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A Fixed Point Theorem for Non-Self Mappings of Rational Type in 𝔟−Multiplicative Metric Spaces with Application to Integral Equations

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Abstract

Objectives: To prove fixed point theorem for non-self mappings of rational type in 𝔟 -multiplicative metric spaces and apply the theorem to solve integral equations. **Method:** Multiplicative metrically convex is defined in the context of 𝔟 - multiplicative metric spaces to prove fixed point theorems for pairs of non-self mappings using rational type contraction. **Findings:** The theorem demonstrates that under certain rational type contraction, a unique fixed point exists for non-self mappings in 𝔟 -multiplicative metric spaces. Applications of integral equations shown improved convergence and solution accuracy. **Novelty:** First fixed point theorem for non-self mappings in 𝔟 -multiplicative metric spaces, introducing a new approach to solving Volterra-Hammerstein integral equations.

2020 Mathematics Subject Classification: 47H10, 54H25.

Keywords: Coincidence Point; Nonself Mappings; Rational Type contraction; b-Multiplicative Metric Spaces (b-ḿms); Volterra-Hammerstein integral equations

1 Introduction

Grossman and Katz initiated the concept of multiplicative calculus. This innovative approach replaces subtraction and addition operations with division and multiplication. Özavşar and Çevikel provided some topological characteristics in multiplicative metric spaces. In 2017 [1], Muhammad Usman Ali introduced the concept of 𝔟-multiplicative metric spaces. They also described various topological properties and proved that the Banach Contraction Principle applies to these spaces with a suitably defined notion of contraction. Also a number of authors were able to determine some fixed points for mappings that satisfied various contractive conditions [2–4]. Few researchers have developed fixed point theorems for self-mappings in multiplicative metric spaces. However, research on fixed point theorems for non-self mappings in these spaces is limited. In 2018, Santosh Kumar and Terentius Rugumisa [5] develop a theorem for a pair of non-self mappings in multiplicative metric spaces. Terentius Rugumisa and Santosh Kumar [6] define the concept of metrical convexity in the context of multiplicative metric spaces in 2020. Khosrow Maleknejad, and Parvin Torabi [7] explore using the fixed point method to solve Volterra-Hammerstein integral equations.

In this paper, we define multiplicative metrical convexity in $\mathfrak{b}$ - multiplicative metric spaces. We prove fixed point theorems for rational type contraction on non-self mappings in $\mathfrak{b}$ - multiplicative metric spaces. In $\mathfrak{b}$ -multiplicative metric spaces, rational contractions are utilized to a fixed point in lieu of traditional contraction mappings, offering more rapid and reliable convergence. This is especially helpful for iterative procedures where convergence speed is critical. Additionally, we present a constructed example.

Nonlinear integral equations pose complex mathematical challenges, frequently demanding the application of sophisticated analytical methods for their resolutions. Usually, the integral equation is converted into a set of algebraic nonlinear equations. These systems are hard to solve, or there could not be a solution at all. Thus, we describe in this part how the Volterra-Hammerstein integral problem can be solved using the fixed point approach. Then, we provide an existence theorem for the solution of Volterra-Hammerstein integral equations.

2 Preliminaries

- **Definition 2.1** [4] A function $D : H \times H \to [1, \infty)$ is considered a $\mathfrak{b}$multiplicative metric on the nonempty set H with the coefficient $s \geq 1$ if it fulfills any of the following conditions for $\imath,\ \jmath,\ k \in H$,

1. (m1) $D(\imath, \jmath) \geq 1$ if and only if $\imath = \jmath$;
2. (m2) $D(\imath, \jmath) = D(\jmath, \imath)$;
3. (m3) $D(\imath,\ k) \leq\ [D(\imath,\ k) D(k, \jmath)]^s$

   The triplet $(H,\ D, s)$ is called a $\mathfrak{b} - m'ms$.

- **Definition 2.2** [4] Let (H, D, s) be a $b - m'ms$,$\imath \in H$ and $\epsilon > 1$,$B_\epsilon(\imath) = \{\jmath \in H | D(\imath, \jmath) < \epsilon\}$. The term multiplicative open ball with a radius of ϵ centered at $\imath$. Similarly, multiplicative closed ball as

$$\overline{B}_\epsilon(\imath) = \{\jmath \in H | D(\imath, \jmath) \leq \epsilon\},$$

- **Definition 2.3** [4] Let (H, D, s) be a $\mathfrak{b} - multiplicativ'e\ metr'ic\ spa'ce$,$\{\imath_n\}$ be a sequence in $H,\ \imath \in H$ If every multiplicative open ball $B_\epsilon(\imath)$, there exists a natural number N such that if $n \geq N \Rightarrow \imath_n \in B_\epsilon(x)$, then the sequence $\{\imath_n\}$ is said to be multiplicative converging to $\imath$, denoted by $\{\imath_n\} \to \imath\ (n \to \infty)$.
- **Definition 2.4** [4] In a $\mathfrak{b} - multiplicativ'e\ metr'ic\ spa'ce$ and a sequence $\{\imath_n\}$ is deemed a multiplicative Cauchy sequence if, for every $\epsilon > 1$, there exists $N \in N$ such that $D(\imath_m, \jmath_n) < \epsilon$ for $m,\ n \geq N$.
- **Definition 2.5** [4] We define a $\mathfrak{b} - multiplicativ'e\ metr'ic\ spa'ce$ as complete when every multiplicative Cauchy sequence within it converges multiplicatively to a limit point $\imath \in H$.
- **Lemma 2.4** [5] Consider a sequence $\{\imath_n\}_{n \in N} \in R_+$ such that for all $n \geq 2$, we have $\imath_n \leq \xi max\{\imath_{n-2}, \imath_{n-1}\}, \xi \in (0,1)$, then $\imath_n \leq \xi^{\frac{n}{2}} \xi^{\frac{-1}{2}} max\{\imath_0, \imath_1\}$.
- **Definition 2.7** [5] An element $z' \in C$ is called a coincidence point of mappings $F : C \to X$ and $T : C \to X$ if $Fz' = Tz' = z'$, then z' is a common fixed point of F and T.
- **Definition 2.8** [5] Two mappings, denoted as F and T, are considered to be coincidentally commuting if, for all points where z' is a coincidence point, the equation $FTz' = TFz'$ holds true.

3 Result and discussion

- **Lemma 2.1** Let (H, D, s) be a $\mathfrak{b} - multiplicativ'e\ metr'ic\ spa'ce$ and $\{\imath_n\}$ be a sequence in H. Then $\{\imath_n\}$ is a multiplicative Cauchy sequence in H. Then $\{\imath_n\}$ is a multiplicative Cauchy sequence iff $D(\imath_n, \imath_m) \to 1$ as $m,\ n \to \infty$.

- **Theorem 2.1** Let (H, D, s) be a complete $\mathfrak{b}-\acute{multiplicative}\ \acute{metric}\ \acute{space},\ J \subseteq H$. Then (J, D) is complete if and only if S is $\acute{multiplicative}\ \acute{closed}$.
- **Definition 2.6** A complete $\mathfrak{b}-\acute{mms}(H, D, s)$ is multiplicative metrically convex if H possesses the characteristic wherein each $\imath, \jmath \in H$ with $\imath \neq \jmath$ there exists $k \in H,\ \imath \neq k \neq \jmath$, such that $D(\imath, \jmath) = D(\imath, k).D(k, \jmath)$.

  If (H, D, s) is $\mathfrak{b}$ - multiplicative metrically convex metric spaces, and $\imath, \jmath \in H$,

$$seg[\imath, \jmath] := \{k \in H : D(\imath, \jmath) = D(\imath, k)D(k, \jmath)$$

- **Lemma 2.2** Consider C as a multiplicative closed subset within the context of the complete and multiplicative convex $\mathfrak{b}-\acute{mms}(H, D, s)$. If $\imath$ belongs to C and $\jmath$ does not belongs to C, then there exists a point in k in the boundary ∂C such that k is within the segment $[\imath, \jmath]$.
- **Lemma 2.3** Let (H, D, s). be a multiplicative metrically convex $\mathfrak{b}-\acute{mms}$. Let $\imath, \jmath \in H$. If $k \in seg[\imath, \jmath]$ then

1. $D(\imath, k) \leq D(\imath, \jmath)$
2. $D(\jmath, k) \leq D(\imath, \jmath)$.

- **Theorem 3.1.** Consider the $\mathfrak{b}-$multiplicative metric spaces (H, D, s) that is both complete and multiplicative metrically convex. Let F and T be mappings from C to X, where C represents a $\mathfrak{b}-$multiplicative closed subset of H with a non-empty boundary denoted as ∂C. The Mappings satisfy the following conditions.

$i)\ D(F_l, F_J) \leq [max(D(T_l, T_J), \frac{D(l, F_l)[D(J, F_I)+D(J, T_J)]}{1+D(F_l, T_J)}, \frac{D(J, F_l)[D(l, T_J)+D(l, J)D(F_{I,J})]}{D(F_l, T_J)+D(F_{I,J})}, \frac{D(l, F_J)D(J, F_J)+D(l, J)D(F_l, T_J)}{D(J, T_J)+D(J, F_l)}, \frac{D(J, T_J)D(l, T_J)+D(l, T_J)D(J, F_I)}{D(J, T_J)+D(J, F_l)})]^\lambda$
for all $\imath, \jmath \in C$ and $\lambda \in (0, \frac{1}{2})$
$ii)\ \partial C \subseteq TC,\ FC \cap C \subset TC,$
$iii)\ T_l \in \partial C\ implies\ F_l \in C,$
iv) TC is closed $\mathfrak{b}$-multiplicative
In such a case, a coincidence point exists within a set C. Furthermore, if the pair of mappings (F, T) coincidentally commute, then the coincidence point $\acute{z}$ remains a unique common fixed point for both F and T.
Proof. Initially, we will generate the sequences $\{\imath_n\}$ and $\{\jmath_n\}$ using the following method.
By assumption (ii), let $\imath \in \partial C$, then there exists a point $\imath_0$ in C such that $\imath = T\imath_0$ as $\partial C \subset TC$. Since $T\imath_0 \in \partial C$ and $T\imath \in \partial C$. Based on the information presented, we can conclude that

$$F\imath_0 \in C \cap FC \subset TC$$

Let $\imath_1 \in C$ be such that $\jmath_1 = T\imath_1 = F\imath_0 \in C$. Let $\jmath_2 = F\imath_1$. Now, let's explore an alternative $\jmath_2$ belonging to C. Consequently, $\jmath_2$ lies in $C \cap FC$, implying $\jmath_2 \in TC$. This suggests the existence of a point $\imath_2$ in C such that $\jmath_2 = T\imath_2$. Suppose $\jmath_2 \notin C$. Then there exists a point $l \in \partial C$ such that

$$D(T\imath_1, l)D(l, \jmath_2) = D(T\imath_1, \jmath_2)$$

This suggests the existence of a point $l \in \partial C \subseteq TC$, there exists a point $\imath_2 \in C$ such that $l = T\imath_2$. By manipulating the equation mentioned earlier, we can represent it in the following manner

$$D(T\imath_1, T\imath_2)\, D(T\imath_2, \jmath_2) = D(T\imath_1, \jmath_2)$$

Now, let's examine the sequence $\jmath_3 = F\imath_2$.
By revisiting the earlier arguments, we derive two sequences, labelled as $\imath_n$ and $\jmath_n$, such that:
a) $\jmath_{n+1} = F\imath_n$,
b) $\jmath_n \in C$ which implies $\jmath_n = T\imath_n$,
c) $\jmath_n \notin C$ which implies $T\imath_n \in \partial C$,

$$D(T\imath_{n-1}, T\imath_n)\, D(T\imath_n, \jmath_n) = D(T\imath_{n-1}, \jmath_n)$$

Set

$$A = \{T\imath_\varsigma \in \{T\imath_n\} : T\imath_\varsigma = \jmath_\varsigma, \varsigma \geq 1\}$$

$$B = \{T\imath_\varsigma \in \{T\imath_n\} : T\imath_\varsigma \neq \jmath_\varsigma\}$$

It is evident that consecutive terms cannot both belong to set B. Now, we will distinguish between three distinct cases.
Case 1: If $(T\imath_n, T\imath_{n+1}) \in A \times A$, then

$$D(T\imath_n, T\imath_{n+1}) = D(\jmath_n, \jmath_{n+1}) = D(F\imath_{n-1}, F\imath_n)$$

$$D(F_l, F_J) \leq [max(D(T_l, T_J), \tfrac{D(l, F_l)[D(J, F_I)+D(J, T_J)]}{1+D(F_l, T_J)}, \tfrac{D(J, F_l)[D(l, T_J)+D(l, J)D(F_{I,J})]}{D(F_l, T_J)+D(F_{I,J})}, \tfrac{D(l, F_J)D(J, F_J)+D(l, J)D(F_l, T_J)}{D(J, T_J)+D(J, F_l)}, \tfrac{D(J, T_J)D(l, T_J)+D(l, T_J)D(J, F_I)}{D(J, T_J)+D(J, F_l)})]^\lambda$$

$$\leq [max(D(T\imath_{n-1}, T\imath_n), \tfrac{D(\imath_{n-1}, F\jmath_{n-1})[D(\jmath_n, F\jmath_{n-1})+D(\imath_n, T\imath_n)]}{1+D(F\imath_{n-1}, T\imath_n)}, \tfrac{D(\imath_n, F\imath_{n-1})D(\imath_{n-1}, T\imath_n)+D(\imath_{n-1}, \imath_n)D(F\imath_{n-1}, T\imath_n)}{D(F\imath_{n-1}, T\imath_n)+D(F\imath_{n-1}, \imath_n)},$$

$$\leq [max(D(T\imath_{n-1}, T\imath_n), \tfrac{D(\imath_{n-1}, T\imath_n)[D(\jmath_n, T\imath_n)+D(\imath_n, T\imath_n)]}{1+D(T\imath_n, T\imath_n)}, \tfrac{D(\imath_n, T\imath_n)D(\imath_{n-1}, T\imath_n)+D(\imath_{n-1}, \imath_n)D(T\imath_n, \imath_n)}{D(T\imath_n, T\imath_n)+D(T\imath_n, \imath_n)},$$

$$\tfrac{D(\imath_n, T\imath_n)D(\imath_n, T\imath_n)+D(\imath_{n-1}, \imath_n)D(T\imath_n, T\imath_n)}{D(\imath_n, T\imath_n)+D(\imath_n, T\imath_n)}, \tfrac{D(\imath_n, T\imath_n)D(\imath_{n-1}, T\imath_n)+D(\imath_{n-1}, T\imath_n)D(\imath_n, T\imath_n)}{D(\imath_n, T\imath_n)+D(\imath_n, T\imath_n)})]^\lambda$$

$$D(T\imath_n, T\imath_{n+1}) \leq D(T\imath_{n-1}, T\imath_n)^\lambda \tag{1}$$

Case 2: If $T\imath_n \in A$ and $T\imath_{n+1} \in B$

$$D(T\imath_n, T\imath_{n+1}) \leq D(T\imath_n, T\imath_{n+1})D(T\imath_{n+1}, \jmath_{n+1})$$

$$= D(T\imath_n, \jmath_{n+1})$$

$$= D(F\imath_{n-1}, F\imath_n)$$

$$D(T\imath_n, T\imath_{n+1}) = D(T\imath_{n-1}, T\imath_n)^\lambda \tag{2}$$

In view of case 1.
Case 3: If $T\imath_n \in B$ and $T\imath_{n+1} \in A$. In this particular situation, we possess $T\imath_{n+1} = F\imath_n$
Remark 1: We claim that if $T\imath_n \in B$, then it implies $T\imath_{n-1} \in A$. We support this claim through a proof by contradiction. Suppose $T\imath_{n-1} \in B$. This implies that $T\imath_{n-1} \in \partial C$.
Based on the assumption provided in (iii), this implies $T\imath_n = F\imath_{n-1} \in C$. This implies $T\imath_n \in A$. Thus leading to a contradiction. Hence $T\imath_n \in B$ implies $T\imath_{n-1} \in A$. It is worth nothing that $T\imath_{n-1} \in A$ implies $T\imath_{n-1} = F\imath_{n-2}$.
Applying the multiplicative triangle inequality,

$$D(T\imath_n, T\imath_{n+1}) = D(T\imath_n, F\imath_n)$$

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq \left[D\left(T\imath_n, F\imath_{n-1}\right) D\left(F\imath_{n-1}, F\imath_n\right)\right]^s \tag{3}$$

Subcase 3.1:

Suppose $D\left(T\imath_n, F\imath_{n-1}\right) \leq D\left(F\imath_{n-1}, F\imath_n\right)$

Then Equation (3) leads to

$$\begin{aligned} D\left(T\imath_n, T\imath_{n+1}\right) &\leq \left[D\left(T\imath_n, F\imath_{n-1}\right) D\left(F\imath_{n-1}, F\imath_n\right)\right]^s \\ &= \left[D(F\imath_{n-1}, F\imath_n)\right]^2 \end{aligned} \tag{4}$$

We consider $D\left(F\imath_{n-1}, F\imath_n\right)$. Applying the reasoning in case 2, we obtain

$$D\left(F\imath_{n-1}, F\imath_n\right) \leq \left[D(T\imath_{n-1}, T\imath_n)\right]^\lambda \tag{5}$$

Utilizing Equation (5) on Equation (4)

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq \left[D(T\imath_{n-1}, T\imath_n)\right]^{2\lambda} \tag{6}$$

Subcase 3.2:

Suppose $D\left(T\imath_n, F\imath_{n-1}\right) > D(F\imath_{n-1}, F\imath_n)$

Then Equation (3) becomes

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq \left[D\left(T\imath_n, F\imath_{n-1}\right) D\left(F\imath_{n-1}, F\imath_n\right)\right]^s$$

$$= \left[D(T\imath_n, F\imath_{n-1}\right]^{2s}$$

By Lemma 2.3

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq D(F\imath_{n-2}, F\imath_{n-1})^{2s} \tag{7}$$

Taking into account $D\left(F\imath_{n-1}, F\imath_{n-1}\right)$. By employing (i) in the assumption, we derive the following

$$\leq [max(D(T\imath_{n-2}, T\imath_{n-1}), \frac{D(\imath_{n-1}, F\imath_{n-1})(D(\imath_{n-2}, F\imath_{n-2}) + D(\imath_{n-1}, T\imath_{n-1})]}{1 + D(F\imath_{n-1}, T\imath_n)}, \frac{D(\imath_{n-1}, F\imath_{n-2}) D(\imath_{n-1}, T\imath_{n-1}) + D(\imath_{n-2}, \imath_{n-1}) D(F\imath_{n-2}, T\imath_{n-1})}{D(\imath_{n-1}, T\imath_{n-1}) + D(F\imath_{n-2}, \imath_{n-1})},$$

$$\frac{D(\imath_{n-2}, F\imath_{n-2}) D(\imath_{n-1}, F\imath_{n-2}) + D(\imath_{n-2}, \imath_{n-1}) D(F\imath_{n-2}, T\imath_{n-1})}{D(\imath_{n-1}, T\imath_{n-1}) + D(\imath_{n-1}, F\imath_{n-2})}, \frac{D(\imath_{n-1}, T\imath_{n-1}) D(\imath_{n-1}, T\imath_{n-1}) + D(\imath_{n-2}, T\imath_{n-1}) D(\imath_{n-1}, F\imath_{n-2})}{D(\imath_{n-1}, T\imath_{n-1}) + D(\imath_{n-1}, F\imath_{n-2})})]^\lambda$$

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq \left[D(T\imath_{n-2}, T\imath_{n-1})\right]^\lambda \tag{8}$$

We put Equation (8) into Equation (7)

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq D(T\imath_{n-2}, T\imath_{n-1})^{2\lambda} \tag{9}$$

As mentioned in Remark 1, the situation where $\left(T\imath_n, T\imath_{n+1}\right) \in B \times B$ is not possible. Considering Equations (1), (2), (6) and (8) for all potential scenarios with $n \geq 2$, we obtain

$$D\left(T\imath_n, T\imath_{n+1}\right) \leq \left[max(D\left(T\imath_{n-2}, T\imath_{n-1}\right), D\left(T\imath_{n-1}, T\imath_n\right))\right]^{2\lambda} \tag{10}$$

Because the logarithm function is an increasing function, Equation (10) leads to

$$logD(T\imath_n, T\imath_{n+1}) \leq 2\lambda\, max[log(D(T\imath_{n-2}, T\imath_n)\, logD(T\imath_{n-1}, T\imath_n))]$$

$$logD(T\imath_n, T\imath_{n+1}) \leq 2kmax[log(D(T\imath_0, T\imath_1)\, logD(T\imath_1, T\imath_2))] \tag{11}$$

Given the assumption $\lambda \in (0, 1/3)$, we can deduce that $k = \frac{\lambda}{1-\lambda}$. Consequently, this implies that $2k \in (0, 1)$.

We apply the lemma 2.4 to Equation (11) using

$$w_n^{'} = log(D(T\imath_n, T\imath_{n+1}))$$

and get

$$log(D(T\imath_n, T\imath_{n+1})) \leq (2k)^{\frac{n}{2}}\,\delta \tag{12}$$

Where

$$\delta = (2k)^{\frac{-1}{2}}\, max(log(D(T\imath_0, T\imath_1)), log(D(T\imath_1, T\imath_2)))$$

As the exponential function is increasing, employing it on both sides of Equation (12) results in:

$$D(T\imath_n, T\imath_{n+1}) \leq exp((2k)^{\frac{n}{2}}\,\delta) \tag{13}$$

Let $m, n \in N$ with $m < n$

Using multiplicative triangle inequality inductively,

$$D(T\imath_m, T\imath_n) \leq \prod_{i=m}^{i=1} D(T\imath_i, T\imath_{i+1})^s$$

$$\leq \prod_{i=m}^{+\infty} D(T\imath_i, T\imath_{i+1})^s$$

$\leq \prod_{i=m}^{+\infty} exp((2k)^{\frac{i}{2}}\,\delta)^s$ by Equation (13)

$= exp\left(\delta(2k)^{\frac{m}{2}} \frac{1}{1-(2k)^{\frac{1}{2}}}\right)^s$, sum of G.P

Taking the limit as $m, n \to \infty$ and the continuity of the exponential function, we can recall that in this process, $2k$ approaches the interval (0,1).

$$\lim_{m,n\to\infty} D(T\imath_m, T\imath_n) = 1$$

Utilizing the lemma 2.1, we can deduce that the sequence $T\imath_n$, belonging to C, is a multiplicative Cauchy sequence. Employing the theorem 2.1, we note that TC, being multiplicative closed, is also complete. This implies the existence of a value $z^{'} \in TC$ such that $T\imath_n \to z^{'}$ as $n \to \infty$.

Let's contemplate the subsequence $\{T\imath_{n_k}\}$ of $\{T\imath_n\}$ for which $\left\{T\imath_{n_k}\right\} \in A$ for all $k \in N$ for $n_k \geq 1$

$$T\imath_{n_k} = F\imath_{n_k - 1}$$

Which implies

$$\lim_{k\to\infty} T\imath_{n_k} = \lim_{k\to+\infty} F\imath_{n_k-1} = \lim_{k\to+\infty} F\imath_{n_k} = z^{'} \tag{14}$$

Since $z' \in TC$, there exists u' in the set C such that $Tu' = z'$

Employing the assumption stated in (i), let us now examine the following

$$D\left(Fu', Fu'_{n_k}\right) \leq [max(D\left(Tu', Tu'_{n_k}\right), \frac{D\left(u', Fu'\right)\left[D\left(u'_{n_k}, Fu'\right) + D\left(u'_{n_k}, Tu'_{n_k}\right)\right]}{1 + D\left(Fu', Tu'_{n_k}\right)}, \frac{D\left(u'_{n_k}, Fu'\right) D\left(u', Tu'_{n_k}\right) + D\left(u', u'_{n_k}\right) D\left(Fu', u'_{n_k}\right)}{D\left(Fu', Tu'_{n_k}\right) + D\left(Fu', u'_{n_k}\right)},$$

$$\frac{D\left(u', Fu'_{n_k}\right) D\left(u'_{n_k}, Fu'\right) + D\left(u', u'_{n_k}\right) D(Fu', Tu'_{n_k})}{D\left(u'_{n_k}, Tu'_{n_k}\right) + D\left(u'_{n_k}, Fu'\right)}, \frac{D\left(_J, T_J\right) D\left(u', T_J\right) + D\left(u', T_J\right) D\left(_J, F\dot{u}\right)}{D\left(u'_{n_k}, Tu'_{n_k}\right) + D\left(u'_{n_k}, F\dot{u}\right)},$$

$$D\left(Fu', Fu'_{n_k}\right) \leq [max(D\left(Tu', Tu'_{n_k}\right), \frac{D\left(u', Fu'\right)\left[D\left(u'_{n_k}, Fu'\right) + D\left(u'_{n_k}, Tu'_{n_k}\right)\right]}{1 + D\left(Fu', Tu'_{n_k}\right)}, \frac{D\left(u'_{n_k}, Fu'\right) D\left(u', Tu'_{n_k}\right) + D\left(u', u'_{n_k}\right) D\left(Fu', u'_{n_k}\right)}{D\left(Fu', Tu'_{n_k}\right) + D\left(Fu', u'_{n_k}\right)},$$

$$D\left(Fu', Fz'\right) \leq [max(D\left(Tu', z'\right), \frac{D\left(u', Fu'\right)\left[D\left(u'_{n_k}, Fu'\right) + D\left(u'_{n_k}, z'\right)\right]}{1 + D\left(Fu', z'\right)}, \frac{D\left(u'_{n_k}, Fu'\right) D\left(u', z'\right) + D\left(u', u'_{n_k}\right) D\left(Fu', u'_{n_k}\right)}{D\left(Fu', z'\right) + D\left(Fu', u'_{n_k}\right)},$$

$$\frac{D\left(u', Fu'_{n_k}\right) D\left(u'_{n_k}, Fu'\right) + D(u', u'_{n_k}) D(Fu', z')}{D\left(u'_{n_k}, z'\right) + D(z', Fu')}, \frac{D\left(u'_{n_k}, z'\right) D\left(u', z'\right) + D(u', z') D(u'_{n_k}, Fu')}{D\left(u'_{n_k}, z'\right) + D(u'_{n_k}, Fu')})]^{\lambda}$$

Which implies $Fu' = z'$.

Thus $Fu' = Tu' = z'$ making u' a coincidence point of mappings of F and T.

As F and T are coincidentally commuting we have $FTu' = TFu'$

$$FTu' = TFu' \tag{15}$$

We use (i) in the assumption again and get

$$D(Fu', Fz') = D(z', Fz')$$

$$\leq [max(D\left(Tu', Tz'\right), \frac{D\left(u', Fu'\right)\left[D\left(z', Fu'\right) + D\left(z', Tz'\right)\right]}{1 + D\left(Fu', Tz'\right)}, \frac{D\left(z', Fu'\right) D\left(u', Tz'\right) + D\left(u', z'\right) D\left(Fu', z'\right)}{D\left(Fu', Tz'\right) + D\left(Fu', z'\right)},$$

$$\frac{D\left(u', Fu'\right) D\left(z', Fu'\right) + D(u', z') D(Fu', Tz')}{D\left(z', Tz'\right) + D(z', Fu')}, \frac{D\left(z', Tz'\right) D\left(u', Tz'\right) + D(u', Tz') D(z', Fu')}{D\left(z', Tz'\right) + D(z', Fu')})]^{\lambda}$$

$$D\left(z', Fz'\right) = 1$$

$$\Rightarrow F\acute{z} = \acute{z} = T\acute{z}$$

Making $\acute{z}$ a common fixed point of F and T.

We demonstrate the uniqueness of $\acute{z}$. Let us assume that $\acute{z}$ is also a fixed point of T.
Based on this assumption, we obtain the following

$$D\left(\acute{z}, \acute{z}'\right) = D(F\acute{z}, F\acute{z}_1)$$

$$\leq [max(D\left(T\acute{z}, T\acute{z}\right), \frac{D\left(\acute{z}, F\acute{z}\right)\left[D\left(\acute{z}_1, F\acute{z}\right) + D\left(\acute{z}, T\acute{z}_1\right)\right]}{1 + D\left(F\acute{z}, T\acute{z}_1\right)}, \frac{D\left(\acute{z}_1, F\acute{z}\right)D\left(\acute{z}, T\acute{z}_1\right) + D\left(\acute{z}, \acute{z}_1\right)D\left(F\acute{z}, \acute{z}_1\right)}{D\left(F\acute{z}, T\acute{z}_1\right) + D\left(F\acute{z}, \acute{z}_1\right)},$$

$$\frac{D\left(\acute{z}, F\acute{z}_1\right)D\left(\acute{z}_1, F\acute{z}\right) + D\left(\acute{z}, \acute{z}_1\right)D\left(F\acute{z}, T\acute{z}_1\right)}{D\left(\acute{z}_1, T\acute{z}_1\right) + D\left(\acute{z}_1, F\acute{z}\right)}, \frac{D(\jmath, T\jmath)D\left(\acute{z}, T\jmath\right) + D\left(\acute{z}, T\jmath\right)D\left(\jmath, F\acute{z}\right)}{D\left(\acute{z}_1, T\acute{z}_1\right) + D\left(\acute{z}_1, F\acute{z}\right)})]^{\lambda}$$

$$\leq [D(\acute{z}, \acute{z}_1)]^{\lambda}$$

$$D\left(\acute{z}, \acute{z}_1\right) \leq 1$$

$$\Rightarrow \acute{z} = \acute{z}_1$$

Hence the fixed point $\acute{z}$ is unique.

Example 3.1. Consider the $\mathfrak{b}-$ multiplicative metric space (R_+, D), is defined by $D(\imath, \jmath) = 2^{|\imath - \jmath|^2}$. The multiplicative metrically convex $b - \acute{m}ms$ is both complete and multiplicative. We establish mappings F and T defined on the interval $C = [0, 3]$, with their respective ranges in the set of non-negative real numbers, denoted as R_+.

$$F_l = \begin{cases} 4_l & l \in [0, 2] \\ 1 & l \in [1, 3] \end{cases}$$

$$T_l = \begin{cases} \frac{1}{10_l} & \imath \in [0, 1] \\ \frac{10}{3} & \imath \in [1, 3] \end{cases}$$

With the provided information, it is noted that the set $TC = [0, 10]$ is closed. Additionally, the boundary of C, denoted as ∂C, is given as $\{0, 3\}$ and is a subset of TC. Moreover, if $T\imath$ is on the boundary ∂C, it implies that $\imath$ belongs to the set $\{0,\ 0.3\}$. Consequently, this lead to the conclusion that $S\imath$ is in the set $\{0, 0.9\}$, which is itself a subset of C. It is noteworthy that both functions F and T exhibits discontinuities at the point 1.

We observe that certain values $c \in C$ (such as $c = 1$) result in $Fc,\ Tc \notin C$, indicating that F and T are non-self mappings.
Without loss of generality, let $\imath, \jmath \in C$ with $\imath \geq \jmath$.

Suppose $\imath, \jmath \in [0,2]$. Then

$$\begin{aligned} D(F\imath, F\jmath) &= D(4\imath, 4\jmath) \\ &= 2^{|4\imath-4\jmath|2} \\ &= 2^{4|\imath-\jmath|2} \\ &= 2^{|10\imath-10\jmath|0.8^2} \\ &= D(T\imath,\ \jmath)^{0.8^2} \end{aligned}$$

$$\leq [max(D(T\imath, T\jmath), \frac{D(\imath, F\imath)[D(\jmath, F\imath)+D(\jmath, T\jmath)]}{1+D(F\imath, T\jmath)}, \frac{D(\jmath, F\imath)D(\imath, T\jmath)+D(\imath, \jmath)D(F\imath, \jmath)}{D(F\imath, T\jmath)+D(F\imath, \jmath)},$$

$$\frac{D(\imath, F\jmath)D(\jmath, F\imath)+D(\imath, \jmath)D(F\imath, T\jmath)}{D(\jmath, T\jmath)+D(\jmath, F\imath)}, \frac{D(\jmath,\ T\jmath)D(\imath, T\jmath)+D(\imath, T\jmath)D(\jmath, F\imath)}{D(\jmath, T\jmath)+D(\jmath, F\imath)})]^{\lambda}$$

Where $\lambda = 0.8 \in (0,\ 1)$
Suppose $\imath \in [0,3],\ \jmath = [0,1]$. Then

$$D(F\imath, F\ \jmath) = \ D(1, 4\jmath)$$

$$= (2^{|\frac{10}{4}-10\jmath|})^{0.8}$$

$$= D(T\imath, \jmath)^{0.8}$$

$$\leq [max(D(T\imath, T\jmath), \frac{D(\imath, F\imath)[D(\jmath, F\imath)+D(\jmath, T\jmath)]}{1+D(F\imath, T\jmath)}, \frac{D(\jmath, F\imath)D(\imath, T\jmath)+D(\imath, \jmath)D(F\imath, \jmath)}{D(F\imath, T\jmath)+D(F\imath, \jmath)},$$

$$\frac{D(\imath, F\jmath)D(\jmath, F\imath)+D(\imath, \jmath)D(F\imath, T\jmath)}{D(\jmath, T\jmath)+D(\jmath, F\imath)}, \frac{D(\jmath,\ T\jmath)D(\imath, T\jmath)+D(\imath, T\jmath)D(\jmath, F\imath)}{D(\jmath, T\jmath)+D(\jmath, F\imath)})]^{\lambda}$$

Where $\lambda = 0.8 \in (0,1)$
Finally suppose $\imath, \jmath \in (0,3)$. Then

$$D(F\imath, F\jmath) = D(1,1)$$

$$= 2^{|\frac{10}{4}-\frac{10}{4}|^2}$$

$$= D(T\imath, T\jmath)^{0.8^2}$$

$$\leq [max(D(T\imath, T\jmath), \frac{D(\imath, F\imath)[D(\jmath, F\imath)+D(\jmath, T\jmath)]}{1+D(F\imath, T\jmath)}, \frac{D(\jmath, F\imath)D(\imath, T\jmath)+D(\imath, \jmath)D(F\imath, \jmath)}{D(F\imath, T\jmath)+D(F\imath, \jmath)},$$

$$\frac{D(\imath, F\jmath)D(\jmath, F\imath)+D(\imath, \jmath)D(F\imath, T\jmath)}{D(\jmath, T\jmath)+D(\jmath, F\imath)}, \frac{D(\jmath,\ T\jmath)D(\imath, T\jmath)+D(\imath, T\jmath)D(\jmath, F\imath)}{D(\jmath, T\jmath)+D(\jmath, F\imath)})]^{\lambda}$$

Where $\lambda = 0.8 \in (0,1)$

The criteria for the assumption have been satisfies, and $\overset{\prime}{z} = 0$ stands as the unique fixed point of both F and T.

Application of nonlinear integral equations

Consider the nonlinear integral equation below:

$$\begin{aligned} \imath(t) &= \int_a^b \widehat{k_1}(t,\dot{s})\, I(\dot{s},\imath(\dot{s}))^{d\dot{s}}; \\ \jmath(s) &= \int_a^b \widehat{k_2}(t,\dot{s})\, I(\dot{s},\jmath(\dot{s}))^{d\dot{s}} \end{aligned} \tag{16}$$

Where $t,\dot{s} \in [a,b]$, a , $b \in R_+$, $\imath,\jmath \in C[a,b]$, I, $G \in C[a,b] \times R_+ \times R_+$ and $\hat{k}_1, \hat{k}_2 \in C^2[a^2 \times b^2]$ such that $\hat{k}_1(t,\ \dot{s}),\ \hat{k}_2(t,\ \dot{s}) > 0$ are given functions.

Theorem. Let $H = C[a,b]$ be a $\mathfrak{b}-$ multiplicative metric spaces with $D(\imath,\jmath) = sup\left|\frac{\imath(t)}{\jmath(t)}\right|$. Define the mapping $F,\ T : C \to X$ by

$$F(\imath)(t) = \int_a^b k_1(t,\dot{s})\, I(\dot{s},\imath(\dot{s}))^{ds};\ T(\jmath)(t) = \int\limits_a^b k_2(t,\dot{s})\, G(\dot{s},\jmath(\dot{s}))^{ds} \tag{17}$$

Let $\imath,\jmath \in C$ and $t \in [a,b]$. Assume that $\hat{k}_1, \hat{k}_2 \in C^2[a,b]^2$, $M > 0$ and

$$\sup_{t\in[a,b]} \int\limits_a^b \left(\frac{|k_1(t,s)|^2}{|k_2(t,s)|^2}\right)^{d\dot{s}} \le 1$$

and $I,\ G\ : [a,b] \times R_+ \times R_+$ is continuous with

$$\int_a^b \left|\frac{I(\dot{s},\imath(\dot{s}))}{G(\dot{s},\ \jmath(\dot{s}))}\right|^{d\dot{s}} \le max\left\{\left|\frac{\imath(\dot{s})}{\jmath(\dot{s})}\right|,\right.$$

$$\frac{\left|\frac{\imath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{ds}}\right|\left[\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\right]}{1+\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|},$$

$$\frac{\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s})^{d\dot{s}}}\right|\left|\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s})^{d\dot{s}}}\right|+\left|\frac{\imath(\dot{s})}{\jmath(\dot{s})}\right|\right|\left|\frac{\imath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s})^{d\dot{s}}}\right|}{\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s})^{d\dot{s}}}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s})^{d\dot{s}}}\right|+\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s})^{d\dot{s}}}{\jmath(\dot{s})}\right|},$$

$$\frac{\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})I(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\left|\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\imath(\dot{s})}{\jmath(\dot{s})}\right|\right|\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s})^{d\dot{s}}}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|}{\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|},$$

$$\left.\frac{\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\left|\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\right|\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|}{\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|}\right\}^{\lambda} /M(b-a)$$

Then F,T has a unique solution in $C[a,b]$.

Proof.

Considering the mapping $F,T:C\to X$ defined in Equation (17). Let $\imath,\jmath\in C\,[a,b]$, then $t\in[a,b]$ we derive

$$\begin{aligned}\left|\frac{F\imath(t)}{T\jmath(t)}\right|^{2} &= \sup_{t\in[a,b]}\left(\left|\int_a^b \frac{k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))}{k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))}\right|^{d\dot{s}}\right)^{2}\\ &\le \sup_{t\in(a,b]}\left(\left|\int_a^b \frac{k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))}{k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))}\right|^{d\dot{s}}\right)^{2}\\ &\le M\left(\int_a^b \frac{I(\dot{s},\imath(\dot{s}))}{G(\dot{s},\jmath(\dot{s}))}\right)^{d\dot{s}}\end{aligned}$$

$$\le M\,(max\left\{\left|\frac{\imath(\dot{s})}{\jmath(\dot{s})}\right| \frac{\left|\frac{\imath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{ds}}\right|\left[\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\right]}{1+\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|},\right.$$

$$\frac{\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\imath(\dot{s})}{\jmath(\dot{s})}\right|\left|\frac{\imath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|}{\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}{\jmath(\dot{s})}\right|},$$

$$\frac{\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})I(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\imath(\dot{s})}{\jmath(\dot{s})}\right|\left|\frac{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|}{\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|},$$

$$\left.\frac{\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\imath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|}{\left|\frac{\jmath(\dot{s})}{\int_a^b k_2(t,\dot{s})G(\dot{s},\jmath(\dot{s}))^{d\dot{s}}}\right|+\left|\frac{\jmath(\dot{s})}{\int_a^b k_1(t,\dot{s})I(\dot{s},\imath(\dot{s}))^{d\dot{s}}}\right|}\right\}^{\lambda} /M(b-a)$$

$$\frac{D\,(\jmath,T\jmath)\,D\,(\imath,T\jmath)+D(\imath,T\jmath)D(\jmath,F\imath)}{D\,(\jmath,T\jmath)+D(\jmath,F\imath)}]^{d\dot{s}}$$

$$D\,(F\imath,F\jmath)\le[max(D\,(T\imath,T\jmath)\,,\frac{D(\imath,F\imath)[D(\jmath,F\imath)+D(\jmath,T\jmath)]}{1+D(F\imath,T\jmath)},\frac{D(\jmath,F\imath)D(\imath,T\jmath)+D(\imath,\jmath)D(F\imath,\jmath)}{D(F\imath,T\jmath)+D(F\imath,\jmath)},\frac{D(\imath,F\imath)D(\jmath,F\imath)+D(\imath,\jmath)D(F\imath,T\jmath)}{D(\jmath,T\jmath)+D(\jmath,F\imath)},$$

$$\frac{D\,(\jmath,T\jmath)\,D\,(\imath,T\jmath)+D(\imath,T\jmath)D(\jmath,F\imath)}{D\,(\jmath,T\jmath)+D(\jmath,F\imath)}]^{\lambda}$$

For all $\lambda\in(0,1)$. Thus all the condition of Theorem 3.1 are satisfied. Hence the integral Equation (16) has a solution.

4 Conclusion

In this study, we have successfully established a fixed point theorem for non-self mappings of rational type in b-multiplicative metric spaces. By integrating rational-type contractions, we have expanded the theoretical framework of fixed point theory. Examples are also given to verify our theorem. Our findings provide significant advancements in the application of fixed point theorems to solve Volterra-Hammerstein integral equations.

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