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RESEARCH ARTICLE

Double Laplace Transform Method for Solving Fractional Fourth-Order Partial Integro-Differential Equations with Weakly Singular Kernel

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Abstract

Objectives: To investigates the solutions of fourth order partial integro-differential equations with high-order non-integer derivatives and weakly singular kernels. **Methods:** Weakly singular kernels present challenges in both analytical and numerical treatments due to their intricate behaviour near singular points. In this article, we introduce a novel approach utilizing the double Laplace transform method to effectively address these challenges. **Findings:** By solving a series of precise and understandable examples, the double Laplace transform clearly transforms the fractional partial integro-differential equation into an algebraic equation that can be solved easily. **Novelty:** The double Laplace transform is an effective instrument for solving fractional partial integro-differential equations, which are commonly used in fields such as physics, fluid mechanics, gas dynamics, and signal processing.
Mathematics Subject Classifications: 35R09, 35R11, 44A10, 44A30.

Keywords: Fractional PIDE; Weakly singular kernel; Double Laplace transform; Inverse double Laplace transform; Exact solution

1 Introduction

The Partial Integro-Differential Equation (PIDE), is a prominent tool for mathematically simulating dynamic systems. Fractional PIDEs with respect to time variable have been increasingly well known and significant because of their appealing applications in a number of significant phenomena in physics, hydrology, chemistry, fluid mechanics, gas dynamics, and signal processing. It is crucial to develop analytical and numerical techniques for solving time-fractional PIDE's. In fact, most of the time it is challenging to find precise solutions. As a result, efforts have been made to suggest an analytical technique for determining the precise solutions to these equations.

Recently, many authors have been given notable attention regarding the study of fractional PIDE /PDE by using single Laplace transform together with the resolvent kernel method [1], residual power series method [2], differential transform method [3] and Sumudu transform method [4]. The Sinc-collocation approach joined with the double exponential transformation [5,6] is used to solve variable order fractional integro-partial differential equations. The spectral collocation technique [7] is employed to obtain numerical approximation of the nonlinear time-fractional PIDE with a weakly singular kernel using new basis functions based on shifted first-kind Chebyshev polynomials.

For numerical solution of fractional PIDE with a weakly singular kernel, the sinc collocation method [8] and Galerkin spectral and finite difference methods [9] are constructed. The Haar Wavelets collocation method [10] and Moving Least Squares (MLS) method [11] are implemented for computing numerical solution of the time-fractional linear/nonlinear PIDE.

We propose double Laplace transform (DLT) method to obtain an exact answer for the following fourth order fractional PIDE of the form:

$$\begin{aligned}&\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}+\int_0^t (t-\tau)^{-b}\frac{\partial^4 v(x,\tau)}{\partial x^4}d\tau = K(x,t),\ \ x,\ t\geq 0,\\ &m-1<\alpha\leq m,\quad m\in N,\ \ 0<b<1.\end{aligned} \tag{1.1}$$

Related with Equation (1.1), we consider the initial conditions

$$\frac{\partial^j v(x,0)}{\partial t^j}=f_j(x),\ \ j=0,1,2,\ldots,m-1,\ \ m=[\alpha]+1, \tag{1.2}$$

and the boundary conditions

$$\frac{\partial^k v(0,t)}{\partial x^k}=g_k(t),\ \ k=0,1,2,3. \tag{1.3}$$

Here $[\alpha]$ indicate the greatest integer less than or equal to α.

Physical Interpretation of Equation (1.1) , Equation (1.2) and Equation (1.3) :

The time-fractional derivative $\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}$ implies that the evolution of the quantity $v(x,t)$ depends not only on the local rate of change of v but also on its past behavior in a non-local manner. The spatial interaction term $\int_0^t (t-\tau)^{-b}\frac{\partial^4 v(x,\tau)}{\partial x^4}d\tau$ signifies that the fourth spatial derivative of v at past times τ influences the current state at time t. The source term $K(x,t)$ represents an external forcing which may represent applied forces, boundary conditions or other external influences on the system.The initial conditions given by Equation (1.2) offer important details about the system's initial state, including the distribution of its derivatives with respect to time. Equation (1.3) specifies the behaviour of the quantity $v(x,t)$ at the boundary $x=0$ for different orders of spatial derivatives up to the third order.

Significance of a weakly singular kernel:

The weakly singular kernel $(t-\tau)^{-b}$ in the integral introduces a long-range interaction effect. This kernel is weakly singular because it approaches infinity as $t-\tau$ approaches zero, but not as rapidly as a strongly singular kernel would. Weakly singular kernels often indicate non-local interactions or memory effects in physical systems. In this case, it suggests that the influence of past spatial configurations decays as time progresses, but not instantaneously. The parameter b determines the strength of this decay.

In this article, for the first time, we use double Laplace transform method to solve weakly singular kernel fourth order fractional PIDE of the type Equation (1.1) subject to Equations (1.2) and (1.3) . The following is how the remaining paper is organized: Section 2 describes the double Laplace transform method and shows how to deduce the general analytical solution for Equation (1.1) step by step. A few numerical examples are also provided in Section 3, which shows how double Laplace transform is applied to a fourth order PIDE with a time fractional derivative. Also in section 4, some conclusions are drawn.

2 Methodology

The double Laplace transform method for obtaining the general solution $v(x,\tau)$ of Equation (1.1) is developed in this segment.

Transforming Equation (1.1) by double Laplace transform, we obtain

$$s^{\alpha}\bar{v}(p,s)-\sum_{j=0}^{m-1}s^{\alpha-1-j}L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\}+L_xL_t\left\{\int_0^t (t-\tau)^{-b}\frac{\partial^4 v(x,\tau)}{\partial x^4}d\tau\right\}=\bar{K}(p,s), \tag{2.1}$$

where $\bar{K}(p,s)=L_xL_t\left[K(x,t)\right].$

Convolution Theorem

If DLT of $f(x,t)$ and single Laplace transform of $g(t)$ are given by $L_x L_t[f(x,t)] = \bar{f}(p,s)$ and $L_t\{g(t)\} = \bar{g}(s)$ then

$$L_x L_t\{g(t) * f(x,t)\} = L_x L_t\left\{\int_0^t g(t-y) f(x,y)\,dy\right\} = \bar{g}(s)\bar{f}(p,s), \tag{2.2}$$

where $g(t) * f(x,t) = \int_0^t g(t-y) f(x,y)\,dy$.

Using convolution theorem Equation (2.2), we get

$$s^{\alpha}\bar{v}(p,s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} + L_t\left[t^{-b}\right] L_x L_t\left\{\frac{\partial^4 v(x,t)}{\partial x^4}\right\} = \bar{K}(p,s). \tag{2.3}$$

The fourth order partial derivatives double Laplace transform formula is

$$L_x L_t\left\{\frac{\partial^4 f(x,t)}{\partial x^4}\right\} = p^4\bar{f}(p,s) - \sum_{j=0}^{3} p^{3-j} L_t\left\{\frac{\partial^j f(0,t)}{\partial x^j}\right\}, \tag{2.4}$$

Using Equation (2.4) in Equation (2.3), we obtain

$$s^{\alpha}\bar{v}(p,s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} + \frac{\Gamma(1-b)}{s^{1-b}}\left[p^4\bar{v}(p,s) - \sum_{k=0}^{3} p^{3-k} L_t\left\{\frac{\partial^k u(0,t)}{\partial x^k}\right\}\right] = \bar{K}(p,s). \tag{2.5}$$

Further, transforming Equations (1.2) and (1.3) by single Laplace transform, we get

$$L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} = \bar{f}_j(p),\; L_t\left\{\frac{\partial^k v(0,t)}{\partial x^k}\right\} = \bar{g}_k(s),\; j = 0,1,2,\ldots,m-1,\; and\; k = 0,1,2,3. \tag{2.6}$$

By putting Equation (2.6) in Equation (2.5) and we obtain by simplifying

$$\bar{v}(p,s) = \frac{1}{\left[s^{\alpha} + \frac{\Gamma(1-b)}{s^{1-b}} p^4\right]}\left[\bar{K}(p,s) + \sum_{j=0}^{m-1} s^{\alpha-1-j}\bar{f}_j(p) + \frac{\Gamma(1-b)}{s^{1-b}}\sum_{k=0}^{3} p^{3-k}\bar{g}_k(s)\right]. \tag{2.7}$$

We obtain the solution of Equation (1.1) by using the inverse DLT to Equation (2.7).

$$v(x,t) = L_x^{-1} L_t^{-1}\left[\frac{1}{\left[s^{\alpha} + \frac{\Gamma(1-b)}{s^{1-b}} p^4\right]}\left[\bar{K}(p,s) + \sum_{j=0}^{m-1} s^{\alpha-1-j}\bar{f}_j(p) + \frac{\Gamma(1-b)}{s^{1-b}}\sum_{k=0}^{3} p^{3-k}\bar{g}_k(s)\right]\right]. \tag{2.8}$$

In this case, we assume that the inverse DLT of Equation (2.8) exists.

3 Results and discussion

In this part, we provide examples to exhibit the applicability of the previous technique.

Example 3.1: By changing $b = \frac{1}{2}$, $m = 1$ and $K(x,t) = x^4 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 32t^{\frac{3}{2}} + 48\sqrt{t}$ in Equation (1.1),

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} + \int_0^t (t-\tau)^{-\frac{1}{2}} \frac{\partial^4 v(x,\tau)}{\partial x^4} d\tau = x^4 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 32t^{\frac{3}{2}} + 48\sqrt{t},\;\; x,\, t \geq 0,\;\; 0 < \alpha \leq 1, \tag{3.1}$$

subject to

$$v(x,0) = f_0(x) = x^4,\; v(0,t) = g_0(t) = 0,\; \frac{\partial v(0,t)}{\partial x} = g_1(t) = 0,\; \frac{\partial^2 v(0,t)}{\partial x^2} = g_2(t) = 0,\; \frac{\partial^3 v(0,t)}{\partial x^3} = g_3(t) = 0. \tag{3.2}$$

Transforming Equation (3.2) by single Laplace transform, we get

$$\bar{f}_0(p) = \frac{24}{p^5},\ \ \bar{g}_k(s) = 0,\ \ k = 0,1,2,3. \tag{3.3}$$

Transforming $K(x,t)$ by double Laplace transform, we get

$$\bar{K}(p,s) = \frac{24}{p^5}\frac{1}{s^{2-\alpha}} + 32\frac{1}{p}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}} + 48\frac{1}{p}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}}. \tag{3.4}$$

Substituting above in Equation (2.8), we get solution of Equation (3.1):

$$v(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{\left[s^\alpha + \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\quad p^4\right]}\left[\frac{24}{p^5}\frac{1}{s^{2-\alpha}} + \quad 32\frac{1}{p}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} \quad + 48\frac{1}{p}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} \quad + s^{\alpha-1}\frac{24}{p^5}\right]\right]. \tag{3.5}$$

Computing, we get desired solution:

$$v(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{\left[s^\alpha + \sqrt{\frac{\pi}{s}}p^4\right]}\left[\left(\frac{1}{s^2} + \frac{1}{s}\right)\frac{24}{p^5}\left[s^\alpha + \sqrt{\frac{\pi}{s}}p^4\right]\right]\right] = L_x^{-1}L_t^{-1}\left[\left(\frac{1}{s^2} + \frac{1}{s}\right)\frac{24}{p^5}\right] = (t+1)x^4. \tag{3.6}$$

Figure 1 shows the exact solution $v(x,t) = (t+1)x^4$ utilizing a variety of values of $0 \le x \le 1$ and $0 \le t \le 15$.

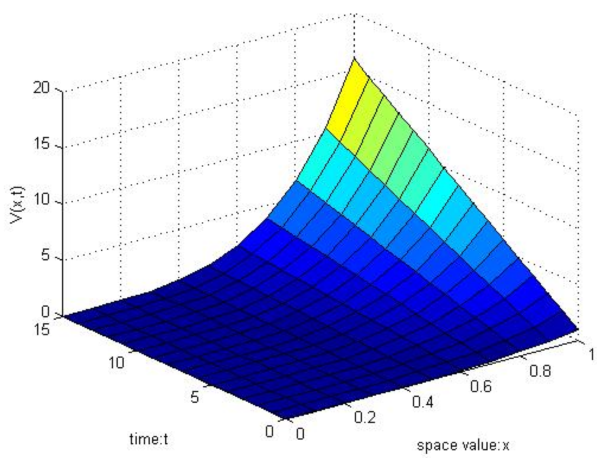

Fig 1. Exact solution $(\mathbf{t} + \mathbf{1})x^{\mathbf{2}}$

Example 3.2: By changing $b = \frac{1}{2}$, $m = 2$ and $K(x,t) = 2x^4\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{128}{5}t^{\frac{5}{2}}$ in Equation (1.1),

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\partial^4 v(x,\tau)}{\partial x^4}d\tau = \quad 2x^4\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \quad + \frac{128}{5}t^{\frac{5}{2}} \quad ,\ x,\ t \ge 0 \quad ,\ 1 < \alpha \le 2 \tag{3.7}$$

subject to

$$v(x,0)=f_0(x)=0,\ \frac{\partial v(x,0)}{\partial t}=f_1(x)=0,\ \ v(0,t)=g_0(t)=0,\ \ \frac{\partial v(0,t)}{\partial x}=g_1(t)=0,$$
$$\frac{\partial^2 v(0,t)}{\partial x^2}=g_2(t)=0,\ \ \frac{\partial^3 v(0,t)}{\partial x^3}=g_3(t)=0. \tag{3.8}$$

Transforming Equation (3.8) by single Laplace transform, we get

$$\bar{f_0}(p)=\bar{f_1}(p)=0,\ \ \bar{g_k}(s)=0,\ \ k=0,1,2,3. \tag{3.9}$$

Transforming $K(x,t)$ by double Laplace transform, we get

$$\bar{K}(p,s)=2\frac{24}{p^5}\frac{1}{s^{3-\alpha}}+\frac{128}{5}\frac{1}{p}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}}. \tag{3.10}$$

Substituting above in Equation (2.8), we get solution of Equation (3.7):

$$v(x,t)=L_x^{-1}L_t^{-1}\left[\frac{1}{\left[s^{\alpha}+\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\quad p^4\right]}\left[2\frac{24}{p^5}\frac{1}{s^{3-\alpha}}\quad+\frac{128}{5}\frac{1}{p}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}}\right]\right]. \tag{3.11}$$

Computing, we get desired solution:

$$v(x,t)=L_x^{-1}L_t^{-1}\left[\frac{\Gamma(5)}{p^5}\frac{\Gamma(3)}{s^3}\right]=x^4t^2. \tag{3.12}$$

Figure 2 shows the exact solution $v(x,t)=x^4t^2$ utilizing a variety of values of $0\le x\le 1$ *and* $0\le t\le 15$.

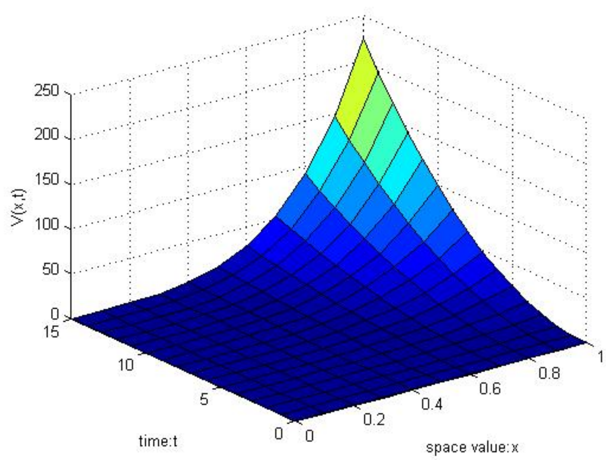

Fig 2. Exact solution x^4t^2

Example 3.3: By changing $b=\frac{1}{2}$, $m=3$ and $K(x,t)=2\left(16t^{\frac{7}{2}}+105\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right)sinx$ in Equation (1.1),

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}+\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\partial^4 v(x,\tau)}{\partial x^4}d\tau=\ 2\left(16t^{\frac{7}{2}}\quad+105\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right)sinx,\quad x,\,t\ge 0,\quad 2<\alpha\le 3, \tag{3.13}$$

subject to

$$f_0(x) = f_1(x) = f_2(x) = 0,\ \ g_0(t) = 0,\ \ g_1(t) = 35t^3,\ \ g_2(t) = 0,\ \ g_3(t) = -35t^3. \tag{3.14}$$

Transforming Equation (3.14) by single Laplace transform, we get

$$\bar{f}_j(p) = 0,\ \ \bar{g}_0(s) = \bar{g}_2(s) = 0,\ \ \bar{g}_1(s) = 35\frac{\Gamma(4)}{s^4},\ \ \bar{g}_3(s) = -35\frac{\Gamma(4)}{s^4}. \tag{3.15}$$

Transforming $K(x,t)$ by double Laplace transform, we get

$$\bar{K}(p,s) = 2\left(16\frac{\Gamma\left(\frac{9}{2}\right)}{s^{\frac{9}{2}}} + 105\frac{1}{s^{4-\alpha}}\right)\frac{1}{p^2+1}. \tag{3.16}$$

Substituting above in Equation (2.8), we get solution of Equation (3.13):

$$v(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{\left[s^{\alpha} + \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\ p^4\right]}\left[2\left(16\frac{\Gamma\left(\frac{9}{2}\right)}{s^{\frac{9}{2}}} + 105\frac{1}{s^{4-\alpha}}\right)\frac{1}{p^2+1} + \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\ p^2 35\frac{\Gamma(4)}{s^4} - \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\ 35\frac{\Gamma(4)}{s^4}\right]\right]. \tag{3.17}$$

Computing, we get desired solution:

$$v(x,t) = L_x^{-1}L_t^{-1}\left[35\frac{\Gamma(4)}{s^4}\frac{1}{(p^2+1)}\right] = 35t^3 sinx. \tag{3.18}$$

Figure 3 shows the exact solution $v(x,t) = 35t^3 sinx$ utilizing a variety of values of $0 \le x \le 1\ and\ 0 \le t \le 15$.

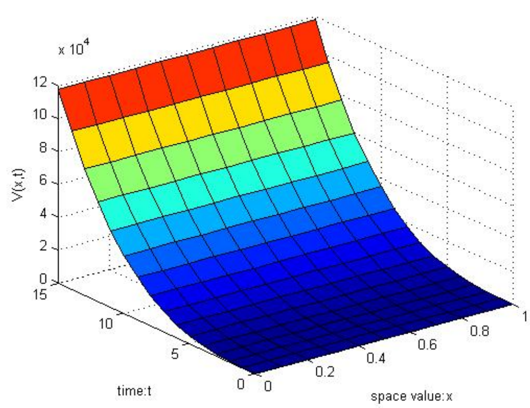

Fig 3. Exact solution $35t^3 sinx$

4 Conclusion

The double Laplace transform is an efficacious technique for solving fourth-order fractional PIDEs with weakly singular kernel. It is concluded that double Laplace Transform improves computing efficiency and solution accuracy by transforming the original issue into a system of algebraic equations. Because of its adaptability, it can address a wide range of problems from several scientific and technological areas, which makes it an invaluable resource for researchers and professionals.

In addition to its existing benefits future studies might include looking into its application to higher-order fractional PIDEs and finding ways to improve its computing efficiency. Moreover, incorporating adaptive strategies will enhance this technique's versatility to address more complex boundary and initial conditions.

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