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RESEARCH ARTICLE

On the Independent Uphill Domination Polynomial of a Graph

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Abstract

Objectives: The concept of independent uphill domination polynomial $IUP(G,\, y)$ is newly defined to deal with the present graph theory in this paper. It is characterized and analyzed within various contexts and types of graphs, its roots are discovered, its relation to different graph transforms is discussed and the study of this graph can help reveal new structures of graphs. **Methods:** The independent uphill domination polynomial is defined as $\sum_{i=0}^{n} iup(G,i)y^i$ is an independent uphill dominating set of size i in G. The uphill paths are used in identifying the criterion of independent uphill dominating sets. The computation of specific values of the polynomial is performed for basic graphs, and the polynomial's response to various flip operations is observed. **Findings:** Checking the standard graphs for some properties of the independent uphill dominance polynomial shows the features of the independent uphill dominance polynomial as an uphill dominance polynomial of a certain type of digraph. Families of polynomials can be best understood when it comes to stability and characteristics by analyzing their roots on different graphical planes. They demonstrate how the structure of the graph and the polynomial change as a result of the different graph operations. **Novelty:** Incorporating uphill paths into independent dominating sets extends common domination polynomials and provides a distinct perspective distinct from previous investigations on graph dominance. Hence, this work enriches the subject and suggests new avenues for studying related polynomials and their uses by encompassing more structural properties of graphs.

Keywords: Domination; Polynomial; Uphill domination; Independent domination; Root; Book graph; Independent uphill domination polynomial

1 Introduction

Let $D(G)$ represent the cardinality of the dominant sets of a graph G. The domination polynomial $D(G,y)$ of graph G is formally defined as [1,2],

$$IUP(G,y) = \sum_{i=\gamma(G)}^{n(G)} D(G,i)y^i$$

is referred to as the domination polynomial of G (see [3]).

All graphs in this article are finite and simple, which implies that the edges in graphs are undirected and do not contain loops or parallel edges. Further, they have a few vertices as compared to the other types of graphs. A dominating set of graphs G is a non-empty set S that is a subset of the vertex set $V(G)$ and every vertex in $S \subseteq V(G)$ is connected with at least one vertex in S[4]. The minimum number of vertices in a dominating set is defined as the domination number of G which is given by $\delta(G)$. For detailed understanding about dominance theory, the reader is referred to reference[4].

An independent set of graphs G is a collection of vertices in G such that any two vertices are not neighbors. An independent number of G, denoted as $\beta(G)$, is the size of the maximal independent set in a G; that is, it is the largest independent set in G. A graph G is said to be a well-covered graph since all maximal independent sets are of equal size[5]. A graph G is well covered if and only if its complement graph G contains no group of size $K(G) + 1$; and each clique of size less than $\beta(G)$ is a sub-clique of some clique of size $\beta(G)$. Hence, a collection of vertices in the graph G is called an independent dominating set iff (if and only if) it is dominating and independent or is maximum independent. The interaction between the characteristics under consideration is quite comprehensible[6,7].

The uphill dominating set, also known as UDS, is a set S that is a subset of V and has the feature that all vertex v in V is located on an uphill path that starts from a certain vertex in S. The uphill domination number of a graph G, indicated as up $\gamma(G)$, is the smallest size of the UDS of G. In a graph $G = (V, E)$, the degree of a vertex $v \in V(G)$, indicated by $deg(v)$, is the count of edges that are incident with v. A root in graph G is a series of different vertices that alternates between adjacent vertices. A root is considered uphill if, for every $1 \leq i \leq k$, the degree of a vertex v_i in addition to one is less than or equal to the degree of the vertex v_{i+1}. The corona of two graphs C_1 and C_2, indicated by $G = C_1 \circ C_2$, is formed by combining one copy of C_1 with n1 copies of C_2. Each vertex in C_1 is connected to all vertex in the corresponding copy of C_2 with an edge[8].

The corona $G \circ K_1$ (specifically) is formed by duplicating each vertex $v \in V\ (G)$ and connecting the duplicate vertex v with a pendant edge vv. The book graph is formed by taking the Cartesian product of the star graph with $m + 1$ vertices, denoted as S, as well as the root graph with two vertices, denoted as P_2. The windmill graph $Wm\ (s,\ k)$ is formed by connecting k complete graphs K_s at a common universal vertex. This construction is valid for $s \geq 2$ and $k \geq 2$. The friendship graph is formed by connecting k copies of the cycle graph C_3. It can be seen that F_k is a specific instance of $D(s,\ k)$. The Finlay firefly graph, denoted as F, is described as a graph with s triangles, t pendent pathways of length 2, and k pendent edges, all joining at same vertex. The graph has $n = 2s + 2t + k + 1$ vertices, where s, t, and k are non-negative integers. For any terms not included here, we direct the reader to reference[9].

Recently, domination polynomials in graph theory have drawn a significant amount of attention, and different types of domination including semi-total domination, independent domination, second domination, and total domination[10–13] have been investigated in numerous researches. However, the concept of dominance has been mainly studied in reference to simple and non-developmental aspects which do not consider the structural development of vertex degrees along routes. Namely, the role of the connection between independent dominant sets and the marsh hierarchy nature in the vertex degree along with the path, or uphill path, has not been investigated to the desired extent. Therefore, the research gap is that the relationship between independent dominating sets and degree-constrained paths to improve the efficiency of the current polynomial solutions has not been investigated and there is no polynomial framework for these studies. However, there is a lack of literature on the domination polynomial of graphs when restricted to vertex sets in the uphill position This research will propose and solve a distinct polynomial known as the independent uphill domination polynomial to enhance the knowledge of domination polynomials in the field of graph theory.

We were inspired by the concept of uphill domination polynomial in a graph[8] and the independent domination polynomial in a graph[13] and to propose and analyze the independent uphill domination polynomial (IUDP) as well as the independent uphill domination roots of a graph.

2 Methodology

Definition 2.1: The IUDP of any graph G with n vertices is defined as follows:

$IUP\,(G,y) = \sum_{i=0}^{n} iup(G,i)y^i$

where the number of independent UDS (IUDS) of size i in G is represented by the function $iup(G,i)$. The independent uphill dominating roots of a graph G are the set of roots of $IUP\,(G,y)$ and are represented by $X_{iup}(G)$.

Example 2.2: The polynomial IUP of Line graph (L) with 4 vertices arranged in a path P_4 is given by

$IUP(P_4,y) = 2y + 2y^2$

Furthermore, $X_{iup}\,(L) = \{0,-1\}$.

Theorem 2.3: Assume that G is a connected graph with n vertices and $n \geq 2$. That if and only if GG is an r-regular graph, then the IUDP of G is given by $IUP\,(G,y) = (1+y)^{n-1}$.

Proof

Suppose $IUP(G,y) = (1+y)^{n-1}$

$IUP(G,y) = 1 + \binom{n-1}{1} y + \binom{n-1}{2} y^2 \cdots + y^{n-1}$

The singleton vertex set $\{v\}$ is an IUDS in G for any $v \in V(G)$, which can be readily confirmed as the polynomial's initial coefficient is 1. Let us assume that the graph G is not an r-regular graph. Thus, there is a vertex $u \in V(G)$ such that $deg(u) = q \neq r$. We now have two instances:

Case 1: The set $\{u\}$ is not an IUDS if $q > r$, which goes against the notion that any set of singletons in G is an IUDS.

Case 2: If $q < r$, then for any $u \neq v$ with $deg(v) = r$, the set $\{v\}$ is not an IUDS, which also goes against the assertion that every set of singletons in G is an IUDS.

Therefore, G must be an r-regular graph.

Conversely, presume that G is an r-regular graph with $n \geq 2$ vertices. We have $\delta_{iup}(G) = 1$, then there exist n IUDS of size one, while for $i = 2$, there are $\left(\begin{array}{c} n \\ 2 \end{array} \right)$ IUDS, and so on. We can thus write the IUDP as follows,

$IUP(G,y) = 1 + \left(\begin{array}{c} n-1 \\ 1 \end{array} \right) y + \left(\begin{array}{c} n-1 \\ 2 \end{array} \right) y^2 \cdots + y^{n-1}$

$= \sum_{i=0}^{n-1} \left(\begin{array}{c} n-1 \\ i \end{array} \right) y^i$

$= (1+y)^{n-1}$

Hence, the proof is completed.

Corollary 2.4: Let G be a graph with q vertices, in the event where G is a cycle $C_q$or a complete graph K_q, then $IUP(G,y) = (1+y)^{n-1}$.

Corollary 2.5: The IUDP for the regular graph $G = C_q \times C_q$ with sk vertices is expressed as $IUP(G,y) = (1+y)^{qk-1}$.

Corollary 2.6: Consider a graph G with m components. Then, the function $IUP(G,y)$ is equal to the product of i from 1 to m. The expression $IUP(G,y)$ represents the product of m terms, where each term is denoted by G_i. It is expressed as follows,

$IUP(G,y) = \sum_{i=1}^{m} IUP(G_i, y)$.

Note: The polynomial for the entire graph is given by the multiplication of the IUDPs, pertaining to components of the graph. The main concept of the proof is that for IUP(G,y) to sum to $(1+y)^{(n-1)}$, then any graph G has the condition that any vertex set V of size one is an induced-dominated set or IUDS. As for this characteristic, it is true if and only if G is a complete graph K_n. This characteristic ensures that each vertex is connected in some way to all the other vertices as is the case in a complete graph.

Preposition 2.7: If a graph G with n vertices consists of m components G_1, G_2, ...G_m, then the intersection polynomial $IUP(G,y)$ is equal to the product of $i = 1\ to\ m$. The expression $IUP(G,y)$ represented as,

$IUP(G,y) = \sum_{i=1}^{m} IUP(G_i, y)$.

Proof

We use mathematical induction based on the number of components, denoted as *m*, in the graph *G*. For *m* = 1, the assertion is true without difficulty since G has just one component, hence $IUP(G_i, y)$ = $IUP(G_1, y)$. Let's assume that the assertion is true for a certain value of *m*, denoted as *k*, i.e.,

$IUP(G,y) = \sum_{i=1}^{k} IUP(G_i, y)$ for a graph G with k components.

Now, we need to show the validity of the proposition for *m* = *k* + 1. Let G be a graph consisting of $k+1$ components, denoted as $G = G_1 \cup G_2 \cup \cdots \cup G_k \cup G_{k+1}$.

By the induction hypothesis, for the first k components, we have:

$IUP(G^{'}, y) = \sum_{i=1}^{k} IUP(G_i, y)$

Where $G^{'} = G_1 \cup G_2 \cup \cdots \cup G_k$.

Now, let's examine the graph G with the inclusion of an extra component G_{k+1}. Our objective is to demonstrate that:

$IUP(G,y) = IUP\left(G^{'}, y\right).IUP(G_{k+1}, y)$

The IUD number $\delta_{iup}(G)$ is defined as the generating function, represented by the polynomial $IUP(G_i, y)$, for the number of IUDS of each size in each component G_i of G. The polynomial $IUP(G,y)$ must consider all possible combinations of IUDS in the components G_1, G_2, ...G_{k+1}. The product of the polynomials $IUP(G_i, y)$ guarantees that the coefficients of y^r in $IUP(G,y)$ represent all possible ways to form IUDS of size r by selecting sets from each G_i. Therefore, by merging the IUDP of $G^{'}$ and G_{k+1}, we achieve:

$IUP(G,y) = IUP\left(G^{'}, y\right).IUP(G_{k+1}, y)$

Through the use of mathematical induction, we have shown that if a graph G with n vertices is composed of m components $G_1, G_2, \ldots G_m$, then:

$IUP(G,y) = \sum_{i=1}^{m} IUP(G_i,y)$

Hence, the preposition is proven.

Theorem 2.8: For any graph G, $IUP(G,y) = y^n$ if and only if, $G \cong K_n$

Proof

Given that $IUP(G,y) = y^n$, this suggests that the polynomial has just one term, the y^n term, with a non-zero coefficient. This suggests that the only IUDS is of size n. A graph must have all n vertices in a single dominating set in order to have an IUDS of size *n*. The only graph where the entire vertex set forms a dominating set (and any subset of the vertex set is also independent and dominating) is the complete graph K_n. Thus, $G \cong K_n$. Nevertheless, if $G \cong K_n$, then by preposition (2.7) we get $IUP(G,y) = y^n$.

Corollary 2.9. If and only if n G ≅ K, graph G has one independent uphill dominating root.

3 Results and Discussion

3.1 Independent uphill dominating sets of some graphs related to the path

The independent dominance polynomial of book graphs can be found using the formula in the lemma that follows.

Lemma 3.1: For $n \geq 2$, the IUDP of the book graph, A_n, is as follows:

$E_u(A_n,y) = (n-1)y^n + y^{n+1}$

Proof

The book graph is denoted by A_n. It is necessary to demonstrate that each IUDS on this graph has size n or $n+1$ and that each of these sets is accounted for precisely once in the assertion above. There are n triangles with a common edge in the book graph A_n. The vertices of $A_n$are denoted as $\{u_1, u_2, v_1, v_2, \ldots v_n, w_1, w_2, \ldots w_n\}$, where $\{u_1, u_2\}$ are the common vertices shared by all n triangles.

There are one of the following forms for the IUDS of each size:

Case 1: Select n vertices of the graph that contains at least one vertex from the set v_i and at least one vertex from the set w_i.

In this case, to gain better domination of the graph, one of the two nodes, belonging to either the $u_1$or u_2, should be taken out. Thereafter, the other vertices or nodes v_i and w_i, must be chosen in a way that the selected set of nodes is an independent set and covers the entire graph uphill. Thus, we choose $n-1$ vertices from the sets $\{v_i\}$ and $\{w_i\}$. Each time, for a particular triangle, we can pick v_i or w_i but not both can be selected from the same triangle. This provides $n-1$ options because one triangle in the kernel has to be dominated by the selected *u* vertex. This contributes to the term $(n-1)y^n$.

Case 2: Either choose all the subscripts $u_1$or u_2 and all the subscripts w_i or the subscripts v_i.

In this instance, if u_1 is selected, then all w_i must be selected (or if u_2 is selected then all v_i). This ensures that each set is independent of each other and also each triangle dominates the other. Since there is only a single way to pick all the vertices in a set u_1 or all the vertices in the set u_2, and then add them to y^{n+1}, the term increases.

Combining these two cases yields the IUDP of the book graph, A_n:

$E_u(A_n,y) = (n-1)y^n + y^{n+1}$

Corollary 3.2: $E_u(A_n,y)$ is unimodal for any natural integer n since it contains only real zeros.

A formula for the independent dominance polynomial of generalized book graphs can be found in the following theorem.

Theorem 3.3: For *n* = 2 and *m* = 3, the IUDP of the generalized book graph $A_{n,m}$is as follows:

$E_u(A_{n,m},y) = (n-1)y^n E_u(T_{m-4},y) + y^{n+1} E_u(T_{m-5},y) + (y^2 + y^{n+1}) E_u(T_{m-6},y)$,

where for all $j \leq 0$, $E_u(T_j,y) = 1$.

Proof

Let the generalized book graph be represented by $A_{n,m}$. It is sufficient to demonstrate that each IUDS in the previous assertion is taken into consideration precisely once. The generalized book graph $A_{n,m}$is made up of a path graph $T_{m-2}$with vertices $\{u_i : 1 \leq i \leq m-2\}$, and each $m-2$is connected to *n* triangles $\{v_i w_i : 1 \leq i \leq n\}$.

There are one of the following forms for the IUDS of each size:

Case 1: Sets that include at least one $v_i$and at least one w_i:

Choose at least one v_i and at least one w_i from the list of choices. From the induced route graph T_{m-4} produced by $\{u_2, u_3, \ldots u_{m-3}\}$, the remaining IUDS is selected. Because each triangle contributes one vertex, guaranteeing the set is independent and uphill dominant, this explains the phrase $(n-1)y^n E_u(T_{m-4},y)$.

Case 2: Sets that include $u_2$and all w_i (or u_{m-3} and all v_i):

The remaining set is selected from the induced path graph T_{m-6}, which is constructed by $\{u_4, \ldots u_{m-3}\}$, (or $\{u_4, \ldots u_{m-5}\}$). This explains the expression $y^{n+1} E_u(T_{m-6}, y)$.

Case 3: Sets that include u_1(or u_{m-2}) and all w_i (or all v_i):

The remaining set of values is selected from the induced path graph T_{m-5}, which is constructed by $\{u_3, \ldots u_{m-3}\}$ (or $\{u_2, \ldots u_{m-4}\}$). This accounts for the term $y^{n+1} E_u(T_{m-5}, y)$.

Case 4: Sets that include both $u_1$and u_{m-2}:

The rest of the set is chosen from the induced path graph T_{m-6}formed by $\{u_3, u_4 \ldots u_{m-4}\}$. The expression $y^2 E_u(T_{m-6}, y)$ is explained by this.

When these instances are combined, we get:

$E_u(A_{n,m}, y) = (n-1)\, y^n E_u(T_{m-4}, y) + y^{n+1} E_u(T_{m-5}, y) + (y^2 + y^{n+1}) E_u(T_{m-6}, y)$

Given that $\delta_{iup}(T_n) = [\frac{n}{2}]$ for any $n \geq 1$, we may arrive at the following conclusion using Theorem 3.3.

Corollary 3.4: The independent uphill domination number of the generalized book graph $A_{n,m}$for$n \geq 2$ and $m \geq 3$ is given by:

$E_u(A_{n,m}) = min\{max\{n, n + \lceil\frac{m-4}{2}\rceil\}, max\{n+1, n+1+\lceil\frac{m-5}{2}\rceil\}, max\{2, 2+\lceil\frac{m-6}{2}\rceil\}\}$

Theorem 3.5:

(a) For $n \geq 2$, the IUDP of the friendship graph, f_n, is given by

$E_u(f_n, y) = y + (2y)^n$

(b) For $n \geq 2$, and $q \geq 4$ the IUDP of the generalized book graph, $A_{q,n,}$ is given by

$E_u(A_{q,n}, y) = y(E_u(T_{q-3}, y))^n + ny(E_u(T_{q-3}, y)\, E_u(T_{q-1}, y))^{n-1}$

Proof

(a) Friendship graph f_n

Consider the friendship graph f_n, which is made up of *n* cycles that share a vertex *v*.

Case 1:$v \in S$

The remaining vertices in S must form IUDS for the cycles minus $X[v]$ (the neighborhood of v) if v is included in the independent uphill dominating set S. Since v is dominant, $X[v]$ does not need any additional vertices in $X[v]$ in S. There are $(2y)^n$ ways to choose the dominating sets in the n cycles without v and its neighbors, leading to the term $(2y)^n$.

Case 2:$v \notin S$

Every cycle must contain at least one vertex in S if v is absent from *S*. In this instance, the term y for each cycle results from each cycle's ability to independently choose one vertex to dominate the cycle. This scenario simplifies to y as it doesn't rely on the selection of v.

Combining these cases, we have:

$E_u(f_n, y) = y + (2y)^n$

(b) Generalized Book Graph $A_{q,n}$

Consider the generalized book graph $A_{q,n}$, which is made up of n copies of the cycle C_q, each sharing a similar edge (or vertex).

Case 1:$v \in S$

If v is included in S, we need to form IUDS for the remaining vertices minus $X[v]$ in each cycle. Each such set can be treated independently as in the path graph T_{q-3}. The term $(E_u(T_{q-3}, y))^n$ is added, in which *v* is the single dominating vertex and the other vertices form IUDS in each cycle.

Case 2:$v \notin S$

At least one neighbor of v (let's assume y) must be in *S* if v is not included in *S*. In this instance, every cycle's other vertex aside from y must form IUDS. This contributes to the term $ny(E_u(T_{q-3}, y)\, E_u(T_{q-1}, y))^{n-1}$, where y is chosen as a dominating vertex in one cycle and the remaining vertices form IUDS in the remaining cycles.

Combining these cases, we have:

$E_u(A_{q,n}, y) = y(E_u(T_{q-3}, y))^n + ny(E_u(T_{q-3}, y)\, E_u(T_{q-1}, y))^{n-1}$

Hence, for the independent uphill dominance of the generalized book graphs A_n, $A_{q,n}$ and friendship graph f_n, we have Theorem 3. 3 and theorem 3. 5 thus providing an orderly and well-structured polynomial that seems to encode some recursive and graphic properties. By including independence, it gives a more restricted and specific outlook in opposition to the general uphill dominating polynomial. The independent self-assertion is the other dominance polynomial, which being independent is comparatively simpler and broader than the former with no hierarchical inclinations.

3.2 Independent uphill dominating polynomials for certain binary operations on graphs

Theorem 3.6 : Let $G \cong C_q \circ K_s$ be a coronagraph with $qs+q$ vertices. Then,

$IUP(G,y) = y^{qs} + qy^{qs+1}$

Proof

An IUDS of size qu in a corona graph G with $qs+q$ vertices is a set in which every vertex from C_q is connected to exactly one vertex from each K_s. Such a set can only be formed in one way. This contributes $1 \times y^{qs}$. We take the IUDS of size qs and add one more vertex from the cycle graph C_q for the phrase y^{qs+1}. Given that C_q has r vertices, there are r ways to choose this extra vertex. This contributes $q \times y^{qs+1}$. When we add together each of these contributions, we obtain:

$IUP(G,y) = y^{qs} + qy^{qs+1}$

Corollary 3.7 : Let $G \cong C_q \circ K_1$ be a corona graph with $2r$ vertices. Then

$IUP(G,y) = y^{q} + qy^{q+1}$

The following conclusion is a generalization of Theorem 3.6.

Theorem 3.8 : If $GG \cong D \circ K_s$ for any nontrivial connected graph D with q vertices, then $IUP(G,y) = y^{qs} + qy^{qs+1}$.

Theorem 3.9 : Let G be a windmill graph $W_g(q,k)$ with vertices equal to $k(q-1)+1$. Then the polynomial of independent uphill dominance $IUP(G,y)$ is provided by:

$IUP(G,y) = \sum_{t=k}^{k(q-2)+1} \left(\sum u_1 + u_2 + \dots + u_{k=t} \binom{q-1}{u_1} \binom{q-1}{u_2} \dots \binom{q-1}{u_k} \right) y^t$

Proof

Let G represent a windmill graph with w as its centre vertex. The minimum uphill dominance set is $\delta_{iup}(G) = k$. One vertex from each copy of K_q must be present in a minimal uphill dominant set, with the exception of the center vertex w. This means that we have $k(q-1)$ vertices from the k copies of K_q, omitting w. To form an IUDS of size $t = k + j$, where $j = 1, 2, \dots, k(q-2)+1$, we need to select u_i vertices from each U_i (the set of vertices in the j-th copy of K_q excluding the central vertex w, as well as we must ensure that $\sum_{i=1}^{k} u_i = 1$ with $u_i \geq 1$ for all i). The number of ways to select u_i vertices for each i[th] copy of K_q is provided by $\binom{q-1}{u_i}$. All k copies of K_q need the selection of this option. The IUDP $IUP(G,y)$ can be expressed as the sum over all possible sizes t of IUDS:

$IUP(G,y) = \sum_{t=k}^{k(q-2)+1} \left(\sum u_1 + u_2 + \dots + u_{k=t} \binom{q-1}{u_1} \binom{q-1}{u_2} \dots \binom{q-1}{u_k} \right) y^t$

This accounts for all possible ways to form IUDS of size t by selecting u_i vertices from each U_i while ensuring no two selected vertices are adjacent and every vertex not in the set is dominated. Hence, the theorem is proved.

Theorem 3.10 : Let G be a firefly graph $T_{q,t,f}$ with vertices $q,\ t,\ f \geq 0$, $n = 2q+2t+f+1$, and $\delta_{iup}(G) = q+t+f = a$. The polynomial of independent uphill dominance $IUP(G,y)$ can be obtained as follows:

$IUP(G,y) = \sum_{h=a}^{n} iup(G,h) y^h$, where $iup(G,h) = 2^q \sum_{j=1}^{h-a+1} \binom{t+1}{j}$ $for\ h\ in\ a \leq h \leq n.$

Proof

Let G be a firefly graph with n vertices, defined as follows: $\delta_{iup}(G) = q+t+f = a$; let $q,\ t,$ and k be the vertices of G. Initially, we split the vertices of G into $q+2$ sets, and we designate $u$as the shared vertex in G. Presume that the vertices of the first triangle excluding u are identified in $S_1 \subset V(G)$, suggesting that S_1 has two vertices, each of degree two. The vertices of the second triangle excluding u are identified in $S_2 \subset V(G)$, and so on for any S_i, where $1 \leq i \leq q$. Let the subset $S_{q+1} \subset V(G)$ include u along with the t vertices of the branch routes next to u. This indicates that S_{q+1} has a cardinality of $t+1$. In last, all of the leaf vertices of G are precisely $t+f$, and $S_{q+2} \subset V(G)$ includes them all. Any IUDS of G must include at least one vertex from each of S_i and all of the vertices of S_{q+2}. Consequently, for $iup(G, a)$, we have:

$iup(G,a) = \left(\prod_{i=1}^{q} \binom{2}{1} \right) \binom{t+1}{1}$

$= 2^q(t+1)$

For $iup(G, a+1)$,

$iup(G, a+1) = \left(\prod_{i=1}^{q} \binom{2}{1} \right) \binom{t+1}{1} + \left(\prod_{i=1}^{q} \binom{2}{1} \right) \binom{t+1}{2}$

$2^q \left(\binom{t+1}{1} + \binom{t+1}{2} \right)$

For $iup(G, a+2)$,

$iup\,(G\ ,\ a+2) = 2^q\left(\binom{t+1}{2}+\binom{t+1}{3}\right)$

In general, for $iup\,(G\ ,\ h)$, where $a+2 \leq h \leq n$, we follow the same condition to find:

$iup\,(G\ ,\ h) = 2^q\sum_{j=1}^{h-a+1}\binom{t+1}{j}$

Finally, the IUDP is,

$IUP\,(G,y) = \sum_{h=a}^{n} iup(G,h)y^h$

Where each $iup(G,h)$ is computed as shown above. This completes the proof.

Theorem 3.11 : Let G be a grid graph with qu vertices as well as $q,\ u\ \geq 4$ and $G \cong P_q \times P_u$. Then,

$IUP\,(G,y) = y^4 + (qu-4)\,y^5 + \cdots + (qu-4)\,y^{qu-1} + y^{qu}$

Proof

Let G be a grid graph with vertices of qu and $q,\ u\ \geq 4$. An IUDS of size four $(k = 4)$ in a grid graph G with qu vertices is a collection of four vertices that are independent and dominate the entire graph. Because of the structure of the grid graph, there is only one set in which four vertices can cover every other vertex in a unique way due to their adjacency. Therefore, the polynomial y^4 is contributed by precisely one IUDS of size 4. For $k = 5$, the graph's fifth vertex can be any of the remaining $qu-4$ vertices or any of the previous four vertices that combine to generate a unique IUDS of size 4. Thus, the number of IUDS of size 5 is $qu-4$, adding $(qu-4)\,y^5$ to the polynomial. For $6\ \leq k\ \leq qu$, the same logic applies to sets of size k. There are $qu-4$ possibilities for each of the remaining vertices in the unique IUDS of size 4, one for each extra vertex added. Consequently, the number of IUDS of size k for k from 5 to $qu-1$ is $qu-4$, adding $(qu-4)\,y^k$ to the polynomial for each k and so on. Thus, we get

$IUP\,(G,y) = y^4 + (qu-4)\,y^5 + \cdots + (qu-4)\,y^{qu-1} + y^{qu}$

The proof is done.

The main difference between the above theorems from uphill domination [8] and independent domination [13] is the condition of adjacency and constraints. Independent uphill dominance is possible when the sets meet the uphill condition (often related to some order or hierarchy in the graph) and a specific criterion of independence that does not allow vertices with proximity. The sets in an uphill dominance can even partially overlap since they only need to satisfy the uphill criterion that has been presented above. The sets in the independent dominance do not necessarily meet the uphill condition to be independent. Independent uphill domination has many benefits when compared to other types of domination since it combines the stability of uphill domination with the efficiency of independent domination. It makes sure that the chosen vertices adhere to the imposed hierarchical characteristics of the graph while not being adjacent to each other or just independent from each other. The enhancement of coverage is made achievable by this approach due to the fact that vertices are spread out across the graph; this minimizes possible weaknesses as well as enhances resource allocation. Also, there is a concern about independent uphill domination which is more relevant in systems that put into consideration both hierarchies and independence such as in the communication networks and the organization's planning among others. This way it manages the constraints necessary and offers a stable solution for applications, which have to be orderly and served without intersections.

4 Conclusion

In this study, we define the $iup(G,i)$ function: the quantity of independent uphill dominant sets of size i, and formulate new theory, in polynomial context, for the description of the most important features of the graphs. Moreover, this work assists enriching the knowledge base on dominance polynomials and diversifying the ability to investigate the connection between independence, domination, and path restrictions in graph theory. While the properties held by the $iup(G,i)$ function provide great thoughts based on the independent uphill dominant sets and the dominance polynomials, the research contains its shortcomings. They can be applied to only a few specific graph families; it can act computationally complex in large graphs. There are problems with generalizing the results and the fact that a number of studies are theoretical and lack real-life applications, more research and real-life experimentation is needed. Possible future work with the $iup(G,i)$, function is the generalization of its scope with other types of graphs, improving its time complexity, and testing on graphs after applying different transformations. Continuing the marriage between the theory founded on empirical evidence and the translation of those findings into easy-to-use user tools will only add further value.

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