceil\frac{\theta}{6}\rceil$ when θ is even.
Example 3.8: The inverse equitable color class domination number of C_{15} and C_8.
For C_{15}, θ is odd.
Therefore, $\gamma^{-1}_{ECC}(C_\theta)=3\lceil\frac{\theta}{9}\rceil$
$\gamma^{-1}_{ECC}(C_{15})=3\lceil\frac{15}{9}\rceil$
$=3(2)=6.$
The way how the equitable dominating nodes and inverse equitable dominating nodes are chosen is shown in Figure 8. The colors blue, green, and red denote α, β, and ω respectively. The circled and squared nodes are ECC dominating nodes and inverse ECC dominating nodes respectively.
The equitable color class dominating set $D=\{v_2,v_5,v_8,v_{11},v_{13},v_{14}\}$.
The inverse equitable color class dominating set $D^{'}=\{v_1,v_4,v_7,v_{10},v_{12},v_{15}\}$.
For C_8, θ is even.
Therefore, $\gamma^{-1}_{ECC}(C_\theta)=2\lceil\frac{\theta}{6}\rceil$
$\gamma^{-1}_{ECC}(C_8)=2\lceil\frac{8}{6}\rceil$
$=2(2)=4.$
The way how the equitable dominating nodes and inverse equitable dominating nodes are chosen is shown in Figure 9. The colors blue and green denote α and β respectively. The circled and squared nodes are ECC dominating nodes and inverse ECC dominating nodes respectively.
The equitable color class dominating set $D=\{v_2,v_3,v_5,v_8\}$.
The inverse equitable color class dominating set $D^{'}=\{v_1,v_4,v_6,v_7\}$.
Remark 3.9: There is no way to find the inverse equitable color class domination number of a complete graph K_θ because in K_θ, $D=V(K_\theta)$ so $V-D=\emptyset$.

4 Conclusion

This article contains some ECC domination number of honeycomb networks and honeycomb cup networks graphs. We want to find more applications of honeycomb networks using this ECC domination number in the future.

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