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RESEARCH ARTICLE

Solving Larger Order Uncertain Differential Equations by Newly Attained Fourth Order R-K Method

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Abstract

Objectives: Deriving a new numerical method to solve uncertain differential equations of larger order. **Method:** Uncertain differential equations of larger order are dealt numerically with Seikkala findings. Runge-Kutta method with linear combinations of means is followed. **Findings:** The fourth-order Runge-Kutta method based on a linear combination of arithmetic mean, geometric mean and contraharmonic mean is applied to determine the numerical solution. **Novelty:** This approach is illustrated by solving second and third order FIVP problems. The comparative analysis of the results between the existing and the proposed method shows that the results produced by the proposed method yield more accurate result than the existing classical RK method of order 4. The findings indicate that the proposed approach is well suited for obtaining the approximate solution of n^{th}-order FIVPs.

Keywords: Fuzzy numbers; FIVP Problems; RungeKutta method; Arithmetic mean; Contra harmonic Mean; Geometric mean; Lipschitz condition

1 Introduction

The arithmetic operations of fuzzy numbers are analysed in [1]. Fuzzy Initial Value Problems are solved by using a new approach in [2]. For Solving Fuzzy Differential Equations, Higher Order Runge-Kutta Method is employed in [3]. Runge-kutta method of order 3 based on linear combination of three means is proposed in [4]. A new method is proposed to solve Fuzzy Initial Value Problem using explicit Runge-Kutta method in [5]. The second order Fuzzy Initial Value Problems are taken and solved in [6]. The Arithmetic operations of Trigonal fuzzy numbers using Alpha cuts are proposed in [7]. For solving ordinary differential equations, the Iterative methods have been proposed in [8]. A detailed review over Fuzzy Differential Equations is made in [9]. In [10], proposed RKM methods to solve higher-order fuzzy differential equations. The traditional Range-Kutta Cash–Karp method is illustrated in [11]. The Numerical Solutions of Delay Differential Equations has been done in [12]. Solving uncertain differential equations is done by Rao T. D and S. Chakraverty in [13]. In [14] solved fuzzy differential equations using orthogonal polynomials. In [15], analysed Computational Techniques Based on Runge-Kutta Method of Various Orders. The significant role of

Convergence of numerical methods in ensuring the accurate results for differential equations of fuzzy conditions is explained in [16].

This study introduces an innovative numerical approach for evaluating linear uncertain initial value problems of larger order and done for nth-order. So far there is no formula for fourth-order Runge-Kutta method with combination of three means. Hence, it will be a novel attempt to solve Fuzzy Initial Value Problems using a combination of means.

The organisation of this paper is as given below: Section 2 provides fundamental insights into fuzzy numbers. Following this, Section 3 delves into fuzzy derivative concepts. Section 4 introduces the new fourth-order Runge-Kutta method combining with a variety of means. Section 5 presents two illustrative examples, while Section 6 offers the comparative analysis of the proposed method with the existing method. Section 7 presents the paper's conclusions.

2 Preliminaries

Definition 2 .1 (fuzzy set):

"A fuzzy set $A = \left\{ \left(x, \mu\widetilde{A}(x)\right) : x\epsilon X, \mu\widetilde{A}(x)\epsilon[0,1] \right\}$. In the pair $\left(x, \mu\widetilde{A}(x)\right)$ the first variable x belongs to the classical set and the second variable $\mu\widetilde{A}(x)$ belongs to the interval [0, 1], known membership function".

Definition 2.2 (α -cut of a fuzzy set):

"The α—level set (or interval of confidence at level α or α—cut) of the fuzzy set $\widetilde{A}$ of X is a crisp set α that includes all the elements of X whose membership values are $\geq \alpha$; that is, $\widetilde{A} = \left\{ x : \mu\widetilde{A}(x) \geq \alpha, x\epsilon X, 0 \leq \alpha \leq 1 \right\}$."

Definition: 2 .3 (Triangular Fuzzy Number):

"The definition of a triangular fuzzy number is the three numbers $b_1 < b_2 < b_3$ where the graph of v(x), the membership function of the fuzzy number v, is a triangle with base on the interval $[b_1, b_3]$ and vertex at x = b_2. We specify v as $(b_1/b_2/b_3)$. The membership function for the triangular fuzzy number v= $(b_1/b_2/b_3)$ is defined as the following

$$\vartheta(x) = \begin{cases} \frac{x-b_1}{b_2-b_1}, & b_1 \leq x \leq b_2 \\ \frac{x-b_3}{b_2-b_3}, & b_2 \leq x \leq b_3 \end{cases}$$

and (1) $\vartheta > 0$ if $b_1 > 0$; (2) $\vartheta \geq 0$ if $b_1 \geq 0$; (3) $\vartheta < 0$ if $b_3 < 0$; and (4) $\vartheta \leq 0$ if $b_3 \leq 0$.

α cut interval of this shape is as follows, $\forall \alpha\epsilon[0,1]$

$A_\alpha = \left[(b_1 - b_2)\alpha + \alpha_1, -(b_4 - b_3)\alpha + b_4\right]$

Let the set of all upper semi continuous normal convex fuzzy numbers with bounded r-level intervals is taken as E. That is if ϑ then r-level set $[\vartheta]_r = \{S/\vartheta(S) \geq r\}, 0 < r \leq 1$, is a closed bounded interval and is denoted by $[\vartheta]_r = \left[\underline{\vartheta}(r), \overline{\vartheta}(r)\right]$. Let a real interval be $X : I \to E'$ where $I \subset R$ is called a fuzzy process and its r-level set is denoted by $[x(t)]_r = \left[\underline{x}(t;r), \overline{x}(t;r)\right], t \in I, r \in [0,1]$.

"Representation of an arbitrary fuzzy number is an ordered pair of functions $(\underline{\omega}(r), \overline{\omega}(r))$ for all $r \in [0,1]$, satisfying the following requirements [1,0]:

(i) $\underline{\omega}(r)$ is a bounded left continuous non-decreasing function over [0,1],

(ii) $\overline{\omega}(r)$ is a bounded left continuous non-increasing function over [0,1],

(iii) $\underline{\omega}(r) \leq \overline{\omega}(r), 0 \leq r \leq 1$

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α—level intervals."

Lemma 2.4

"Let $[\underline{\varsigma}(\alpha), \overline{\varsigma}(\alpha), \alpha\epsilon[0,1]]$ be a given family of non-empty intervals. If

(i) $\left[\underline{\varsigma}(\alpha), \overline{\varsigma}(\alpha)\right] \supset \left[\underline{\varsigma}(\beta), \overline{\varsigma}(\beta)\right]$ for $0 < \alpha \leq \beta$, and

(ii) $\left[\lim_{k\to\infty} \underline{\varsigma}(\alpha_k), \lim_{k\to\infty} \overline{\varsigma}(\alpha_k)\right] = \left[\underline{\varsigma}(\alpha), \overline{\varsigma}(\alpha)\right]$

Whenever (α_k) is a non-decreasing sequence converging to $\alpha \in [0,1]$, then the family $\left[\underline{\varsigma}(\alpha), \overline{\varsigma}(\alpha)\right], \alpha\epsilon[0,1]$,, represent the α— level set of fuzzy number v in E. Conversely if $\left[\underline{\varsigma}(\alpha), \overline{\varsigma}(\alpha)\right], \alpha\epsilon[0,1]$ are α— level set of fuzzy number $v \in E$ then the conditions (i) and (ii) hold true."

3 Fuzzy Derivatives

Definition 3.1

"Let Ibe a real interval. A mapping $\varsigma : I \to E$ is called a fuzzy process and we denoted the $\alpha-$level set by $[\varsigma(t)]_{\alpha} = \left[\underline{\varsigma}(t,\alpha), \overline{\varsigma}(t,\alpha)\right]$. The Seikkala derivative $\varsigma^{'}(t)$ of ς is defined by$[\varsigma'(t)]_{\alpha} = \left[\underline{\varsigma}'(t,\alpha), \overline{\varsigma'}(t,\alpha)\right]$ a equation defines a fuzzy number $\varsigma'(t)\epsilon E$".

Definition 3.2

"Suppose $u\ and\ v$ are fuzzy sets in E. Then their Hausdorff distance is as follows.

$D : E \times E \to R_{+} \bigcup \{0\}, D(u,v) = \underset{\alpha\epsilon[0,1]}{sup}\ max\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}$

i.e D(u,v) is maximal distance between α level sets of $u\ and\ v$.

Fuzzy Initial Value Problem 3.3

"Let's examine the initial value problem.

$\left(p^{(n)}(t) = \psi\left(t, p, p^{'}, \ldots.p^{(n-1)}\right), p(0) = a_1, \ldots.p^{(n-1)}(0) = a_n\right.$

Here, p represents a continuous function mapping from R to R and α and β are fuzzy numbers in R. We modify the Nth-order fuzzy differential equation by altering variables. $q_1(t) = p(t),\ q_2(t) = p'(t), \ldots, q_n(t) = p^{(n-1)}(t),$

converts to the following fuzzy system

$$\left\{\begin{array}{c} q_1'(t) = f_1(t, q_1, \ldots.., q_n) \\ q_{n'}(t) = f_n(t, q_1, \ldots.., q_n) \\ q_1(0) = q_1^{[0]} = q_1, \ldots.q_n(0) = q_n^{[0]} = a_n \end{array}\right. \tag{3.2}$$

where $f_i(1 \leq i \leq n)$ are continuous mapping from $R_{+} \times R^n$ into ${y_i}^{[0]}$ are fuzzy numbers in E with α - level intervals.

$[q_i^{[0]}]_{\alpha} = \left[\underline{q}_i^{(0]}(\alpha), \overline{q}_i^{(0]}(\alpha)\right]$ for i=1,.......n and 0≤ $\alpha \leq 1$.

We call $q = (q_1, \ldots.q_n)^T$ is a fuzzy solution of Equation (3.2) on an interval I, if

$$\begin{array}{c} \underline{q}_i^{'}(t,\alpha) = min\left\{ f_i(t, u_1, \ldots.., u_n); u_j\epsilon\left[\underline{q}_j(t,\alpha), \overline{q}_j(t,\alpha)\right]\right\} = \underline{f}_i(t, q(t,\alpha)) \\ \overline{q}_i^{'}(t,\alpha) = max\left\{ f_i(t, u_1, \ldots.., u_n); u_j\epsilon\left[\underline{q}_j(t,\alpha), \overline{q}_j(t,\alpha)\right]\right\} = \overline{f_i}(t, q(t,\alpha) \end{array} \tag{3.3}$$

and

$$\underline{q}_i^{'}(t,\alpha) = \underline{q}_i^{[0]}(\alpha), \overline{q}_i^{'}(t,\alpha) = \overline{q}_i^{[0]}(\alpha) \tag{3.4}$$

Therefore, for a fixed t, we get a system of initial value problems in x. Once solved uniquely, we merely need to confirm that the intervals $[\alpha(t), \beta(t)]$ define a fuzzy number. $q_i(t) \in E$ Now let $\underline{q}^{[0]}(\alpha) = (\underline{q}_1^{[0]}(\alpha), \ldots\ldots, \underline{q}_n^{[0]}(\alpha))^T$ and

$\overline{q}^{[0]}(\alpha) = (\overline{q}_1^{[0]}(\alpha), \ldots\ldots, \overline{q}_n^{[0]}(\alpha))^T$ with respect to the above-mentioned indicators, system Equation (3.2) can be written as with assumption

$$\left\{\begin{array}{l} q^{'}(t) = F(t, q(t)), \\ q(0) = q^{[0]} \in E^n \end{array}\right. \tag{3.5}$$

With assumption,

$$\begin{array}{l} \underline{q}(t,\alpha) = \left[\underline{q}(t,\alpha), \ldots\ldots, \underline{q}(t,\alpha)\right]^T, \\ q(t,\alpha) = \left[\underline{q}(t,\alpha), \overline{q}(t,\alpha)\right]\ and\ q'(t,\alpha) = \left[\underline{q}'(t,\alpha), \overline{q}'(t,\alpha)\right] \end{array} \tag{3.6}$$

$$\overline{q}(t,\alpha) = [\overline{q}(t,\alpha), \ldots\ldots, \overline{q}(t,\alpha)]^T, \tag{3.7}$$

$$\underline{q}'(t,\alpha) = \left[\underline{q}'(t,\alpha), \ldots\ldots, \underline{q}'(t,\alpha)\right]^T, \tag{3.8}$$

$$\overline{q'}(t,\alpha) = [\overline{q}'(t,\alpha), \ldots\ldots, \overline{q}'(t,\alpha)]^T, \tag{3.9}$$

and under the assumption that
$F(t,q(t,\alpha)) = \left[\underline{F}(t,q(t,\alpha), \overline{F}(t,q(t,\alpha))\right]^n$, where

$$\underline{F}(t,q(t,\alpha)) = \left[\underline{f}_1(t,q(t,\alpha))), \ldots\ldots.\underline{f}_n(t,q(t,\alpha))\right]^n \tag{3.10}$$

$$\overline{F}(t,q(t,\alpha)) = [\overline{f_1}(t,q(t,\alpha))), \ldots\ldots.\overline{f_n}(t,q(t,\alpha))]^n \tag{3.11}$$

q(t) constitutes a fuzzy solution of Equation (3.5) across an interval I for all ∈[0,1] α∈[0,1], provided that

$$\begin{cases} \underline{q}'(t,\alpha)) = \underline{F}(t,q(t,\alpha)); \\ \overline{q}'(t,\alpha) = \overline{F}(t,\alpha)) \\ \underline{q}(0,\alpha) = \underline{q}^{[0]}(\alpha), \overline{q}(0,\alpha) = \overline{q}^{[0]}(\alpha) \end{cases} \tag{3.12}$$

Or

$$\begin{cases} q'(t,\alpha) = F(t,q(t,\alpha)), \\ q(0,\alpha) = q^{[0]}(\alpha). \end{cases} \tag{3.13}$$

4 FRK4AMGMCONTHM Formula for Solving Fuzzy Initial Value Problems

4.1. Runge – Kutta Method for Problems with Initial Values

The problem with initial values is,

$$\begin{cases} y'(t) = f(t,y(t)); a \le t \le b \\ y(a) = \alpha, \end{cases}$$

The basis of all Runge-Kutta method is to express the difference between the value of y at

$$sy_{n+1} - y_n = \sum_{i=1}^{m} w_{ik_i} \tag{3.2}$$

where for i =1, 2, …,m, wi's are constants and

$$k_i = hf(t_n + c_i h, y_n + \sum_{j=1}^{i-1} a_{ij} k_j) \tag{3.3}$$

Equation (3.2) is to be exact for powers of h through h^m, because it is to be coincident with Taylor series of order m.

4.2. RK4AmGmContHm for IVPs

The newly developed fourth-order Runge-Kutta method, which relies on a linear combination of the Arithmetic Mean, Geometric Mean, And Contra-Harmonic Mean for solving initial value problems is presented here,

$$y_{n+1}=y_n+\frac{h}{135}\left[7(\kappa_1+2\kappa_2+2\kappa_3+\kappa_4)-(\sqrt{\kappa_1+\kappa_2}+\sqrt{\kappa_2+\kappa_3}+\sqrt{\kappa_3+\kappa_4})+32(\frac{\kappa_1^2+\kappa_2^2}{\kappa_1+\kappa_2}+\frac{\kappa_2^2+\kappa_3^2}{\kappa_2+\kappa_3}+\frac{\kappa_3^2+\kappa_4^2}{\kappa_3+\kappa_4})\right]$$

Where

$$\begin{gathered}\kappa_1=f(x_n,y_n)\\ \kappa_2=f(x_n+0.5h,y_n+0.5h\kappa_1)\\ \kappa_3=f(x_n+0.5h,y_n+(0.09028)h\kappa_1+0.4097\kappa_2)\\ \kappa_4=f(x_n+h,y_n+(0.18055)h\kappa_1-0.48046h\kappa_2+1.2991h\kappa_3)\end{gathered} \tag{4.1}$$

with the grid points $a=c_0\leq c_1\leq ...\leq c_N=b$ and $h=\frac{(b-a)}{N}=c_{i+1}-c_i$

4.3. Procedure for Solving system of FIVPs using RK4AmGmContHm (FRK4AMGMCONTM)

For finding the unique solution of Equation (3.2) and approximate solution of Equation (3.2) by applying the proposed method (FRK4AMGMCONTM), let us proceed as follows:

$$\begin{gathered}\underline{q}(s_{n+1};\alpha)-\underline{q}(s_n;\alpha)=\sum_{i=1}^{4}\mu_i\underline{\kappa}_i(s_n,q(s_n;\alpha),h),\\ \overline{q}(s_{n+1};\alpha)-\overline{q}(s_n;\alpha)=\sum_{i=1}^{4}\mu_i\overline{\kappa}_i(s_n,q(s_n;\alpha),h)\end{gathered} \tag{4.2}$$

where the w_i's are constants and

$$\begin{gathered}[\kappa_i(s,q(s_n;\alpha),h)]_r=\left[\underline{\kappa}_i(s,q(s_n;\alpha),h),\overline{\kappa}_i(s,q(s_n;\alpha),h)\right],i=1,2,3,4\\ \underline{\kappa}_i(s_n,q(s_n;\alpha),h)=f(s_n+c_ih,\underline{q}(s_n)+\sum_{j=1}^{i-1}a_{ij}\underline{k}_j(s_n,q(s_n;\alpha),h)),\\ \overline{\kappa}_i(s_n,q(s_n;\alpha),h)=f(s_n+c_ih,\overline{q}(s_n)+\sum_{j=1}^{i-1}a_{ij}\overline{\kappa}_j(s_n,q(s_n;\alpha),h)),\end{gathered} \tag{4.3}$$

$$\underline{\kappa}_{ij}(s,q(s_n;\alpha),h)=min\left\{f_i(s_n,X_{1,.......},X_n)\ \ x_j\in\left[\underline{q}_j(s;\alpha),\overline{q}_j(s;\alpha)\right]\right\},(1\leq i,j\leq n)$$

$$\overline{\kappa}_{ij}(s,q(s_n;\alpha),h)=max\left\{f_i(s_n,X_{1,.......},X_n)\ \ x_j\in\left[\underline{q}_j(s;\alpha),\overline{q}_j(s;\alpha)\right]\right\},(1\leq i,j\leq n)$$

$$\underline{\kappa}_{i2}(s,q(s_n;\alpha),h)=min\left\{f_i\left(s+\frac{h}{2},X_{1,.......},X_n\right)\ \ x_j\in\left[\underline{m}_{j1}(s;\alpha),\overline{m}_{j1}(s;\alpha)\right]\right\},$$

$$\overline{\kappa}_{i2}(s,q(s_n;\alpha),h)=max\left\{f_i\left(s+\frac{h}{2},X_{1,.......},X_n\right)\ \ x_j\in\left[\underline{m}_{j1}(s;\alpha),\overline{m}_{j1}(s;\alpha)\right]\right\},$$

$$\underline{\kappa}_{i3}(s,q(s_n;\alpha),h)=min\left\{f_i\left(s+\frac{h}{2},X_{1,.......},X_n\right)\ \ x_j\in\left[\underline{m}_{j2}(s;\alpha),\overline{m}_{j2}(s;\alpha)\right]\right\},$$

$$\overline{\kappa}_{i3}\left(s,q\left(s_n;\alpha\right),h\right)=max\left\{f_i\left(s+\frac{h}{2},X_{1,\dots\dots,}X_n\right)\ \ x_j\in\left[\underline{m}_{j2}\left(s;\alpha\right),\overline{m}_{j2}\left(s;\alpha\right)\right]\right\},$$

$$\underline{\kappa}_{i4}\left(s,q\left(s_n;\alpha\right),h\right)=min\left\{f_i\left(s+h,X_{1,\dots\dots,}X_n\right)\ \ x_j\in\left[\underline{m}_{j3}\left(s;\alpha\right),\overline{m}_{j3}\left(s;\alpha\right)\right]\right\},$$

$$\overline{\kappa}_{i4}\left(s,q\left(s_n;\alpha\right),h\right)=max\left\{f_i\left(s+h,X_{1,\dots\dots,}X_n\right)\ \ x_j\in\left[\underline{m}_{j3}\left(s;\alpha\right),\overline{m}_{j3}\left(s;\alpha\right)\right]\right\} \tag{4.4}$$

Such that

$$\underline{m}_{j1}\left(s,q\left(s;\alpha\right),h\right)=\underline{q}_j\left(s,\alpha\right)+0.5h\underline{\kappa}_{j1}\left(s,q\left(s;\alpha\right),h\right)$$

$$\overline{m}_{j1}\left(s,q\left(s;\alpha\right),h\right)=\overline{q}_j\left(s,\alpha\right)+0.5h\overline{\kappa}_{j1}\left(s,q\left(s;\alpha\right),h\right)$$

$$\underline{m}_{j2}\left(s,q\left(s;\alpha\right),h\right)=\underline{q}_j\left(s,\alpha\right)+0.09028h\underline{\kappa}_{j1}\left(s,q\left(s;\alpha\right),h\right)+0.4097h\underline{\kappa}_{j2}\left(s,q\left(s;\alpha\right),h\right)$$

$$\overline{m}_{j2}\left(s,q\left(s;\alpha\right),h\right)=\overline{q}_j\left(s,\alpha\right)+0.09028h\overline{\kappa}_{j2}\left(s,q\left(s;\alpha\right),h\right)+0.4097h\overline{\kappa}_{j2}\left(s,q\left(s;\alpha\right),h\right)$$

$$\begin{aligned}&\underline{m}_{j3}\left(s,q\left(s;\alpha\right),h\right)\\&\quad=\underline{q}_j\left(s,\alpha\right)+0.18055h\underline{\kappa}_{j1}\left(s,q\left(s;\alpha\right),h\right)-48046h\underline{\kappa}_{j2}\left(s,q\left(s;\alpha\right),h\right)+1.29991h\underline{\kappa}_{j3}\left(s,q\left(s;\alpha\right),h\right)\end{aligned}$$

$$\begin{aligned}&\overline{m}_{j3}\left(s,q\left(s;\alpha\right),h\right)\\&\quad=\overline{q}_j\left(s,\alpha\right)+0.18055h\overline{\kappa}_{j3}\left(s,q\left(s;\alpha\right),h\right)-48046h\overline{\kappa}_{j2}\left(s,q\left(s;\alpha\right),h\right)+1.29991h\overline{\kappa}_{j3}\left(s,q\left(s;\alpha\right),h\right)\end{aligned} \tag{4.5}$$

now consider the following relations

$$\begin{aligned}F_i\left(s,q\left(s;\alpha\right)\right)=&\Bigg[7((\underline{\kappa}_{i1}\left(s,q\left(s;\alpha\right),h\right)+2(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)+2(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)+\left(\underline{\kappa}_{i4}\left(s,q\left(s;\alpha\right),h\right)\right)-\\&\left(\sqrt{\underline{\kappa}_{i1}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)}+\sqrt{\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)}+\sqrt{\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i4}\left(s,q\left(s;\alpha\right),h\right)}\right)\\&+32\left(\frac{(\underline{\kappa}^2_{i1}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}^2_{i2}\left(s,q\left(s;\alpha\right),h\right)}{(\underline{\kappa}_{i1}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)}+\frac{(\underline{\kappa}^2_{i2}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}^2_{i3}\left(s,q\left(s;\alpha\right),h\right)}{(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)}+\frac{(\underline{\kappa}^2_{i3}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}^2_{i4}\left(s,q\left(s;\alpha\right),h\right)}{(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i4}\left(s,q\left(s;\alpha\right),h\right)}\right)\Bigg]\end{aligned}$$

$$\begin{aligned}G_i\left(s,q\left(s;\alpha\right)\right)=&\Bigg[7((\underline{\kappa}_{i1}\left(s,q\left(s;\alpha\right),h\right)+2(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)+2(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)+\left(\underline{\kappa}_{i4}\left(s,q\left(s;\alpha\right),h\right)\right)\\&-\left(\sqrt{\underline{\kappa}_{i1}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)}+\sqrt{\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)}+\sqrt{\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i4}\left(s,q\left(s;\alpha\right),h\right)}\right)\\&+32\left(\frac{(\underline{\kappa}^2_{i1}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}^2_{i2}\left(s,q\left(s;\alpha\right),h\right)}{(\underline{\kappa}_{i1}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)}+\frac{(\underline{\kappa}^2_{i2}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}^2_{i3}\left(s,q\left(s;\alpha\right),h\right)}{(\underline{\kappa}_{i2}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)}+\frac{(\underline{\kappa}^2_{i3}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}^2_{i4}\left(s,q\left(s;\alpha\right),h\right)}{(\underline{\kappa}_{i3}\left(s,q\left(s;\alpha\right),h\right)+(\underline{\kappa}_{i4}\left(s,q\left(s;\alpha\right),h\right)}\right)\Bigg]\end{aligned}$$

5 Numerical Example

Example 5.1

Following is a second order Fuzzy Differential Equation $\begin{cases} q''(s) - 4q'(s) + 4q(s) = 0, (s \geq 0) \\ q(0) = (2+r, 4-r) \\ q'(0) = (5+r, 7-r) \end{cases}$ $with\ Fuzzy\ Initial\ Value$

The exact solution to this problem is outlined below

$\underline{q}(s,\alpha) = (2+\alpha)e^{2s} + (1-\alpha)se^{2s}$

$\overline{q}(s,\alpha) = (4-\alpha)e^{2s} + (\alpha-1)se^{2s}$

Figure 1 represents the absolute error of the fourth order Runge-Kutta Method based on the linear combination of RK4AM, RK4GM, RK4CONTM (FRK4AMGMCONTM), for Example 5.1 with h=0.1,s=0.1.

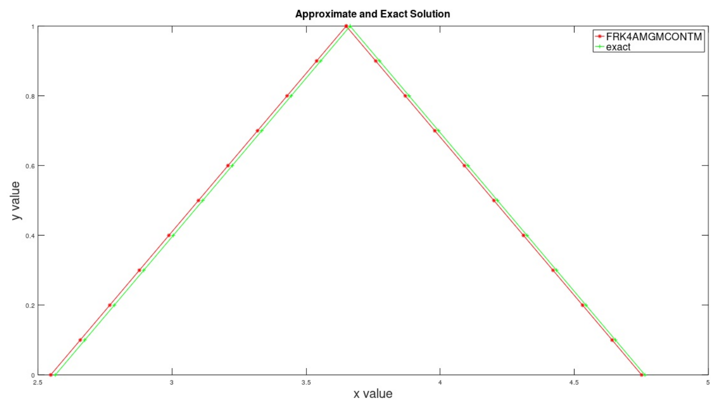

Fig 1. Comparison between Approximate and Exact Solution of example 5.1

Example 5.2 :

Following is a greater order FIVP $\begin{cases} p'''(s) = 2p''(s) + 3p'(s), (0 \leq s \leq 1) \\ p(0) = (3+r, 5-r) \\ p'(s) = (-3+r, -1-r) \\ p''(s) = (8+r, 10-r) \end{cases}$

The analytical solution of this problem is as follows :

$$p(s,\alpha) = \left[\left(-\frac{1}{3} + \frac{7}{4}\right)e^{3s} + \left(\frac{11}{4} + \alpha\right)e^{-s}, \left(-\frac{1}{3} + \frac{7}{4}\right)e^{3s} + \left(\frac{19}{4} - \alpha\right)e^{-s}\right]$$

The approximate and exact solution for example 5.2 using FRK4AMGMCONTM with h=0.1,s=0.1 is displayed here.

Table 1. The Approximate and Exact solution for Example 5.2

r	S	Approximate value		Exact value		Absolute Error for Lower Cut of y	Absolute Error for Upper Cut of y
0	0.1	2.913520	2.942387	4.7078a37	4.752062	2.886731e-02	4.422495e-02
0.1	0.1	3.003441	3.032871	4.618194	4.661578	2.942950e-02	4.338451e-02
0.2	0.1	3.093305	3.123355	4.528548	4.571095	3.004944e-02	4.254658e-02
0.3	0.1	3.183126	3.213838	4.438899	4.480611	3.071242e-02	4.171146e-02
0.4	0.1	3.272914	3.304322	4.349248	4.390127	3.140836e-02	4.087948e-02
0.5	0.1	3.362676	3.394806	4.259592	4.299643	3.213009e-02	4.005104e-02
0.6	0.1	3.452417	3.485290	4.169933	4.209160	3.287238e-02	3.922660e-02
0.7	0.1	3.542142	3.575773	4.080269	4.118676	3.363134e-02	3.840672e-02
0.8	0.1	3.631853	3.666257	3.990600	4.028192	3.440399e-02	3.759204e-02

Continued on next page

Table 1 continued

0.9	0.1	3.721553	3.756741	3.900925	3.937708	3.518803e-02	3.678336e-02
1	0.1	3.811243	3.847225	3.811243	3.847225	3.598163e-02	3.598163e-02

6 Comparative Analysis

The problem given in example 5.2 is solved by both the classical Range-Kutta method of order 4 and by the proposed RK4AMGMCONTM methods. We find that, the newly attained method FRK4AMGMCONTM to solve FIVPs gives nearer values to the exact value than the classical Range-Kutta method of order 4. Thus, the newly proposed method gives the minimal error value and thus produces good accuracy in the result.

Table 2 represents the comparison between the newly attained FRK4AMGMCONTM method with the existing classical FRK4 method for example 5.2 with h = 0.1, s = 0.1.

Table 2. Comparative Analysis between The Classical RK4 Method and the Proposed Method for Example 5.2

		FRK4AMGMCONTM		FRK4	
R	**S**	**Absolute Error for Lower Cut of y**	**Absolute Error for Upper Cut of y**	**Absolute Error for Lower Cut of y**	**Absolute Error for Upper Cut of y**
0	0.1	2.886731e-02	4.422495e-02	1.380767e-01	3.002923e-01
0.1	0.1	2.942950e-02	4.338451e-02	1.475510e-01	2.928623e-01
0.2	0.1	3.004944e-02	4.254658e-02	1.567360e-01	2.853960e-01
0.3	0.1	3.071242e-02	4.171146e-02	1.656807e-01	2.778905e-01
0.4	0.1	3.140836e-02	4.087948e-02	1.744226e-01	2.703426e-01
0.5	0.1	3.213009e-02	4.005104e-02	1.829911e-01	2.627486e-01
0.6	0.1	3.287238e-02	3.922660e-02	1.914097e-01	2.551044e-01
0.7	0.1	3.363134e-02	3.840672e-02	1.996975e-01	2.474053e-01
0.8	0.1	3.440399e-02	3.759204e-02	2.078704e-01	2.396460e-01
0.9	0.1	3.518803e-02	3.678336e-02	2.159414e-01	2.318204e-01
1	0.1	3.598163e-02	3.598163e-02	2.239216e-01	2.239216e-01

7 Conclusion

The attempt of solving FIVPs of order n has been a continuous focus of research, employing wide range of methodologies. Initially, the FIVPs of order n are transformed into a system of fuzzy equations. Then, these systems of equations are solved by using Runge-Kutta method of order four which relies on a linear combination of the Arithmetic Mean, Geometric Mean, and Contra-Harmonic Mean.

By comparing the absolute error values at s=0.1 and s=0.2 from the numerical examples 5.1 and 5.2, and also based on the data presented in Table 1 and from Figure 1, the efficiency of the discussed method is clearly demonstrated. The obtained results revealed that the proposed method is more efficient than the classical RK method of order four.

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