INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Received: 21-05-2024

Accepted: 25-06-2024

Published: 23-08-2024

Citation: Shanmuganandham P, Ramachandran R (2024) A Study on the Sum of 'm+1' Consecutive Cullen Numbers. Indian Journal of Science and Technology 17(34): 3496-3501. https://doi.org/10.17485/IJST/v17i34.1696

* **Corresponding author**.

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

A Study on the Sum of 'm+1' Consecutive Cullen Numbers

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Abstract

Objectives: To discover new formulae for the matrix form of the sum of m+1 Cullen numbers. This is an effort to explain the Recursive Matrix form formula and a few of its uses. **Methods**: Conclusions derived from theorems and the matching matrix representations of them. Additionally, a few apps are offered. The terminology of Cullen Numbers are utilised to demonstrate the main theorems. Additionally, algebraic simplifications and mathematical computations are used to derive the findings. **Findings:** A lemma is used to construct the formula for the Sum of m+1 consecutive Cullen Numbers. Here, the recursive and sum forms are obtained in matrix form. It also obtains the matrix representation of the sums of m+1 successive Cullen Numbers. Several intriguing correlations among Special Numbers, Cullen Numbers, and Carol Numbers are provided in the application section. **Novelty:** New formulas are obtained for the analysis. Research has recently discovered matrix representation with its recursive versions. Furthermore, there are many relationships between Cullen Numbers along with other peculiar numbers.

Keywords: Cullen Numbers; Carol Numbers; Woodall Numbers; Sum of Cullen Number

1 Introduction

The general term of the Cullen Number is $C(\tau) = \tau(2^{\tau}) + 1$. The first ten terms of the sequence [1] are 1,3,9,25,65,161,385,897,2049,4609….. There has been discussion on the generalized Cullen numbers in linear recurrence sequences of greater order [2]. Many authors have searched for special properties of Cullen numbers and their generalizations. We take into consideration [3] as their greatest common divisor and [4–6] as the results of primality for these numbers. The problem of finding Cullen numbers that correspond to other known sequences has attracted a lot of attention in the last 20 years.

Advancement of Number Theory: Every study that contributes to the understanding of number sequences like Cullen numbers enriches number theory as a whole. It expands the toolkit of mathematicians working in this field and may inspire further investigations into related sequences or problems.

Potential for Applications: While Cullen numbers themselves may not have direct practical applications, the methods and insights developed through their study can be applied to other areas of mathematics and even beyond, such as cryptography or algorithm design.

Educational Value: Results from such studies can be incorporated into educational curricula, enriching the learning experience of students and demonstrating the depth and beauty of mathematical research.

Contributions to Mathematical Literature: Publishing findings on the sums of m+1 consecutive Cullen numbers adds to the body of mathematical literature, providing a reference point for future researchers and enriching the understanding of these sequences for the broader mathematical community.

The mathematical equation for the summation of squares of Woodall numbers ('m') and its matrix version have been studied and communicated. Additionally, there are some explicit associations among Woodall numbers along with various unique numbers[7].

In 2022, researchers examined generalized Woodall sequences as a result of extended Woodall, modified Cullen, Woodall, and Cullen sequences, along with matrices and identities associated with these sequences[8].

Soykan et al. explored closed forms of the sum formulas for generalized Hexanacci numbers with the particular case of summation formulae of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences[9]. Two specific cases—the Mersenne and Mersenne-Lucas sequences—have been addressed by the introduction of the generalized Mersenne number[10]. They spoke about the summation formulae for various sequences, generating functions, and Simson formulas, including Binet's formulas. The closed sorts of the sum formulae pertaining to squares of generalized Tribonacci numbers were examined by Soykan et al. The additional third-order recurrence relations and the summing formulae for the Tribonacci, Tribonacci-Lucas, Padovan, and Perrin numbers were examined as special instances[11].

Cletus Emmanuel studies the Carol number first. Shanmuganandham et al. (2023) looked at the sum of squares of "m+1" consecutive Carol numbers and their matrix representation[12]. It has also been proved that the sum of squares of *n* consecutive Carol numbers possesses a recursive structure. The number of Cullens has not been extensively studied thus far. By exploiting matrix relations, this article determines the indistinguishable identities for Fermat, Triangular, Heptagonal, Mersenne, and Nasty numbers. Additionally, the sum of the recursive matrix of m+1 successive Cullen numbers as well as an integer solution is shown.

While the immediate practical applications of studying sums of consecutive Cullen numbers may not be obvious, the study contributes significantly to the advancement of mathematics, computational methods, and educational resources. Its impact lies in deepening theoretical understanding, fostering algorithmic developments, and inspiring future research across various mathematical disciplines. Thus, the study of sums of m+1 consecutive Cullen numbers serves as a cornerstone in the broader landscape of mathematical exploration and application.

2 Methodologies

- **Definitions 2.1:** $C(\tau) = \tau(2^{\tau}) + 1$ is the Cullen number for all $\tau \geq 1$.
- **Definitions 2.2:** $(Fer)_{\tau} = 2^{\tau} + 1$ is the Fermat number for all $\tau \geq 1$.
- **Definitions 2.3:** Ca(τ)=$(2^{\tau}+1)^2 + 2$ is the carol number for all $\tau \geq 1$.
- **Definitions 2.4 :** H (7,τ)=$\tau(2\tau - 1)$ is the Heptagonal number for all $\tau \geq 1$.
- **Definitions 2.5:**$(Tri)_{\tau} = \frac{\tau(\tau+1)}{2}$ is the Triangular number for all $\tau \geq 1$.
- **Definitions 2.6 :** $\omega(\tau)$= $\tau 2^{\tau} - 1$ is the Woodall number for all $\tau \geq 1$.
- **Definitions 2.7:** $M_{\tau} = 2^{\tau} - 1$ is the Mersenne number for all $\tau \geq 1$.
- **Definition 2.8:** $(n)_{\tau}$=6(α^2) is the Nasty number for all $\tau \geq 1$.

3 Results and Discussion

Theorem 3.1:

For all $\tau \geq 1$, the following equality holds,

$\sum_{k=0}^{m}(\tau+k)$=$\sum_{k=0}^{m}[2(k-1)L-(\tau-2)]$X+$\frac{2}{k}(Tri)m$, where X=$2^{\tau}$; K=$\tau$+k and L=$2^{k}$.

The following lemma is needed to fulfill the derivable pace in the proof of the main theorem.

Lemma: For all $\tau \geq 1, \sum_{k=0}^{m} c(\tau+k)$ =$\tau(2^{\tau}) + \tau(2^{\tau})\sum_{k=1}^{m} 2^{k} + 2^{\tau}\sum_{k=1}^{m} 2^{k} + \sum_{k=0}^{m} 1$.

Proof of the lemma: As in Section I, this lemma has two different types of proof. One is direct from the definition and another is by induction on k=0, 1, 2,….m.

Proof (1): By the definition, C(τ)=$\tau(2^{\tau}) + 1$

Therefore, C$(\tau+1)$= $(\tau+1)\left(2^{\tau+1}\right)+1$=$2\tau\left(2^{\tau}\right)+2\left(2^{\tau}\right)+1$
Similarly, $C(\tau+2)$= $(\tau+2)\left(2^{\tau+2}\right)+1$=$4\tau\left(2^{\tau}\right)+2\left(2^{2}\right)\left(2^{\tau}\right)+1$
We extend this result up to $(m+1)^{th}$ term, at last, we arrive at the result as follows,
$C(\tau+m)$= $(\tau+m)\left(2^{\tau+m}\right)+1$=$2^{m}\tau\left(2^{\tau}\right)+m\left(2^{m}\right)\left(2^{\tau}\right)+1$
Adding all the above terms,

$$\sum_{k=0}^{m} C(\tau+k)=\tau\left(2^{\tau}\right)\left(2+2^{2}+2^{3}+\cdots+2^{k}\right)+\left(1.2+2.2^{2}+3.2^{2}+\cdots+k.2^{2}\right)\left(2^{\tau}\right)+(k+1) \tag{1}$$

Hence, $\sum_{k=0}^{m} C(\tau+k)$= $\tau\left(2^{\tau}\right)+\ \tau\left(2^{\tau}\right)\sum_{k=1}^{m} 2^{k}+2^{\tau}\sum_{k=1}^{m} 2^{k}+\sum_{k=0}^{m} 1$
Proof (2): ConsiderEquation (1)**,** By the induction on k, we can give the alternate proof of the lemma.
When k=1, the Equation (1) becomes, C$(\tau+1)$= $(\tau+1)\left(2^{\tau+1}\right)+1$=$2\tau\left(2^{\tau}\right)+2\left(2^{\tau}\right)+1$
Therefore, the lemma is true When k=1. By induction, we assume that the lemma is true for k=m-1.
i.e,

$$\sum_{k=1}^{m-1} C(\tau+k-1)=\tau\left(2^{\tau}\right)\left(2+2^{2}+2^{3}+\cdots+2^{k-1}\right)+\left(1.2+2.2^{2}+3.2^{2}+\cdots+(k-1).2^{k-1}\right)\left(2^{\tau}\right)+k \tag{2}$$

By replacing k by k+1 inEquation (2) and by using induction hypothesis, we proved lemma.
Hence, $\sum_{k=0}^{m} C(\tau+k)$= $\tau\left(2^{\tau}\right)+\ \tau\left(2^{\tau}\right)\sum_{k=1}^{m} 2^{k}+2^{\tau}\sum_{k=1}^{m} 2^{k}+\sum_{k=0}^{m} 1$
Proof of the main theorem:
In RHS ofEquation (2)**,** first term consists of a geometric series whose common ratio is 2.
Therefore, is sum being equal to $2^{k}-1$. Then we can easily prove the sum of the series $\left(1.2+2.2^{2}+3.2^{2}+\cdots+(k-1).2^{k-1}\right)$ is equal to $\left[(k-1)2^{k+1}+2\right]$.
Equation (2) modified as follows,
$\sum_{k=0}^{m} C(\tau+k)$=$\tau\left(2^{\tau}\right)2^{k+1}-\tau\left(2^{\tau}\right)+\left[(k-1)2^{k+1}\right]\left(2^{\tau}\right)+2^{\tau+1}+(k+1)$
Rearranging the terms, $\sum_{k=0}^{m} C(\tau+k)=\sum_{k=0}^{m}\left\{\left[(\tau+k-1)2^{k+1}-\tau+2\right]\left(2^{\tau}\right)+(k+1)\right\}$
By suitable substitution, we have $\sum_{k=0}^{m}(\tau+k)$ = $\sum_{k=0}^{m}[2(k-1)L-(\tau-2)]$X + $\frac{2}{K}(Tri)_{m}$
Where X=2^{τ}; K=$\tau+k$and L= 2^{k}
Hence, the theorem.

Corollary 3.2:

The sum of consecutive m+1 Woodall numbers is
For all $\tau\geq 1$, the following equality holds,
$\sum_{k=0}^{m}(\tau+k)$=$\sum_{k=0}^{m}[2(k-1)L-(\tau-2)]$ X+$(Tri)\,m$, where X=2^{τ}; K=τ+k and L=2^{k}.

Matrix form of sum of consecutive Cullen numbers

Theorem 3.3:

For all $\tau\geq 1$, and k=1, 2, 3,…m, $C(\tau)$= $\begin{bmatrix} P & (k-2)P & 1\\ Q & (k-1)Q & 1\\ R & kB & 1\end{bmatrix}\begin{bmatrix} X_1\\ X_2\\ 1\end{bmatrix}$

Where P=2^{K-2}, Q=2^{K-1}, and R=2^{k}, X_1=$\tau\left(2^{\tau}\right)$, X_2=$\left(2^{\tau}\right)$ and C(τ)= $\begin{bmatrix} c(\tau+k-2)\\ c(\tau+k-1)\\ c(\tau+k)\end{bmatrix}$.

Proof: By the definition, C(τ)=$\tau 2^{\tau}+1$
Also, the next Cullen number is, C$(\tau+1)$= $\tau\left(2^{\tau+1}\right)+\ \left(2^{\tau+1}\right)+1$ = $2\tau\left(2^{\tau}\right)+2\left(2^{\tau}\right)+1$
In the similar way we have, C$(\tau+2)$ = $\tau\left(2^{\tau+2}\right)+\ \left(2^{\tau+2}\right)+1$ = $4\tau\left(2^{\tau}\right)+2.2^{2}\left(2^{\tau}\right)+1$
The above three equations can revise in matrix form as $C_{1,1}(\tau)$=$D_{1,1}(\tau)\,Y_{1,1}(\tau)$, Where

$$C_{1,1}(\tau)=\begin{bmatrix} c(\tau)\\ c(\tau+1)\\ c(\tau+2)\end{bmatrix}\ D_{1,1}(\tau)=\begin{bmatrix} 1 & 0 & 1\\ 2 & 2 & 1\\ 4 & 8 & 1\end{bmatrix}\text{And}Y_{1,1}(\tau)=\begin{bmatrix} \tau\left(2^{\tau}\right)\\ 2^{\tau}\\ 1\end{bmatrix}$$

Indistinguishable $C_{1,2}(\tau)$ and $D_{1,2}(\tau)$ can be acquired by put back (τ) by $(\tau+1)$ in the respective $C_{1,2}(\tau)$ matrix, keeping $Y_{1,1}(\tau)$ is fixed and $D_{1,2}(\tau)=\begin{bmatrix}2 & 2 & 1\\ 4 & 8 & 1\\ 8 & 24 & 1\end{bmatrix}$. In this fashion $C_{1,k-1}(\tau)$ and $D_{1,k-1}(\tau)$ gets the following matrix depiction as $C_{1,k-1}(\tau) = D_{1,k-1}(\tau)Y_{1,1}(\tau)$ Where

$C_{1,k-1}(\tau)=\begin{bmatrix}c(\tau+k-2)\\ c(\tau+k-1)\\ c(\tau+k)\end{bmatrix}$ $D_{1,k-1}(\tau)=\begin{bmatrix}2^{k-2} & (k-2)2^{k-2} & 1\\ 2^{k-1} & (k-1)2^{k-1} & 1\\ 2^{k} & K2^{k} & 1\end{bmatrix}$ And$Y_{1,1}(\tau)=\begin{bmatrix}\tau(2^{\tau})\\ 2^{\tau}\\ 1\end{bmatrix}$

By the appropriate substitution for P=2^{K-2}, Q=2^{K-1}, and R=2^{k}, the matrix representation reduced to

C(τ) = $\begin{bmatrix}P & (k-2)P & 1\\ Q & (k-1)Q & 1\\ R & kB & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ 1\end{bmatrix}$

Where C(τ) = $\begin{bmatrix}c(\tau+k-2)\\ c(\tau+k-1)\\ c(\tau+k)\end{bmatrix}$

$Y_{1,1}(\tau)=\begin{bmatrix}X_1\\ X_2\\ 1\end{bmatrix}$

And X_1=$\tau(2^{\tau})$, X_2=(2^{τ}).

Hence, the theorem.

Corollary 3.4:

The matrix form of sums of m+1 consecutive Woodall number is,

For all $\tau \geq 1$, and k=1, 2, 3,…m, $\omega(\tau)$= $\begin{bmatrix}P & (k-2)P & 1\\ Q & (k-1)Q & 1\\ R & kB & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ 1\end{bmatrix}$

Where P=2^{k-2} , Q=2^{k-1} , and R=2^{k} , X_1=$\tau(2^{\tau})$, X_2=(2^{τ}) and $\omega(\tau)$= $\begin{bmatrix}\omega(\tau+k-2)\\ \omega\quad(\tau+k-1)\\ \omega(\tau+k)\end{bmatrix}$.

Recursive Matrix form:

Theorem 3.5:

For all n$\geq$ 0 the recursive coefficient matrix has the form

C (n) = $\begin{bmatrix}a_{11}^{n} & na_{12}^{n} & -a_{13}\\ a_{21}^{n+1} & (n+1)a_{22}^{n+1} & -a_{23}\\ a_{31}^{n+2} & (n+2)a_{32}^{n+2} & -a_{33}\end{bmatrix}$ Where $a_{13}=a_{23}=a_{33}=1$.

Proof: Consider the initial coefficient matrix, from $D_{1,1}(\tau)$ of theorem 3.3 as given below,

C (n=1) = $\begin{bmatrix}1 & 0 & 1\\ 2 & 2 & 1\\ 4 & 8 & 1\end{bmatrix}$ = $\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{23} & a_{33}\end{bmatrix}$

The elements of the next order matrix C (n=2) depend on the earlier order elements in C (n=1) excluding the elements of the last column.

C (n=2) = $\begin{bmatrix}4 & 8 & 1\\ 8 & 24 & 1\\ 16 & 64 & 1\end{bmatrix}$ = $\begin{bmatrix}a_{11}^{2} & 2a_{12}^{2} & a_{13}\\ a_{21}^{2} & 3a_{22}^{3} & a_{23}\\ a_{31}^{4} & 4a_{32}^{4} & a_{33}\end{bmatrix}$

In this way the elements of n^{th} order matrix C (n) can be written as follows,

C (n) = $\begin{bmatrix}a_{11}^{n} & na_{12}^{n} & a_{13}\\ a_{21}^{n+1} & (n+1)a_{22}^{n+1} & a_{23}\\ a_{31}^{n+2} & (n+2)a_{32}^{n+2} & a_{33}\end{bmatrix}$

Where $a_{13}=a_{23}=a_{33}=1$, then the theorem concludes.

Corollary 3.6:

The recursive matrix form of the sum of Woodall numbers is,

For all n$\geq$ 0 the recursive coefficient matrix has the form

W (n) = $\begin{bmatrix} a_{11}^{n} & na_{12}^{n} & a_{13} \\ a_{21}^{n+1} & (n+1)a_{22}^{n+1} & a_{23} \\ a_{31}^{n+2} & (n+2)a_{32}^{n+2} & a_{33} \end{bmatrix}$ Where $a_{13} = a_{23} = a_{33} = 1$.

4 Applications

- **Statement 1**

The sum of three consecutive Cullen number is a linear combination of Woodall number: Mersenne number and Fermat number.

Proof

C(τ) + C$(\tau+1)$ + C$(\tau+2)$ = $2^{\tau}\,(\tau+2\tau+2+4\tau+8)+3$

=$7\tau 2^{\tau}+8\,(2^{\tau})+3$

=$7(\tau 2^{\tau}-1+1)+8\,(2^{\tau}-1+1)+3$

=7W$(\tau)+7+8(Mer)_{\tau}+8+3$

= 7W$(\tau)+8M\,(\tau)+F\,(4)+1$

- **Statement 2**

C$(\tau)\,W\,(\tau)-\tau^{2}M\,(\tau)\,F\,(\tau)=\frac{2}{\tau}\,(\tau+1)\,(Tri)_{(\tau-1)}$

Proof

C$(\tau)\,W\,(\tau)$= $\tau^{2}2^{2\tau}-1$

C$(\tau)\,F\,(\tau)$= $2^{2\tau}-1$

$\tau^{2}M\,(\tau)\,F\,(\tau)=\tau^{2}2^{2\tau}-\tau^{2}$

C$(\tau)\,W\,(\tau)-\tau^{2}M\,(\tau)\,F\,(\tau)=\tau^{2}-1$

=2$(Tri)_{(\tau-1)}+\frac{1}{\tau}\tau\,(\tau-1)$

= 2(1+ $\frac{1}{\tau}$) $(Tri)_{(\tau-1)}$

C$(\tau)\,W\,(\tau)-\tau^{2}M\,(\tau)\,F\,(\tau)=\frac{2}{\tau}\,(\tau+1)\,(Tri)_{(\tau-1)}$

- **Corollary**

1. C (τ)+ C$(\tau+1)$+ C$(\tau+2)$ -7W$(\tau)-8M(\tau)\equiv 0(Mod\,2)$
2. C(τ)+ C$(\tau+1)$+ C$(\tau+2)$-7W$(\tau)-8M(\tau)+6$is a Nasty Number.

5 Conclusion

By using a lemma matrix form for the sum, a formula for the sum of m+1 consecutive Cullen numbers is constructed, and recursive versions are obtained here. It also obtains the matrix form of Sums of m+1 consecutive Cullen Numbers. Several intriguing correlations involving Special Numbers, Cullen Numbers, and Carol Numbers are provided in the application section. One may expand this study to include other unique numbers.

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