INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS
Received: 22-07-2024
Accepted: 02-08-2024
Published: 20-08-2024

A Finite Capacity Finite Source Single Server Markovian Queueing System with Discouraged Arrivals Retention of Reneged Customers and Controllable Arrival Rates with Feedback

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Citation: Bebittovimalan A, Thiagarajan M (2024) A Finite Capacity Finite Source Single Server Markovian Queueing System with Discouraged Arrivals Retention of Reneged Customers and Controllable Arrival Rates with Feedback. Indian Journal of Science and Technology 17(33): 3456-3466. https://doi.org/10.17485/IJST/v17i33.2354

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Funding: DST-FIST, Government of India

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: The purpose of this study is to (i) introduce a finite capacity finite source of interdependent queueing model and retention of reneged customers with feedback controllable arrival rates, (ii) compute the mean quantity of clients within the system and determine the anticipated waiting time of those clients, (iii) deal with the model descriptions, steady-state equations, and characteristics, which are expressed in terms and (iv) analyze the probabilities of the queueing system and its characteristics with numerical verification of the obtained results. (v) Observe graphical representation for L_s & W_s.**Methods :** The Poisson process is used to manage the arrival rates through quicker and slower arrival rates while delivering the input. The service additionally offers an exponential distribution. The service is supplied by the server using FCFS. In this article, two types of models are used: one that represents the system's conditions by the number of units present in the queue, and another that distributes all probabilities based on the speed of arrival using this concept. Then, the steady-state probabilities are computed using a recursive approach. **Findings:** This paper explores the average number of clients using the system and the expected number of clients in the system, based on the probability derived from the steady-state calculation. Little's formula is used to derive the expected waiting period of the clients in the system. **Novelty:** There are articles connected to the finite capacity of failed service in functioning and malfunctioning, but this takes the initiative to provide a link in connection with the rates of the controllable arrivals and interdependency in the arrival and service processes. This is the new model of M/M/1/K/N controllable arrival rates derived.

Mathematics Subject allocation: 60K25, 68M20, 90B22.

Keywords: Finite Capacity; Finite Source; Interdependent Controllable; Arrival and Service rates; Reverse balking; Single Server; Bivariate Poisson process and feedback

1 Introduction

Queueing models are used in a variety of real-world contexts to facilitate the effective design, analysis, and assessment of various technological systems and their behavior, including client waiting times, estimated customer counts, etc. Queueing models are more practical for analyzing situations in real life. These explanations assist researchers in carefully choosing the conditions and maintaining an appropriate estimate to accurately determine the stream.

When a consumer enters a line and later chooses to depart it without being served is called reneging. A customer who may choose to come again to the queue until the service is completed is said to be feedback. A reneged customer has a chance of being induced to remain in the queue for more services with a probability of q_1, and he has a chance of leaving the queue without obtaining the service with a probability of p_1. A customer again goes to the queue due to unsatisfied previous service with the probability q_2 and he is satisfied by having a chance of acquiring the service p_2.

Lines filled with dejected newcomers have computer programs that process jobs. When the system is utilized often, job submissions are discouraged, and arrivals are represented as a Poisson process with a state-dependent arrival rate. Morse considered dismay in which, based on a declining arrival rate the inverse exponential law. Haight[(1)] was the first to examine queuing with reneging. He waited in line and focused on how to make a decision that would have the least number of negative consequences. Ancker and Gafarian have studied "M/M/1/N queueing systems with balking and reneging and performing its steady state analysis".

Srinivasa Rao. K and Shobha. T has discussed "the M/M/1 interdependent queueing model with a controllable arrival rate". Srinivasan and Thiagarajan have analyzed "the M/M/1/K interdependent queueing model with controllable arrival rates and M/M/C/K/N interdependent queueing model with controllable arrival rates, balking, reneging and spares". Kapodistria is educated in "a single server Markovian queue with impatient customers and considered the situations, where customers abandoned the system concurrently".

Kumar and Sharma have utilized "the M/M/1/N queueing system with retention of reneged customers". Rakesh Kumar and S K Sharma have analysed "a single server Markovian queueing system with discouraged arrivals and retention of reneged customers". Thiagarajan and Premalatha have elaborated on "a single server Markovian queueing system with discouraged arrivals retention of reneged customers and controllable arrival rates".

Nithya, N., Anbazhagan, N., Amutha, S., Jeganathan, K., Park, G. C., Joshi, G. P., & Cho, W. (2023)[(2)] from this paper, how the controlled arrivals on the retrial queueing inventory system which is an essential interruption for vacation server. Sasikala, S., and V. Abinaya[(3)] helps to know "The M/M/C/N Interdependent Inter Arrival Queueing Model with Controllable Arrival Rates, Reverse Balking and Impatient Customers" by using the old method for Reverse Balking. Sindhu, S., Achyutha Krishnamoorthy, and Dmitry Kozyrev[(4)] "On Queues with Working Vacation and Interdependence in Arrival and Service Processes" how vacation works an independence in process of arrival and service. Yue, Rui, Guangchuan Yang, Yichen Zheng, and Zong Tian[(5)] "Capacity estimation at all-way stop-controlled intersections considering pedestrian crossing effects" helped to know about pedestrian crossing effect on the finite capacity.

Yang, Yaoqi, Bangning Zhang, Daoxing Guo, Weizheng Wang, Jiangtian Nie, Zehui Xiong, Renhui Xu, and Xiaokang Zhou[(6)] "Stochastic geometry-based age of information performance analysis for privacy preservation-oriented mobile crowdsensing" helps to know about "performance analysis for privacy preservation-oriented mobile crowdsensing". Mahanta, Snigdha, Nitin Kumar, and Gautam Choudhury[(7)] "An analytical approach of Markov modulated Poisson input with feedback queue and repeated service under N-policy with setup time" helps to find the feedback queue repeated service using N-policy with time. Xie, Qian, Jiayi Wang, and Li Jin[(8)] help to know this feedback-controlled queue "Strategic Defense of Feedback-Controlled Parallel Queues against Reliability and Security Failures".

Gayathri, S., and G. Rani[(9)] "An FM/M/C interdependent stochastic feedback arrival model of transient solution and busy period analysis with interdependent catastrophic effect" helps to know about the "feedback arrival model of transient solution using catastrophic effect". Cherfaoui, Mouloud, Amina Angelika Bouchentouf, and Mohamed Boualem[(10)] help the "Modelling and simulation of Bernoulli feedback queue with general customers". Rahim, K. H., and M. Thiagarajan[(11)] "M/M (a, b)/1 Model of Interdependent Queueing with Controllable Arrival Rates" helps to find multi servers using controllable arrival rates.

Therasal SN, Thiagarajan M,[(12)] "Poisson Input and Exponential Service Time Finite Capacity Interdependent Queueing Model with Breakdown and Controllable Arrival Rates" helps to know about the breakdown with Controllable Arrival Rates. Chettouf, A., Bouchentouf, A.A. & Boualem, M. A[(13)] "Markovian Queueing Model for Telecommunications Support Center with Breakdowns and Vacation Periods" helps to know the telecommunication center for Breakdowns and Vacation Periods. M. Seenivasan, R. Abinaya,[(14)] "Markovian queueing model with single working vacation and catastrophic, Materials Today" studied using the paper for a single server working vacation with catastrophic and Materials. Ahmed, F. I., Alemu, S. D., & Tilahu, G. T[(15)] "A Single Server Markovian Queue with Single Working Vacation, State-Dependent Reneging and Retention of Reneged Customers" helps to know about Vacation and Retention of Reneged Customers.

R. Kalyanaraman and A. Sundaramoorthy[16] “A Markovian single server working vacation queue with server state-dependent arrival rate, balking and with breakdown” helps to know about balking and with breakdown using a single server working vacation queue. Dudin, A. N., Chakravarthy, S. R., Dudin, S. A., & Dudina, O. S.[17] “Queueing system with server breakdowns and individual customer abandonment” helps to know about “server breakdowns and individual customer abandonment”. Mahanta, Snigdha, Nitin Kumar, and Gautam Choudhury[18] ”An analytical approach of Markov modulated Poisson input with feedback queue and repeated service under N-policy with setup time” studied about “an analytical approach of Markov modulated Poisson input with feedback queue”. Subhapriya SP, Thiagarajan M[19] “M/M/1/K Loss and Delay Interdependent Queueing Model with Vacation and Controllable Arrival Rates” helps to find “Loss and Delay Interdependent Queueing Model with Vacation in M/M/1/K”. Subhapriya, S. P., & Thiagarajan, M[20] “M/M/1 Interdependent Queuing Model with Vacation and Controllable Arrival Rates” helps to know about “Vacation and Controllable Arrival Rates in M/M/1”.

Keerthana, M., Sangeetha, N. & Sivakumar, B. (2023)[21] “Optimal service rates of a queueing inventory system with finite waiting hall, arbitrary service times and positive lead times” helps to know about the optimal service in a finite capacity waiting hall with positive time inventory system. Basak, A., Choudhury, A. Bayesian (2024)[22] “Estimation of finite buffer size in single server Markovian queuing system” helps with the estimation value of buffering in finite capacity in Markovian Queue for the single server.

Bebittovimalan A, Thiagarajan M. (2024)[23] “An Infinite Capacity Single Server Markovian Queueing System With Discouraged Arrivals Retention Of Reneged Customers and Controllable Arrival Rates With Feedback” helps to elaborate the finite capacity in the finite source of this paper in the feedback queueing model.

2 Methodology

2.1 Description of the model

For this model, one server with infinite capacity of the system retention of reneged customers with discouraged arrivals derived. Customer arrivals are independent, identically, and exponentially distributed with the rate according to the Poisson flow of rate λ_0 and λ_1, which relies on the number of customers in the system and service time μ.

According to the arrival order, customers are waiting in the queue. Each consumer must wait a specific amount of time before receiving service after joining the queue. If it doesn’t start by then, he will get anxious and either leave the line without receiving service with probability p_1 or stay in line to receive with probability q_1.

The consumer can either exit the system with probability p_2 or join it as a feedback customer with a probability q_2 after receiving the service. The reneging time with feedback follows an exponential distribution with parameter ξ.

$$P[X_1(t)] = x_1,\, X_2(t) = x_2$$

$$= e^{-(\lambda_i+\mu-\epsilon)t} \sum_{j=0}^{min(x_1,x_2)} \frac{(\epsilon t)^j\,[(\lambda_i-\epsilon)\,t]^{x_1-j}\,[(\mu-\epsilon)\,t]^{x_2-j}}{j!\,(x_1-j)!\,(x_2-j)!}$$

where $x_1, x_2 = 0,1,2,\ldots$ $\lambda_i > 0$, $i = 0,1$; $\mu > 0$, $0 \le \epsilon < min(\lambda_i, \mu)$, $i = 0,1$

$X_1(t), X_2(t)$ represents the arrival process and the service process respectively.

- When the system size increases to R from below the arrival rate which was λ_0 until $R-1$, decrease to λ_1 and remains as λ_1 for subsequent upward movement of the system size.
- When the system size decreases to $r(0 < r < R)$ from above, the arrival rate which was λ_1 until $r+1$, increase to λ_0 and remain as λ_0 both for the subsequent downward movement to 0 and for the subsequent upward movement up to $R-1$. This process is repeated.

Clearly the process $\{N(t); t > 0\}$, where $N(t)$ is the system size in time t, is a discrete Markov chain with state space $\{0,1,2,\ldots,r-1,r,r+1,\ldots R-1,R,R+1,\ldots K\}$.

The postulates of the model are:

- The probability that there is one arrival with retention of reneged customers and no service completion during a small interval of time h, at state n, when the system is in a faster rate of arrival is $(N-n)\,\frac{(\lambda_0-\epsilon)h}{n+1} + o(h)$.
- The probability that there is one arrival with retention of reneged customers and no service completion during a small interval of time h, at state n, when the system has in slower rate of arrival is $(N-n)\,\frac{(\lambda_1-\epsilon)h}{n+1} + o(h)$.

- The probability that there is no arrival with retention of reneged customers and no service completion during a small interval of time h, at state n, when the system is in a faster rate of arrival is $1-\left[(N-n)\frac{(\lambda_0-\epsilon)h}{n+1}+(\mu-\epsilon)\right]h+o(h)$.
- The probability that there is no arrival with retention of reneged customers and no service completion during a small interval of time h, at state n, when the system is in a slower rate of arrival is $1-\left[(N-n)\frac{(\lambda_1-\epsilon)h}{n+1}+(\mu-\epsilon)\right]h+o(h)$.
- The probability that there is no arrival with retention of reneged customers and one service completion during a small interval of time h, at state n, when the system either in faster or slower rate of arrival is $(\mu-\epsilon)h+o(h)$.
- The probability that there is one arrival with retention of reneged customers and one service completion during a small interval of time h, at state n, when the system either faster or slower rate of arrival is $\epsilon h+o(h)$.

3 Results and Discussion

3.1 Steady State Probabilities

We observe that $n=0,1,2,\ldots,r-1,r$ exists for $P_n(0)$, $n=r+1,r+2,\ldots R-2,R-1$ exists for both $P_n(0)$ & $P_n(1)$, $n=R,R+1,\ldots,K$ exists for $P_n(1)$. When $n>K$, $P_n(0)=P_n(1)=0$ exists.

$$-N(\lambda_0-\epsilon)+p_2(\mu-\epsilon)P_1(0)=0 \tag{3.1.1}$$

$$\begin{aligned}&-\left[(N-n)\frac{(\lambda_0-\epsilon)}{(n+1)}+(p_2(\mu-\epsilon)+(n-1)\xi p_1]\right]P_n(0)+(N-n+1)\frac{(\lambda_0-\epsilon)}{n}P_{n-1}(0)+\\&[p_2(\mu-\epsilon)+n\xi p_1]P_{n+1}(0)=0 \qquad 1\le n\le r-1\end{aligned} \tag{3.1.2}$$

$$\begin{aligned}&-\left[(N-r)\frac{(\lambda_0-\epsilon)}{(n+1)}+(p_2(\mu-\epsilon)+(r-1)\xi p_1]\right]P_r(0)+(N-r+1)\frac{(\lambda_0-\epsilon)}{n}P_{r-1}(0)+\\&[p_2(\mu-\epsilon)+n\xi p_1]P_{r+1}(0)+(p_2(\mu-\epsilon)+n\xi p_1]P_{r+1}(1)=0\end{aligned} \tag{3.1.3}$$

$$\begin{aligned}&-\left[(N-n)\frac{(\lambda_0-\epsilon)}{(n+1)}+(p_2(\mu-\epsilon)+(n-1)\xi p_1]\right]P_n(0)+(N-n+1)\frac{(\lambda_0-\epsilon)}{n}P_{n-1}(0)+\\&[p_2(\mu-\epsilon)+n\xi p_1]P_{n+1}(0)=0\end{aligned} \tag{3.1.4}$$

$$-\left[(N-R+1)\frac{(\lambda_0-\epsilon)}{(R)}+(p_2(\mu-\epsilon)+(R-2)\xi p_1]\right]P_{R-1}(0)+(N-R+2)\frac{(\lambda_0-\epsilon)}{R-1}P_{R-2}(0)=0 \tag{3.1.5}$$

$$-\left[(N-r-1)\frac{(\lambda_1-\epsilon)}{(r+2)}+(p_2(\mu-\epsilon)+r\xi p_1]\right]P_{r+1}(1)+[p_2(\mu-\epsilon)+(r+1)\xi p_1]P_{r+2}(1)=0 \tag{3.1.6}$$

$$\begin{aligned}&-\left[(N-n)\frac{(\lambda_1-\epsilon)}{(n+1)}+(p_2(\mu-\epsilon)+(n-1)\xi p_1]\right]P_n(1)+(N-n+1)\frac{(\lambda_1-\epsilon)}{n}P_{n-1}(1)+\\&[p_2(\mu-\epsilon)+n\xi p_1]P_{n+1}(1)=0 \qquad n=r+2,r+3,\ldots,R-1\end{aligned} \tag{3.1.7}$$

$$\begin{aligned}&-\left[(N-R)\frac{(\lambda_1-\epsilon)}{(R+1)}+(p_2(\mu-\epsilon)+(R-1)\xi p_1]\right]P_R(1)+(N-R+1)\frac{(\lambda_1-\epsilon)}{R}P_{R-1}(1)+\\&(N-R-1)\frac{(\lambda_o-\epsilon)}{R}P_{R-1}(0)+[p_2(\mu-\epsilon)+R\xi p_1]P_{R+1}(1)=0\end{aligned} \tag{3.1.8}$$

$$\begin{aligned}&-\left[(N-n)\frac{(\lambda_1-\epsilon)}{(n+1)}+(p_2(\mu-\epsilon)+(n-1)\xi p_1]\right]P_n(1)+(N-n+1)\frac{(\lambda_1-\epsilon)}{n}P_{n-1}(1)+\\&[p_2(\mu-\epsilon)+n\xi p_1]P_{n+1}(1)=0 \qquad n=R+1,R+2,\ldots,K-1\end{aligned} \tag{3.1.9}$$

$$-[p_2(\mu-\epsilon)P_K(1)+\frac{(\lambda_1-\epsilon)}{K}P_{K-1}=0 \tag{3.1.10}$$

From Equation (3.1.1) we get,

$$P_1(0)=N\left(\frac{\lambda_0-\epsilon}{p_2(\mu-\epsilon)}\right)P_0(0)$$

Let

$$S_6=\frac{\lambda_0-\epsilon}{p_2(\mu-\epsilon)},\, T_6=\frac{\lambda_1-\epsilon}{p_2(\mu-\epsilon)}\, and\, E_6=\frac{\xi p_1}{p_2(\mu-\epsilon)}$$

$$\Rightarrow P_1(0)=NS_6P_0(0)$$

Using the above result in Equation (3.1.2) we get,

$$P_2(0)=\frac{N(N-1)}{2!}\frac{S_6^2}{[1+E_6]}P_0(0)$$

And recursively we get,

$$P_n(0)=\frac{(N)_n}{(r+1)!}\frac{(S_6)^{r+1}}{\prod_{l=0}^{n-1}[1+lE_6]}P_0(0),\qquad n=1,2,\ldots,r \tag{3.1.11}$$

Where $(N)_n=N(N-1)(N-2)\ldots(N-n+1)$.

Using Equation (3.1.11) in Equation (3.1.3) we get,

$$P_{r+1}(0)=\frac{N(N-1)}{2!}\frac{(S_6)^{r+1}}{\prod_{l=0}^{r}[1+lE_6]}P_0(0)-P_{r+1}(1)$$

Using the above result and Equation (3.1.11) in Equation (3.1.4) we get,

$$P_n(0)=\frac{(N)_n}{n!}\frac{(S_6)^n}{\prod_{l=0}^{n-1}[1+lE_6]}P_0(0)-\frac{P_{r+1}(1)}{\prod_{l=0}^{n-1}[1+lE_6]} \left[\frac{(s_6)^{n-r-1}}{nP_{n-r-1}}(N-r-1)_{n-r-1}+\frac{(s_6)^{n-r-2}}{nP_{n-r-2}}(N-r-2)_{n-r-2}[1+rE_6]+\cdots+[1+rE_6][1+[r+1)E_6]\ldots[1+(n-2)E_6]\right] \tag{3.1.12}$$

Using Equation (3.1.12) in Equation (3.1.5) we get,

$$P_{r+1}(1)=\frac{(N)_{R-1}[N-(R-1)]\frac{(S_6)^r}{r!}\frac{1}{\prod_{l=0}^{r}[1+lE_6]}P_0(0)}{\left\{\frac{(s_6)^{R-r-1}}{nP_{R-r-1}}(N-r-1)_{R-r-1}+\frac{(s_6)^{R-r-2}}{nP_{R-r-2}}(N-r-2)_{R-r-2}[1+E_6]+\cdots+\frac{S_6}{R}(N-R+1)[1+rE_6]\ldots[1+(R-3)E_6]\right\}} \tag{3.1.13}$$

From Equation (3.1.6) we get,

$$P_{r+2}(1)=\frac{P_{r+1}(1)}{[1+(r+1)E_6]}\left[\frac{T_6}{(r+2)}(N-r-1)+[1+rE_6]\right]$$

Using Equation (3.1.13) in Equation (3.1.7) recursively we get,

$$P_n(1) = \frac{P_{r+1}(1)}{\prod_{l=r+1}^{n-1}[1+lE_6]} \left[\frac{(T_6)^{n-r-1}}{nP_{n-r-1}}(N-r-1)_{n-r-1} + \frac{(T_6)^{n-r-2}}{nP_{n-r-2}}(N-r-2)_{n-r-2}[1+rE_6] + \cdots + [1+rE_6][1+(r+1)E_6]\ldots[1+(n-2)E_6]\right] \quad (3.1.14)$$
$$n = r+1, r+2, \ldots R-1, R$$

Where $P_{r+1}(1)$ is given in Equation (3.1.13)

Using Equation (3.1.14) in Equation (3.1.8) we get,

$$P_{R+1}(1) = \frac{P_{r+1}(1)}{\prod_{l=r+1}^{R}[1+lE_6]} \left[\frac{(T_6)^{R-r}}{nP_{R-r}}(N-r-1)_{R-r} + \frac{(T_6)^{R-r-1}}{nP_{R-r-1}}(N-r-1)_{R-r-1}[1+rE_6] + \ldots + \frac{T_6}{R+1}(N-R)[1+rE_6][1+(r+1)E_6]\ldots[1+(R-2)E_6]\right] \quad (3.1.15)$$

Using the above result in Equation (3.1.9) and Equation (3.1.10), we recursively derive

$$P_n(1) = \frac{P_{r+1}(1)}{\prod_{l=r+1}^{n-1}[1+lE_6]} \left[\frac{(T_6)^{n-r-1}}{nP_{n-r-1}}(N-R-1)_{n-R-1} + \frac{(T_6)^{n-r-2}}{nP_{n-r-2}}(N-R-2)_{n-R-2}[1+rE_6] + \ldots + \frac{(T_6)^{n-R}}{nP_{n-R}}(N-R-1)_{n-R-1}[1+rE_6][1+(r+1)E_6]\ldots[1+(R-2)E_6]\right] \quad (3.1.16)$$

Where $P_{r+1}(1)$ is given in Equation (3.1.13)

Thus, from Equations (3.1.11), (3.1.12), (3.1.13), (3.1.14) and (3.1.15), we find that all the steady-state probabilities are expressed in terms of $P_0(0)$.

3.2 Qualities of the model

The probability that the system has in faster rate of arrival is

$$P(0) = \sum_{n=0}^{N} P_n(0)$$

$$P(0) = \sum_{n=0}^{r} P_n(0) + \sum_{n=r+1}^{R-1} P_n(0) + \sum_{n=R}^{N} P_n(0)$$

Since $P_n(0)$ exists only when $n = 0, 1, 2, \ldots, r-1, r, r+1, \ldots R-2, R-1$, we get

$$P(0) = P(0) = \sum_{n=0}^{r} P_n(0) + \sum_{n=r+1}^{R-1} P_n(0) \quad (3.2.1)$$

From Equation (3.1.11) and Equation (3.1.12) we get,

$$P(0) = \sum_{n=0}^{R-1}\left[\frac{(S_6)^n}{n!}\frac{(N)_n}{\prod_{l=0}^{n-1}[1+lE_6]}\right]P_0(0) + \sum_{n=r+1}^{R-1}\left[\frac{(S_6)^n}{n!}\frac{(N)_n}{\prod_{l=0}^{n-1}[1+lE_6]}\right]P_0(0)$$
$$-\sum_{n=r+1}^{R-1}\left[\frac{P_{r+1}(1)}{\prod_{l=r+1}^{n-1}[1+lE_6]}\right.$$
$$\left.\left[\frac{(s_6)^{n-r-1}}{nP_{n-r-1}}(N-r-1)_{n-r-1} + \frac{(s_6)^{n-r-2}}{nP_{n-r-2}}(N-r-2)_{n-r-2}[1+rE_6] + \cdots + [1+rE_6][1+(r+1)E_6]\ldots[1+(n-2)E_6]\right]\right]$$

Where $P_{r+1}(1)$ is given in Equation (3.1.13)

$$P(0)=\sum_{n=0}^{r}\left[\frac{(S_6)^n}{n!}\frac{(N)_n}{\prod_{l=0}^{n-1}[1+lE_6]}\right]P_0(0)-\sum_{n=r+1}^{R-1}\left[\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}[1+lE_6]}\left(\frac{A_6}{B_6}\right)\right]P_0(0) \tag{3.2.2}$$

Where

$$\begin{aligned}A_6 &= \frac{(s_6)^{n-r-1}}{nP_{n-r-1}}(N-r-1)_{n-r-1}+\frac{(s_6)^{n-r-2}}{nP_{n-r-2}}(N-r-2)_{n-r-2}[1+rE_6]+\ldots\\ &+[1+rE_6][1+(r+1)E_6]\ldots[1+(n-2)E_6]\end{aligned}$$

$$\begin{aligned}B_6 &= \frac{(s_6)^{R-r-1}}{nP_{R-r-1}}(N-r-1)_{R-r-1}+\frac{(s_6)^{R-r-2}}{nP_{R-r-2}}(N-r-2)_{R-r-2}[1+E_6]+\ldots\\ &+\frac{S_6}{R}(N-R+1)[1+rE_6]\ldots[1+(R-3)E_6]\end{aligned}$$

The probability that the system is in slower rate of arrival is

$$P(1)=\sum_{n=0}^{K}P_n(1)$$

$$P(1)=\sum_{n=0}^{r}P_n(1)+\sum_{n=r+1}^{R-1}P_n(1)+\sum_{n=R}^{K}P_n(1) \tag{3.2.3}$$

Since $P_n(1)$ exists only when $n=r+1,r+2,\ldots R-2,R-1\ldots K$ we get,

$$P(1)=\sum_{n=r+1}^{R}P_n(1)+\sum_{n=R+1}^{K}P_n(1) \tag{3.2.4}$$

From Equations (3.1.14) and (3.1.15) and Equation (3.2.4) we get

$$P(1)=\sum_{n=r+1}^{R}\left[\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}[1+lE_6]}\left(\frac{C_6}{B_6}\right)\right]P_0(0)+\sum_{n=r+1}^{R-1}\left[\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}[1+E_6]}\left(\frac{D_6}{B_6}\right)\right]P_0(0) \tag{3.2.5}$$

Where

$$\begin{aligned}C_6 &= \frac{(T_6)^{n-r-1}}{nP_{n-r-1}}(N-r-1)_{n-r-1}+\frac{(T_6)^{n-r-2}}{nP_{n-r-2}}(N-r-2)_{n-r-2}[1+rE_6]+\ldots\\ &+[1+rE_6][1+(r+1)E_6]\ldots[1+(n-2)E_6\end{aligned}$$

$$\begin{aligned}D_6 &= \frac{(T_6)^{K-r-1}}{KP_{K-r-1}}(N-R-1)_{n-R-1}+\frac{(T_6)^{K-r-2}}{nP_{K-r-2}}(N-R-2)_{n-R-2}(1+rE_6]+\ldots\\ &+\frac{(T_6)^{K-R}}{KP_{K-R}}(N-R-1)_{n-R}[1+rE_6][1+(r+1)E_6]\ldots[1+(n-2)E_6\end{aligned}$$

The probability $P_0(0)$ that the system is empty can be calculated from the normalizing condition

$$P(0)+P(1)=1$$

From Equation (3.2.2) and Equation (3.2.3) we get,

$$\begin{aligned}P(0)+P(1) &= \sum_{n=0}^{R-1}\left(\frac{(S_6)^n}{n!}\frac{(N)_n}{\prod_{l=0}^{n-1}[1+lE_6]}\right]P_0(0)-\sum_{n=r+1}^{R-1}\left(\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}[1+lE_6]}\left(\frac{A_6}{B_6}\right)\right]P_0(0)\\ &+\sum_{n=r+1}^{R}\left(\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}[1+lE_6]}\left(\frac{C_6}{B_6}\right)\right]P_0(0)+\sum_{n=r+1}^{R-1}\left(\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}[1+E_6]}\left(\frac{D_6}{B_6}\right)\right]P_0(0)\end{aligned}$$

$$P_0(0)=\frac{1}{1+\sum_{n=0}^{R-1}\left[\frac{(S_6)^n}{n!}\frac{(N)_n}{\prod_{l=0}^{n-1}(1+lE_6]}\right]-\sum_{n=r+1}^{R-1}\left[\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}(1+lE_6]}\left(\frac{A_6}{B_6}\right)\right]+\sum_{n=r+1}^{R}\left[\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}(1+lE_6]}\left(\frac{C_6}{B_6}\right)\right]+\sum_{n=R+1}^{K}\left[\frac{(S_6)^R}{R!}\frac{(N)_R}{\prod_{l=0}^{n-1}(1+E_6]}\left(\frac{D_6}{B_6}\right)\right]}$$

Now we calculate the expected number of customers in the system. Let L_s denote the average number of customers in the system that we have,

$$L_s = L_{s_0} + L_{s_1} \tag{3.2.6}$$

$$L_{s_0} = \sum_{n=o}^{r} nP_n(0) + \sum_{n=r+1}^{R-1} nP_n(0) \tag{3.2.7}$$

$$L_{s_1} = \sum_{n=r+1}^{R-1} nP_n(1) + \sum_{n=R}^{K} nP_n(1) \tag{3.2.8}$$

From Equation (3.2.7) and Equation (3.2.8) we get,

$$L_s = \sum_{n=0}^{R-1}\left[\frac{(S_6)^n}{n!}\frac{n(N)_n}{\prod_{l=0}^{n-1}[1+lE_6]}\right]P_0(0) - \sum_{n=r+1}^{R-1}\left[\frac{(S_6)^R}{R!}\frac{n(N)_R}{\prod_{l=0}^{n-1}[1+lE_6]}\left(\frac{A_6}{B_6}\right)\right]P_0(0)$$
$$+\sum_{n=r+1}^{R}\left[\frac{(S_6)^R}{R!}\frac{n(N)_R}{\prod_{l=0}^{n-1}[1+lE_6]}\left(\frac{C_6}{B_6}\right)\right]P_0(0) + \sum_{n=r+1}^{R-1}\left[\frac{(S_6)^R}{R!}\frac{n(N)_R}{\prod_{l=0}^{n-1}[1+E_6]}\left(\frac{D_6}{B_6}\right)\right]P_0(0)$$

Using Little's Formula the expected waiting time of the customer in the system is calculated as

$W_s = \frac{L_s}{\bar{\lambda}}$

where $\bar{\lambda} = \lambda_0 P(0) + \lambda_1 P(1)$

3.3 Numerical Illustration Table

For the various values of $\lambda_0, \lambda_1, \mu, \epsilon$ and the fixed values of r, R, K, N and $p_1 = 1, p_2 = 1, \xi = 1$ computed and get the values of $P_0(0),\ P(0), P(1),\ L_s\ and\ W_s$ are tabulated in Table 1 and Table 2.

Table 1. $P(0)\ and\ P(1)$ **values**

r	R	K	N	λ_0	λ_1	μ	ϵ	$P_0(0)$	$P(0)$	$P(1)$
5	7	10	12	5	3	5	1	0.005390	0.6245	0.3755
5	7	10	12	6	3	5	1	0.081437	0.8972	0.1028
5	7	10	12	7	3	5	1	0.028071	0.8003	0.1997
5	7	10	12	8	3	5	1	0.005146	0.5963	0.4037
5	7	10	12	7	3	6	0.5	0.001796	0.4335	0.5665
5	7	10	12	7	4	6	0.5	0.000832	0.3758	0.6242
5	7	10	12	7	5	6	0.5	0.003009	0.5268	0.4732
5	7	10	12	7	6	6	0.5	0.008999	0.7511	0.2489
5	7	10	12	8	4	5	1	0.006871	0.6865	0.3135
5	7	10	12	8	4	6	1	0.012130	0.7338	0.2622
5	7	10	12	8	4	7	1	0.006616	0.6612	0.3388
5	7	10	12	8	4	8	1	0.022507	0.8485	0.1515
5	7	10	12	6	3	5	0.2	0.000962	0.4346	0.5654
5	7	10	12	6	3	5	0.3	0.002521	0.4858	0.5142
5	7	10	12	6	3	5	0.6	0.052321	0.8952	0.1048
5	7	10	12	6	3	5	0.8	0.062758	0.9121	0.0879

Table 2. $L_s\ and\ W_s$ **Values**

r	R	K	N	λ_0	λ_1	μ	ϵ	L_s	W_s
5	7	10	12	5	3	5	1	3.85820	1.06924
5	7	10	12	6	3	5	1	2.01522	1.10987
5	7	10	12	7	3	5	1	2.98094	0.80217
5	7	10	12	8	3	5	1	3.95982	0.99507
5	7	10	12	7	3	6	0.5	4.49610	0.90083

Continued on next page

Table 2 continued

5	7	10	12	7	4	6	0.5	4.76348	0.89377
5	7	10	12	7	5	6	0.5	4.21204	1.19944
5	7	10	12	7	6	6	0.5	3.44242	0.92436
5	7	10	12	8	4	5	1	3.66175	0.99888
5	7	10	12	8	4	6	1	3.42411	0.92608
5	7	10	12	8	4	7	1	3.74837	0.94333
5	7	10	12	8	4	8	1	2.01491	1.03974
5	7	10	12	6	3	5	0.2	4.41170	1.02577
5	7	10	12	6	3	5	0.3	3.44242	0.92436
5	7	10	12	6	3	5	0.6	2.35899	1.28004
5	7	10	12	6	3	5	0.8	2.10822	0.91772

3.4 Graphical representation

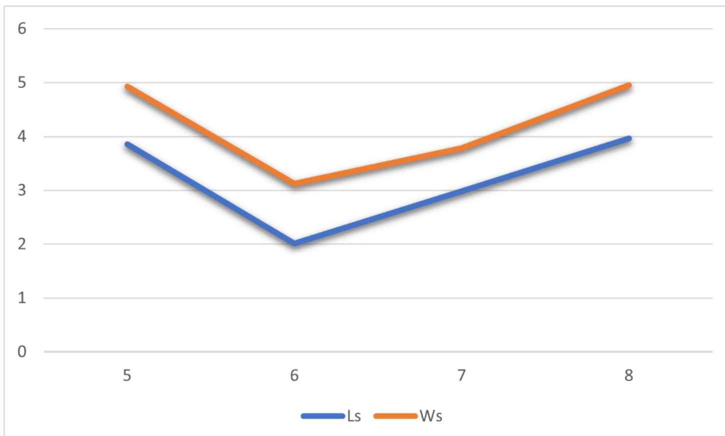

Fig 1. Number of consumers using the system on average L_s and the anticipated length of the customer's wait W_s by varying faster rate of arrival λ_0 and other parameters kept fixed

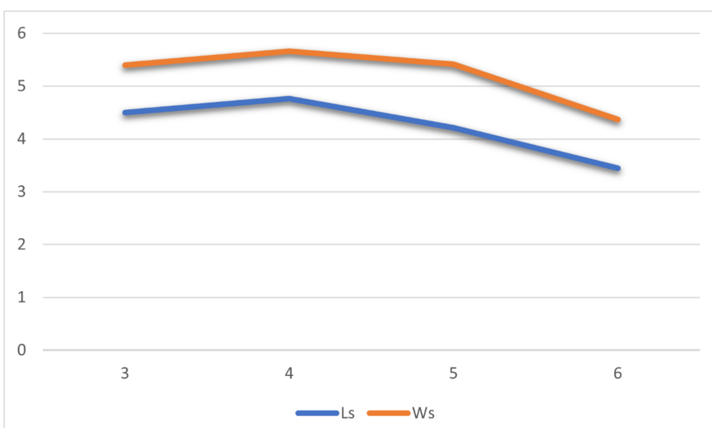

Fig 2. Number of consumers using the system on average L_s and the anticipated length of the customer's wait W_s by varying faster rate of arrival λ_1 and other parameters kept fixed

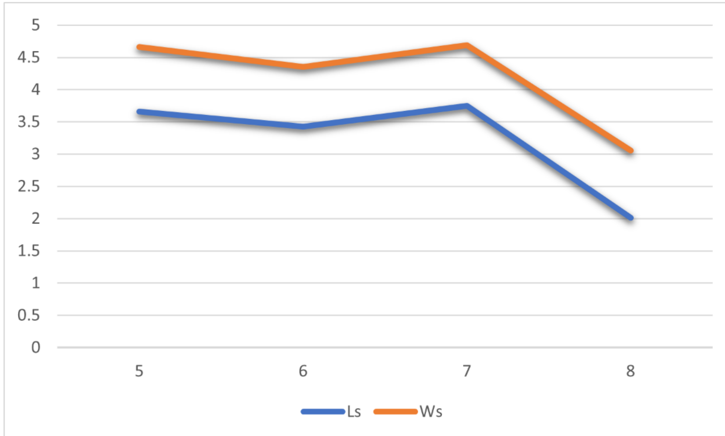

Fig 3. Number of consumers using the system on average L_s and the anticipated length of the customer's wait W_s by varying faster rate of arrival μ and other parameters kept fixed

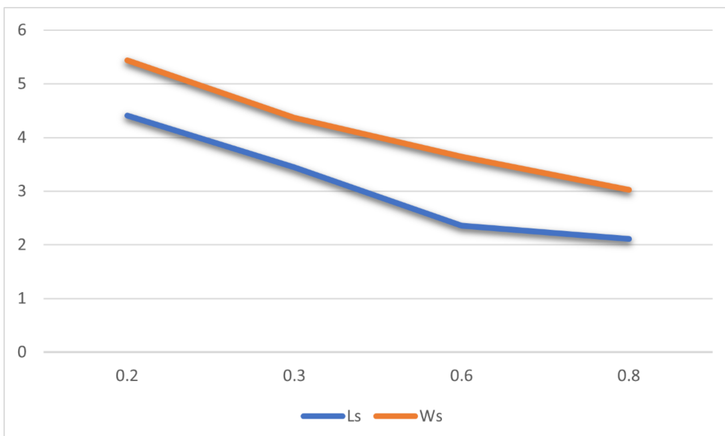

Fig 4. Number of consumers using the system on average L_s and the anticipated length of the customer's wait W_s by varying faster rate of arrival ϵ and other parameters kept fixed

4 Conclusion

This study shows that an innovative technique that combines both slower and quicker rates may be used to regulate arrival rates in a feedback service. The average number of clients in the system and the average wait time for clients in the system are examined using a steady-state probability analysis. The probability and characteristics of the queueing system are analysed, and the collected data and graphs are validated numerically. These models can be used to analyse real-world scenarios in phone centres with a limited number of waiting rooms, banks, hospital emergency rooms with a limited number of beds, and computer systems with a limited amount of buffers. This creative study provides a unique method for controlling varying arrival rates, including slower and faster rates, in certain feedback service settings.

From the Table 1 and Table 2, we observe that

- When the mean service rate increases and the other parameters are kept fixed, $P_0(0)$, $P(0)$ increase and $P(1)$ decrease.
- When the arrival rate decreases and the other parameters are kept fixed $P_0(0)$, $P(0)$ increase and $P(1)$ decrease.
- When the mean dependence rate increases and the other parameters are kept fixed, $P_0(0)$, $P(0)$ increase and $P(1)$ decrease.
- When the value of ϵ increases and the other parameters are kept fixed, $P_0(0)$, $P(0)$ increase and $P(1)$ decrease.
- When the feedback service time increases and the other parameters are kept fixed, $P_0(0)$, $P(0)$ increases and $P(1)$ increases for the final service completion.

Acknowledgment

This article does not contain any studies with human participants or animals performed by any of the authors. The authors thank, DST-FIST, Government of India for funding towards infrastructure facilities St. Joseph's College (Autonomous), Tiruchirapalli - 620 002.

References

1) Haight FA. Queueing with reneging. *Metrika*. 1959;2:186–196. Available from: http://dx.doi.org/10.1007/BF02613734.
2) Nithya N, Anbazhagan N, Amutha S, Jeganathan K, Park GC, Joshi GP, et al. Controlled arrivals on the retrial queueing-inventory system with an essential interruption and emergency vacationing server. *Mathematics*. 2023;11(16). Available from: https://doi.org/10.3390/math11163560.
3) Sasikala S, Abinaya V. Interdependent Inter Arrival Queueing Model with Controllable Arrival Rates, Reverse Balking and Impatient Customers. *Arya Bhatta Journal of Mathematics and Informatics*. 2023;15:1–10. Available from: http://dx.doi.org/10.5958/2394-9309.2023.00001.X.
4) Sindhu S, Krishnamoorthy A, Kozyrev D. On Queues with Working Vacation and Interdependence in Arrival and Service Processes. *Mathematics*. 2023;11:2280–2280. Available from: https://www.mdpi.com/2227-7390/11/10/2280.
5) Yue R, Yang G, Zheng Y, Tian Z. Capacity estimation at all-way stop-controlled intersections considering pedestrian crossing effects. *Transportation research record*. 2023;2677:583–595. Available from: https://doi.org/10.1177/03611981221116368.
6) Yang Y, Zhang B, Guo D, Wang W, Nie J, Xiong Z, et al. Stochastic geometry-based age of information performance analysis for privacy preservation-oriented mobile crowdsensing. *IEEE Transactions on Vehicular Technology*. 2023. Available from: https://doi.org/10.1109/TVT.2023.3252167.
7) Mahanta S, NKumar, Choudhury G. An analytical approach of Markov modulated Poisson input with feedback queue and repeated service under N-policy with setup time. *Quality Technology & Quantitative Management*. 2024;21(2):257–285. Available from: https://doi.org/10.1080/16843703.2023.2193767.
8) Xie Q, Wang J, Jin L. Strategic Defense of Feedback-Controlled Parallel Queues against Reliability and Security Failures. *arXiv preprint* . 2023. Available from: https://doi.org/10.48550/arXiv.2301.11530.
9) Gayathri S, Rani G. AN FM/M/c INTERDEPENDENT STOCHASTIC FEEDBACK ARRIVAL MODEL OF TRANSIENT SOLUTION AND BUSY PERIOD ANALYSIS WITH INTERDEPENDENT CATASTROPHIC EFFECT. *Advances and Applications in Statistics*. 2023;86:19–45. Available from: https://doi.org/10.17654/0972361723014.
10) Cherfaoui M, Bouchentouf AA, Boualem M. Modelling and simulation of Bernoulli feedback queue with general customers' impatience under variant vacation policy. *International Journal of Operational Research*. 2023;46:451–480. Available from: https://doi.org/10.1504/IJOR.2023.129959.
11) Rahim KH, Thiagarajan M. M/M (a, b)/1 Model Of Interdependent Queueing With Controllable Arrival Rates. *Indian Journal of Science and Technology*. 2023;16:3100–3109. Available from: https://doi.org/10.17485/IJST/v16i37.1259.
12) Therasal SN, Thiagarajan M. Poisson Input and Exponential Service Time Finite Capacity Interdependent Queueing Model with Breakdown and Controllable Arrival Rates. *Indian Journal of Science and Technology*. 2024;17(12):1167–1179. Available from: https://doi.org/10.17485/IJST/v17i12.2852.
13) Chettouf A, Bouchentouf AA, Boualem M. A Markovian Queueing Model for Telecommunications Support Center with Breakdowns and Vacation Periods. *Oper Res Forum*. 2024;54:22–22. Available from: https://doi.org/10.1007/s43069-024-00295-y.
14) Seenivasan M, Abinaya R. Markovian queueing model with single working vacation and catastrophic. *Materials Today: Proceedings*. 2022;51(8):2348–54. Available from: https://doi.org/10.1016/j.matpr.2021.11.572.
15) Ahmed FI, Alemu SD, Tilahu GT. A Single Server Markovian Queue with Single Working Vacation, State Dependent Reneging and Retention of Reneged Customers. *Ethiopian Journal of Education and Sciences*. 2022;17(1):51–69. Available from: https://journals.ju.edu.et/index.php/ejes/article/view/3976.
16) Kalyanaraman R, Sundaramoorthy A. A Markovian single server working vacation queue with server state dependent arrival rate, balking and with breakdown. *Malaya Journal of Matematik*. 2019;S(1):160–166. Available from: https://doi.org/10.26637/MJM0S01/0037.
17) Dudin AN, Chakravarthy SR, Dudin SA, Dudina OS. Queueing system with server breakdowns and individual customer abandonment. *Quality Technology & Quantitative Management*. 2023;p. 1–20. Available from: https://doi.org/10.1080/16843703.2023.2215630.
18) Mahanta S, Kumar N, Choudhury G. An analytical approach of Markov modulated Poisson input with feedback queue and repeated service under N-policy with setup time. *Quality Technology & Quantitative Management*. 2024;21:257–285. Available from: https://doi.org/10.1080/16843703.2023.2193767.
19) Subhapriya SP, Thiagarajan M. M/M/1/K Loss and Delay Interdependent Queueing Model with Vacation and Controllable Arrival Rates. *Indian Journal Of Science And Technology*. 2024;17(6):487–493. Available from: https://dx.doi.org/10.17485/ijst/v17i6.1691.
20) Subhapriya SP, Thiagarajan M. M/M/1 Interdependent Queuing Model with Vacation and Controllable Arrival Rates. *Indian Journal of Science and Technology*. 2024;17(14):1497–1506. Available from: https://dx.doi.org/10.17485/ijst/v17i6.1691.
21) Keerthana M, Sangeetha N, Sivakumar B. Optimal service rates of a queueing inventory system with finite waiting hall, arbitrary service times and positive lead times. *Annals of Operations Research*. 2023;331(2):739–762. Available from: https://dx.doi.org/10.1007/s10479-022-04901-2.
22) Basak A, Choudhury A. Bayesian estimation of finite buffer size in single server Markovian queuing system. *Int J Syst Assur Eng Manag*. 2024;15:2366–2373. Available from: https://doi.org/10.17485/IJST/v17i14.251.
23) Bebittovimalan A, Thiagarajan M. An Infinite Capacity Single Server Markovian Queueing System With Discouraged Arrivals Retention Of Reneged Customers and Controllable Arrival Rates With Feedback. *Indian Journal Of Science And Technology*. 2024;17(30):3100–3108. Available from: https://dx.doi.org/10.17485/ijst/v17i30.1900.