INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Unique Strong Isolate Semitotal Domination in Graphs

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OPEN ACCESS

Received: 25-05-2024

Accepted: 29-07-2024

Published: 20-08-2024

Citation: Mutharasu S, Nithya D (2024) Unique Strong Isolate Semitotal Domination in Graphs. Indian Journal of Science and Technology 17(33): 3442-3446. https://doi.org/10.17485/IJST/v17i33.1785

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objective: This study introduces a new domination parameter called "Unique strong isolate semitotal domination". **Methods**: A unique strong isolated semitotal dominating set(USISTD-set) D of a graph G is an isolated semitotal dominating set(ISTD-set) in which there exists exactly one vertex $v \in D$ such that $N_2(v) \cap D = \varphi$. **Findings**: This study gives the USISTD-number of some special graphs. Also, it discusses about the non-existence of USISTD-number in some graphs. **Novelty**: A few findings of a new domination parameter called unique strong islolate semitotal domination of some special graphs have been established.

AMS subject classification: 05C69

Keywords: Isolate semitotal domination; Unique strong isolate semitotal domination; Comb graph; Helm graph; Flower graph; Excellent graph

1 Introduction

This study focuses on the strengthening of domination. A set S of vertices in G is a semi total dominating set (STD-set) of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S[1]. Many authors have published many papers related to semitotal domination[2–5]. I. Shahul Hamid and S. Balamurugan initiate the study on isolate domination in graphs[6].

A dominating set S of vertices of G is said to be an isolated semi total dominating set, abbreviated ISTD-set of G, if it is an isolate dominating set of G and every vertex in S is within distance 2 of another vertex of S. $N_k(v)$ is the set of vertices which is at a distance less than or equal to k from v. A unique strong isolate semitotal dominating set S, abbreviated USISTD-set of G is an isolate semitotal dominating set of G in which there exist a vertex $v \in D$ such that $N_2(v) \cap D = \varphi$.

A unique strong isolate semi total domination number of G denoted by $\gamma_{t_2}^{U,s,0}(G)$is the minimum cardinality of an isolate semitotal dominating set. In this paper we obtained unique strong isolate semi total domination number of some families of graphs and also its non-existence in some graphs.

2 Methodology

Using the definition listed below, we obtained the following results.

Definition 2.1: Comb is a graph obtained by joining a single pendant edge to each vertex of a path.

Definition 2.2: The **Helm graph** $H_{\mathbf{n}}$ is the graph with $2n + 1$ vertices obtained from a n−wheel graph by adjoining a pendant edge at each node of the cycle[7].

Definition 2.3: A **sun graph** n−sun is a graph formed from a cycle on n−vertices by attaching a pendant edge to each vertex[7].

Definition 2.4: A **flower graph** F_n is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm[8].

Definition 2.5: Bistar graph $B_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end and n pendant edges to the other end of K_2[8].

Definition 2.6: Book graph is a Cartesian product of a star and single edge, denoted by B_m. The m-book graph is defined as the graph Cartesian product $B_m = S_{m+1} \circ P_2$, where S_m is a star graph and P_2 is the path graph on two nodes.

3 Results and Discussion

Since every USISTD-set is a dominating set, the next result follows:

Theorem 3.1. For any graph G which admits USISTD-set, $\gamma(G) \leq \gamma_{t_2}^{U,s,0}(G)$.

Theorem 3.2. For the comb graph $P_n * K_1 (n \geq 3)$, we have $\gamma_{t_2}^{U,s,0}(P_n * K_1) = n$.

Proof : Let $P_n * K_1$ be a comb graph and let $V(P_n * K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and
$E(P_n * K_1) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$.

Since each v_i is pendant, either v_i or u_i must be in every dominating set and so $\gamma(G) \geq n$. Thus, by Theorem 3.1 $n \leq \gamma_{t_2}^{U,s,0}(P_n * K_1)$.

Since the set $D = \{v_1, v_2, u_3, u_4, ..., u_n\}$ is an USISTD-set with n elements and v_1 is the strong isolated vertex of D, $\gamma_{t_2}^{U,s,0}(P_n * K_1) \leq n$.

Theorem 3.3. For the Helm graph H_n , $n \geq 4$, we have $\gamma_{t_2}^{U,s,0}(H_n) = n$.

Proof : Let H_n be a Helm graph and let $V(H_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u\}$ and $E(H_n) = \{u_i v_i, uu_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1 u_n\}$. Since each v_i is pendant, either v_i or u_i must be in every dominating set and so $\gamma(H_n) \geq n$. Thus by Theorem 3.1, $n \leq \gamma_{t_2}^{U,s,0}(H_n)$. Since the set $D = \{v_1, v_2, u_3, u_4, ..., u_{n-1}, v_n\}$ is an USISTD-set with n elements and v_1 is the strong isolated vertex of D, $\gamma_{t_2}^{U,s,0}(H_n) \leq n$.

Corollary 3.1. The flower graph does not admit USISTD-set.

Theorem 3.4. For the sun graph S_n, $n \geq 4$, we have $\gamma_{t_2}^{U,s,0}(S_n) = n$.

Proof : Let S_n be a sun graph and let $V(S_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$
and $E(S_n) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1 u_n\}$.

Since each v_i is pendant, either v_i or u_i must be in every dominating set for all $1 \leq i \leq n$ and so $\gamma(S_n) \geq n$. Thus by Theorem 3.1, $n \leq \gamma_{t_2}^{U,s,0}(S_n)$.

Since the set $D = \{v_1, v_2, u_3, u_4, ..., u_{n-1}, v_n\}$ is an USISTD-set with n elements and v_1 is the strong isolated vertex of D, $\gamma_{t_2}^{U,s,0}(S_n) \leq n$.

Theorem 3.5. Let C_n be a cycle of n vertices, $n \geq 7$. Then $\gamma_{t_2}^{U,s,0}(C_n) = 1 + \left[\frac{2(n-3)}{5}\right]$.

Proof: Let $V(C_n) = \{v_1, v_2, ..., v_n\}$.

Case 1: Suppose $n - 3 = 5k + j$ for $j = 0$ *and* $k \geq 1$.

Let $l = \left[\frac{n-3}{5}\right]$.

Let D be $\gamma_{t_2}^{U,s,0}-$ set of C_n and without loss of generality, let v_1 be the strong isolated vertex in <D>. Note that the vertex v_1 can dominate a maximum of 2 vertices, namely v_n and v_2. Thus, the remaining $n-3$ vertices of C_n must be dominated by $D - \{v_{n-1}, v_n, v_1, v_2, v_3\}$.

Note that the remaining un-dominated vertices forms a path $P : v_3, v_4, ..., v_{n-1}$ on $n-3$ vertices. For each $1 \leq i \leq l$, to dominate the consecutive 5 vertices $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i$, D must include at least two out of these 5 vertices.

Thus, D must have $2\left[\frac{n-3}{5}\right] + 1$ vertices. Thus, $\gamma_{t_2}^{U,s,0}(C_n) \geq 2\left[\frac{n-3}{5}\right] + 1 = 1 + \left[\frac{2(n-3)}{5}\right]$. Further, the set $\{v_1\} \cup \{v_{5i-1}, v_{5i+1} : i = 1, 2, ...l\}$ is an USISTD-set with $1 + \left[\frac{2(n-3)}{5}\right]$ vertices and so $\gamma_{t_2}^{U,s,0}(C_n) \leq 1 + \left[\frac{2(n-3)}{5}\right]$.

Case 2: Suppose $n - 3 = 5k + j$ for $j = 4$ and $k \geq 1$.

Let $l = \left[\frac{n-3}{5}\right]$.

Let D be $\gamma_{t_2}^{U,s,0}$-set of C_n and without loss of generality, let v_1 be the strong isolated vertex in <D>. Note that the vertex v_1 can dominate a maximum of 2 vertices, namely v_n and v_2. Thus, the remaining $n-3$ vertices of C_n must be dominated by $D-\{v_{n-1}, v_n, v_1, v_2, v_3\}$.

Note that the remaining un-dominated vertices forms a path $P:\ v_3, v_4, \ldots, v_{n-1}$ on $n-3$ vertices. For each $1\ \leq i\ \leq l-1$, to dominate the consecutive 5 vertices $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i$, D must include at least two out of these 5 vertices. Note that at last, we have to dominate the 4 vertices $v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}$.

Since $v_{n-1} \notin D$, to dominate v_{n-1}, we must have v_{n-2} in D. If both the vertices v_{n-3} and v_{n-4} are not in D, then v_{n-2} will be another strong isolated vertex, a contradiction. Thus, D must include at least two vertices from the set of vertices $v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}$. Thus, $|D| \geq 1+2(l-1)+2 = 2\left[\frac{n-3}{5}\right]+1$ and so $\gamma_{t_2}^{U,s,0}(C_n) \geq 2\left[\frac{n-3}{5}\right]+1 = 1+\left[\frac{2(n-3)}{5}\right]$.

Further, the set $\{v_1\} \cup \{v_{5i-1}, v_{5i+1}\ :\ i = 1,2,\ldots l-1\} \cup \{v_{n-3},\ v_{n-2}\}$ is an USISTD-set with $1+\left\lceil\frac{2(n-3)}{5}\right\rceil$ vertices and so $\gamma_{t_2}^{U,s,0}(C_n) \leq 1+\left[\frac{2(n-3)}{5}\right]$.

Case 3: Suppose $n-3 = 5k+j$ for $j = 3$ and $k \geq 1$.

Let $l = \left\lceil\frac{n-3}{5}\right\rceil$.

Let D be $\gamma_{t_2}^{U,s,0}$-set of C_n and without loss of generality, let v_1 be the strong isolated vertex in <D>. Note that the vertex v_1 can dominate a maximum of 2 vertices, namely v_n and v_2. Thus, the remaining $n-3$ vertices of C_n must be dominated by $D-\{v_{n-1}, v_n, v_1, v_2, v_3\}$.

Note that the remaining un-dominated vertices forms a path $P:\ v_3, v_4, \ldots, v_{n-1}$ on $n-3$ vertices. For each $0 \leq i \leq l-2$, to dominate the consecutive 5 vertices $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i$, D must include at least two out of these 5 vertices. Note that at last, we have to dominate the 3 vertices $v_{n-1}, v_{n-2}, v_{n-3}$.

Since $v_{n-1} \notin D$, to dominate v_{n-1}, we must have v_{n-2} in D. If both the vertices v_{n-3} and v_{n-4} are not in D, then v_{n-2} will be another strong isolated vertex, a contradiction. Thus, D must include at least two vertices from the set of vertices $v_{n-2}, v_{n-3}, v_{n-4}$. Suppose $v_{n-2}, v_{n-3} \in D$, then nothing to prove.

Suppose $v_{n-3} \notin D$, then $v_{n-4} \in D$ (otherwise v_{n-2} is strong isolated, a contradiction). Suppose there are three vertices from the set $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i,\ (i = l-2)$ in D, then there is nothing to prove. If not, then $v_{(n-2)-5}$ must be in D. Since D is semitotal, $v_{(n-4)-5} \in D$.

Suppose there are three vertices from the set $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i, (i = l-3)$ in D, then there is nothing to prove. If not, then $v_{(n-2)-2(5)}$ must be in D. Since D is semitotal, $v_{(n-4)-5} \in D$. Proceeding like this, we get $v_7 \in D$.

Suppose there are three vertices from the set $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i,\ (i = 0)$ in D, then there is nothing to prove. If not, then v_4 must be in D. In this case, the vertex v_4 becomes another strong isolated vertex, a contradiction. Thus, $v_{n-3} \in D$ and so $\gamma_{t_2}^{U,s,0} = 2\left[\frac{n-3}{5}\right]+1$. Thus, $\gamma_{t_2}^{U,s,0}(C_n) \geq 2\left[\frac{n-3}{5}\right]1 = 1+\left[\frac{2(n-3)}{5}\right]$.

Further, the set $\{v_1\} \cup \{v_{5i-1}, v_{5i+1}\ :\ i = 1,2,\ldots l-2\} \cup \{v_{n-3},\ v_{n-2}\}$ is an USISTD-set with $1+\left[\frac{2(n-3)}{5}\right]$ vertices and so $\gamma_{t_2}^{U,s,0}(C_n) \leq 1+\left[\frac{2(n-3)}{5}\right]$.

Case 4: Suppose $n-3 = 5k+j$ for $1 \leq j \leq 2$ and $k \geq 1$.

Let $l = \left[\frac{n-3}{5}\right]$.

Let D be $\gamma_{t_2}^{U,s,0}$-set of C_n and without loss of generality, let v_1 be the strong isolated vertex in <D>. Note that the vertex v_1 can dominate a maximum of 2 vertices, namely v_n and v_2. Thus, the remaining $n-3$ vertices of C_n must be dominated by $D-\{v_{n-1}, v_n, v_1, v_2, v_3\}$.

Note that the remaining un-dominated vertices forms a path $P:\ v_3, v_4, \ldots, v_{n-1}$ on $n-3$ vertices. For each $0 \leq i \leq l-2$, to dominate the consecutive 5 vertices $3+5i,\ 4+5i, 5+5i,\ 6+5i,\ 7+5i$, D must include at least two out of these 5 vertices. Since $1 \leq j \leq 2$, to dominate the remaining vertices (or vertex) D must include one more vertex in it and so D must have $2\left[\frac{n-3}{5}\right]$ vertices. Thus, $\gamma_{t_2}^{U,s,0}(C_n) \geq 2\left[\frac{n-3}{5}\right] = 1+\left[\frac{2(n-3)}{5}\right]$.

Further, the set $\{v_1\} \cup \{v_{n-2}\} \cup \{v_{5i-1}, v_{5i+1}\ :\ i = 1,2,\ldots l\}$ is an USISTD-set with $1+\left[\frac{2(n-3)}{5}\right]$ vertices and so $\gamma_{t_2}^{U,s,0}(C_n) \leq 1+\left[\frac{2(n-3)}{5}\right]$.

As proved above, we can prove the following result.

Theorem 3.6. Let P_n be a path of n vertices for $n \geq 5$. Then $\gamma_{t_2}^{U,s,0}(P_n) = 1+\left[\frac{2(n-3)}{5}\right]$.

A graph G is said to be **USIST-excellent** if for any given vertex v of G, there is a $\gamma_{t_2}^{U,s,0}$-set of G which contains v.

Remark 3.1. If the eccentricity of a vertex v of a graph G is less than or equal to 2, then the vertex is not strong isolated and such vertex not belongs to any USISTD-set of G.

The next result follows from Remark 3.1.

Theorem 3.7. If the eccentricity of a vertex v of a graph G is less than or equal to 2 for some $v \in V(G)$, then G is not USIST-excellent.

Theorem 3.8. The cycle C_n is an USIST-excellent graph.

Proof : Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and v_j be any vertex in C_n.

Let D be a $\gamma_{t_2}^{U,s,0}$-set of C_n and without loss of generality, assume that $v_1 \in D$. If $v_j \in D$, then there is nothing to prove.

Suppose $v_j \notin D$. Then $D' = \{v_{k \oplus_n (j-1)} : v_k \in D\}$ is also a $\gamma_{t_2}^{U,s,0}$-set of C_n and $v_j \in D'$, where $\oplus_n$ is the operation addition modulo n.

Theorem 3.9. Bistar graph $B_{m,n}$ admits USISTD-set if and only if $m = 1; n \geq 2$ or $m \geq 2;\ n = 1$.

Proof : Let a and b be the end vertices of K_2 in the bistar graph $B_{m,n}$ and let $a_1, a_2, ..., a_m$ be the vertices adjacent to a and $b_1, b_2, ..., b_n$ be the vertices adjacent to b. Suppose there exists a minimum USISTD-set D of $B_{m,n}$ and let x be the strong isolated vertex in <D>

Case (i): Suppose $m,\ n \geq 2$. By the Remark 3.1, a and b are not strong isolated. Suppose $x = a_i$ for some $1 \leq i \leq m$. Since $m \geq 2$, there exists a vertex a_j such that $j \neq i$ and $a_j \notin D$. In this case, D will not dominate a_j, a contradiction to D is a dominating set.

Similarly, we get the same contradiction when $x = b_i$.

Case (ii): Suppose $m = n = 1$. Then $B_{m,n}$ is a path $a_1 - a_, - b - b_1$. By Remark 3.1, $a_, b \neq x$. If $x = a_1$, then $a_, b \notin D$ and so $b_1 \in D$. In this case, <D> has two strong isolated vertices, a contradiction. Similarly, we get the same contradiction when $x = b_1$.

From cases (i) and (ii) we can conclude that $m = 1; n \geq 2$ or $m \geq 2;\ n = 1$.

Conversely suppose $m = 1; n \geq 2$ or $m \geq 2;\ n = 1$. Without loss of generality assume that $m = 1; n \geq 2$. In this case $\{a_1\} \cup \{b_1, b_2, ..., b_n\}$ is an USISTD-set.

Theorem 3.10. The book graph $B_m,\ m \geq 2$ admits USISTD-set and $\gamma_{t_2}^{U,s,0}(B_m) = m$.

Proof : Let a and b be the center vertices of two copies of S_{m+1} in the Cartesian product $S_{m+1} \circ P_2$. Let $a_1, a_2, ..., a_m$ be the vertices adjacent to a and $b_1, b_2, ..., b_m$ be the vertices adjacent to b. By the Remark 3.1, a and b are not in any USISTD-set of B_m. Thus, for each $1 \leq i \leq m$, to dominate the vertex a_i, either a_i or b_i must be in any USISTD-set of B_m and so $\gamma_{t_2}^{U,s,0}(B_m) \geq m$. Since the set $\{a_1\} \cup \{b_2, b_3, ..., b_m\}$ is an USISTD-set with m vertices, $\gamma_{t_2}^{U,s,0}(B_m) \leq m$.

Theorem 3.11. Any tree T with $diam(T) = 2$ does not admit USISTD-set.

Proof : Let T be a tree with $diam(T) = 2$. Let u be the strong isolated vertex. Since $diam(T) = 2$, there exists a path $P : x - y - z$. If $u = x$, then by the definition of USISTD-set, $y, z \notin D$ and so T does not admit USISTD-set.

Corollary 3.2 The complete graph K_n, complete bipartite graph $K_{m,n}$, wheel graph W_n and the star graph S_n do not admit USISTD-set.

4 Conclusion

Unique Strong Isolate Semitotal Domination number of comb graph, helm graph, flower graph, sun graph, bistar graph and book graph has been found and also USIST excellent graph is defined. Additionally, for cycle and path the USISTD number is found. Also, the non-existence of USISTD number is discussed. These are helpful in applying semitotal domination theory to actual circumstances. The results can be extended upon by the readers in further work.

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