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On the Ternary Cubic Diophantine Equation $x^3+y^3=2(z+w)^2(z-w)$

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Abstract

The theory of Diophantine equation offers a rich variety of fascinating problems. There are Diophantine problems, which involve cubic equations with four variables. The cubic Diophantine equation given by $x^3+y^3 = 2(z+w)^2(z-w)$ is analyzed for its patterns of non-zero distinct integral solutions. **Objectives:** The objective of this paper is to explore the integral solutions of cubic equation $x^3+y^3 = 2(z+w)^2(z-w)$ by using suitable methodologies. A few interesting relations between the solutions and special numbers are exhibited. **Method:** Solving Diophantine equation is obtained by the method of Decomposition. The structure of decomposition: $f(u_1,u_2,u_3,\ldots u_n) = 0$ like $f_1\ (u_1,u_2,u_3,\ldots u_n),\ f_2\ (u_1,u_2,u_3,\ldots u_n),\ldots,f_k\ (u_1,u_2,u_3,\ldots u_n) = a$, where $f_1,\ f_2,\ldots,f_k\ \in Z[u_1,u_2,\ldots u_n]$ and $a\ \in$ Z. By the decomposing method in primary terms of a, we achieve a countable number of decompositions in k full factors $a_1,a_2,\ldots a_k$. Each decomposition of this kind leads to a system of equations similar to: $f_1\,(u_1,u_2,\ldots u_n) = a_1,\ldots f_n\,(u_1,u_2,\ldots u_n) = a_n$. We get multitude of solutions for a given equation, by determining the system of equations. **Findings:** By the method of linear transformations, the ternary cubic equation with four unknowns is solved for its integral solutions. The equation is researched for its attributes and correlation among the solutions for its non – zero unique integer points. In each of the transformations taken, the cubic equation yields different solutions. The properties of the solutions and their relationship with the special numbers are also exhibited. **Novelty:** Mathematician's interest towards solving Pell's equation has been so much not because they approximate with a value for $t-\sqrt{t}$. The main importance of the Pell's equation is due to that most of the common questions have answers in this equation which can be sorted by 2 variables in the Quadratic equations. This document is about the research on higher degree Cubic Diophantine equation which gives the integral solutions of this equation, taken into consideration.

Keywords: Integral solutions; Ternary Cubic; Oblong number; Polygonal number

1 Introduction

The Cubic Equation offers an unlimited field for research because of their variety[1]. For an extensive review of various problems, one may refer[2–6]. A polynomial equation which talks about the integer solutions with more than one unknown variable is known as Diophantine Equation. The concept called indefinite or undetermined equations when there is a hypothesis about Diophantine equation for more unsure cases as measured to the number of forms. This theory in Math is carried out to acquire the algebraic equations to extend the rational p-acidic Q-field number. In this paper, we focus on exploring the methodologies and strategies employed in solving Diophantine quadratic equations of the form, by $x^3+y^3 = 2(z+w)^2(z-w)$. This class of equation stands out for its intricate interplay between Algebraic techniques and number theory.

By investigating through various approaches, including factorization methods modular arithmetic and bounds on solutions, we aim to uncover systematic ways to determine the existence and uniqueness of solutions, thereby contributing to the broader understanding of Diophantine equations. This communication concerns with yet another interesting Ternary Quadratic equation by $x^3+y^3 = 2(z+w)^2(z-w)$ representing a homogenous cone for determining it's infinitely may non-zero integral solutions. Also, a few interesting relations among the solutions have been presented. The ternary quadratic Diophantine equation is studied for it's non-trivial distinct integral solutions[6]. Diophantine equations may have finite, infinite or no solutions in integers[7]. Also, a few interesting relations among the solutions have been presented.

2 Notations

- obl $_n$ - Oblong number of rank 'n'
- t$_{m,n}$ - Polygonal number of rank 'n' with sides' m

3 Methodology

Solving cubic Diophantine equations can be approached through several methodologies, depending on the specific form and complexity of the equation. Sometimes, equations can be simplified or transformed by factoring techniques, especially when dealing with cubic equations that involve integer coefficients or solutions. Some cubic Diophantine equations can be transformed into special parametric forms that allow for systematic solution finding, such as equations reducible to elliptic curves or Mordell equations. These methodologies often complement each other and may be combined depending on the nature of the cubic Diophantine equation under investigation. The choice of methodology often depends on factors such as the form of the equation, the availability of known results in the literature, and the computational resources available for exploring potential solutions.

4 Results and Discussion

There are various Diophantine equations with uncertain values along with a specific sequence of eccentric forms for evaluating non-zero integer solutions. In this paper, the non-trivial unique integer solutions for the non-homogeneous three dimensional forms are discussed. To conclude, it needs to be surfed for options for outputs to this three-dimension forms with single or more variables. Similar to the second-degree equations, endless sets of non-zero unique integer solutions to the cubic equations are calculated. With the help of this calculation, it requires further analysis to determine the integer solutions to Diophantine equations with a higher degree and several variables.

5 Method of Analysis

The ternary cubic equation with four unknowns to be solved for getting non – zero integral solution is by

$$x^3+y^3 = 2(z+w)^2(z-w) \tag{1}$$

On substituting the linear transformations

$$x = u+v,\ y = u-v,\ z = u+p,\ w = u-p$$

$$u \neq p \; and \; v \neq p \tag{2}$$

Then

by $x^3+y^3 = 2(z+w)^2(z-w)$ becomes

$$(u+v)^3+(u-v)^3 = 2[(u+p)+(u-p)]^2[(u+p)-(u-p)]$$

$$2u\left(u^2+3v^2\right)=8u^2 2p$$

$$\left(u^2+3v^2\right)=8up$$

It can be written as

$$(u-4p)^2+3v^2=16p^2 \tag{3}$$

We obtain three different patterns of integral solutions to Equation (1) through solving Equation (3) which are illustrated as follows:

5.1 Pattern 1:

In Equation (3) take

$$p=a^2+3b^2 \tag{4a}$$

and write '16' as

$$16=(2+2i\sqrt{3})(2-2i\sqrt{3}) \tag{4b}$$

Substituting Equations (4a) and (4b) in Equation (3) and employing the method of factorization,

$$(u-4p)^2+3v^2=(2+2i\sqrt{3})(2-2i\sqrt{3})(a^2+3b^2)^2$$

which is equivalent to the system of equations

$$\left(u-4p+i\sqrt{3}v\right)=(2+2i\sqrt{3})(a+i\sqrt{3b})^2$$

$$\left(u-4p-i\sqrt{3}v\right)=(2-2i\sqrt{3})(a-i\sqrt{3b})^2$$

Equating the real and imaginary parts, we have

$$u-4p=2a^2-6b^2-12\,ab \tag{5}$$

$$v=2a^2-6b^2+4ab \tag{6}$$

substituting Equation (4a) in Equation (5), we get

$$u-4(a^2+3b^2)=2a^2-6b^2-12\,ab$$

$$u=6a^2+6b^2-12\,ab \tag{7}$$

From Equations (4a), (6), (7) and (2), we get

$$x = u + v = 8\left(a^2 - ab\right)$$

$$y = u - v = 4\left(a^2 + 3b^2 - 4ab\right)$$

$$z = u + p = 7(a^2 + 9b^2 - 12ab)$$

$$w = u - p = 5(a^2 + 3b^2 - 12ab)$$

The distinct integral solutions of Equation (1) are expressed by,

$$x = x(a,b) = 8\left(a^2 - ab\right)$$

$$y = y(a,b) = 4\left(a^2 + 3b^2 - 4ab\right)$$

$$z = z(a,b) = 7(a^2 + 9b^2 - 12ab)$$

$$w = w(a,b) = 5(a^2 + 3b^2 - 12ab)$$

5.1.1 Properties

- $x(1,b) + y(1,b)\ is\ a\ nasty\ number$
- $z(a,b) + w(a,b)\ \equiv 0\ (mod\ 2)$
- $y(a,b) + w(a,b)\ \equiv 0\ (mod\ 3)$
- $3x(a,1) + 4z(a,b)\ \equiv 0\ (mod\ 3)$

5.2 Pattern 2:

Taking

$$u - 4p = 4X,\ v = 4\,V \tag{8}$$

in Equation (3), we have

$$X^2 + 3V^2 = p^2 \tag{8a}$$

which is satisfied by

$$X = a^{\,2} - 3b^2\,,\ V = 2ab,\ P = a^2 + 3b^2 \tag{9}$$

On substituting, we get

$$u - 4p = 4X$$

$$u = 4X + 4p$$

$$= 4\left(a^2 - 3b^2\right) + 4(a^2 + 3\,b^2)$$

$$= 8\,a^2 \tag{10a}$$

$$v = 4V$$

$$= 4\left(2ab\right) = 8\,ab \tag{10b}$$

From Equations (9), (10a), (10b) and (2), we get

$$x = u + v = 8\,\left(a^2 + ab\right)$$

$$y = u - v = 8\,\left(a^2 - ab\right)$$

$$z = u + p = 3\,\left(3a^2 + b^2\right)$$

$$w = u - p = 7\,\left(a^2 - 3b^2\right)$$

The distinct integral solutions of Equation (1) are given by,

$$x = x\left(a,b\right) = 8\,\left(a^2 + ab\right)$$

$$y = y\left(a,b\right) = 8\,\left(a^2 - ab\right)$$

$$z = z\left(a,b\right) = 3(3a^2 + b^2)$$

$$w = w\left(a,b\right) = 7(a^2 - 3b^2)$$

5.2.1 Properties:

- $x\left(a,b\right) + y\left(a,b\right) \equiv 0 \;(mod\, 8)$
- $z\left(a,1\right) + w\left(a,1\right) \equiv 0 \;(mod\, 4)$
- $y\left(a,1\right) + z\left(a,1\right) - t_{36,\;a} \equiv 0 \;(mod\, 8)$
- $y\left(a,1\right) + w\left(a,1\right) - t_{32,\;a} \equiv 0 \;(mod\, 3)$
- $x\left(a,1\right) + 2z\left(a,1\right) - t_{36,\;a} \equiv 0 \;(mod\, 3)$
- $3z\left(a,b\right) - w\left(a,b\right) \equiv 0 \;(mod\, 2)$

5.3 Pattern 3:

Rewrite Equation (8a) as

$$p^2 - 3v^2 = x^2 * 1 \tag{11}$$

and write '1' as

$$1 = \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) \; and \; X = a^2 - 3b^2 \tag{12}$$

Substituting Equations (12) and (11) and using method of factorization, define

$$p^2 - 3v^2 = \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) * \left(a^2 - 3b^2\right)^2$$

$$\left(p+\sqrt{3}\right)\left(p-\sqrt{3}\right)V = \left(\left(a+\sqrt{3b}\right)^2\left(a-\sqrt{3b}\right)^2\right)\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$$

Equating rational and irrational parts on both sides, we obtain

$$\left(p+\sqrt{3}\right) = \left(\left(a+\sqrt{3b}\right)^2\right)\left(2+\sqrt{3}\right)$$

$$\left(p-\sqrt{3}\right) = \left(\left(a-\sqrt{3b}\right)^2\right)\left(2-\sqrt{3}\right)$$

Hence,

$$p = 2a^2 + 6b^2 + 6ab \tag{13}$$

$$v = a^2 + 3b^2 + 4ab \tag{14}$$

From Equations (8), (13) and (14), we have

$$u - 4p = 4X$$

$$u = 4X + 4p$$

$$= 12a^2 + 12b^2 + 24ab \tag{15}$$

$$v = 4V$$

$$= 4a^2 + 12b^2 + 16ab \tag{16}$$

$$p = 2a^2 + 6b^2 + 6ab$$

On substituting Equations (13), (15), (16) and (2), we get

$$x = u + v = 8\left(2a^2 + 3b^2 + 5ab\right)$$

$$y = u - v = 8\,(a^2 + ab)$$

$$z = u + p = 2\,(7a^2 + 9b^2 + 15ab)$$

$$w = u - p = 2\,(5a^2 + 3b^2 + 9ab)$$

The distinct integral solutions of Equation (1) are expressed by,

$$x = x\,(a,b) = 8\,\left(2a^2 + 3b^2 + 5ab\right)$$

$$y = y(a,b) = 8\,(a^2 + ab)$$

$$z = z(a,b) = 2\,(7a^2 + 9b^2 + 15ab)$$

$$w = w(a,b) = 2\,(5a^2 + 3b^2 + 9ab)$$

5.3.1 Properties:

- $x\,(a,b) - y\,(a,b)\ \equiv 0\ (mod\ 8)$
- $x\,(a,1) - y\,(a,1) -\ t_{8,\ a}\ \equiv 0\ (mod\ 3)$
- $z\,(a,b) - w\,(a,b) \equiv 0\ (mod\ 4)$
- $x\,(a,1) - 2y\,(a,1) - 34a\ is\ a\ nasty\ number$
- $10y\,(a,1) - 8w\,(a,1)\ \equiv 0\ (mod\ 8)$

6 Conclusion

Ideally, this exercise and experiments greatly helped in determining the unique combinations of design patterns of solutions in correlation with its attributes. To conclude, one may search for other patterns of solutions to the equation under consideration. Real time problems can also be modeled as linear equations. Problems identified in semi-urban areas in accessing the internet facilities thereby it will be difficult for many of the students to continue their learning can be written as mathematical problem and solved[7]. In real life, it is used in biometric for passport in Europe. Also in USA it is equipped with a chip which describes with the owner's identity which is secured by the elliptical curve. It is also used in solving data involving social economic problems[8]. Cubic equations have many applications in engineering[9].

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