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RESEARCH ARTICLE

Investigation on Pebbling Energy for Standard Graphs

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Abstract

Objectives: Introducing the idea of energy pebbling to path, star, tree, and cycle graphs is the main objective of this study. The sum of the absolute values of the eigenvalues in the adjacency matrix of a pebbling network is its energy. We then construct the upper and lower bounds for the aforementioned graphs in this discussion. **Methods:** Removing two pebbles from the first vertex and placing one pebble on the neighboring vertex illustrates a pebbling move. The pebbling graph's adjacency matrix is computed as $A_p(G) = \begin{cases} e(v_i, v_j), if\, i\, and\, j\, are\, adjacent \\ 0\ otherwise \end{cases}$ where $e(v_i, v_j)$ is the edge value of v_i is adjacent to v_j. The energy of the graph is calculated using $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_i' s$ are the eigenvalues of the graph G. Analyses are made for the cycle, path, star, and tree graphs' lower and upper bounds. **Findings:** In regards to the path, star, cycle, and tree pebbling graph, the relationship between energy, and lower and upper bounds was discovered and tabulated. **Novelty:** Pebbling path, star, cycle, and tree graphs were subjected to the energy idea, and a relationship was found between the lower bound, upper bound, and energy of pebbling graphs in general.

Keywords: Pebbling; Adjacency energy; Lower bound; Energy

1 Introduction

Graph pebbling is a fun hobby that has fascinated scientists all around the world. Graph pebbling has its roots in number theory, which goes back to the early 1900s. Graph pebbling was initially proposed in 1989 by Fan R.K. Chung [1]. Think of a graph that has two vertices. To illustrate a pebbling move, take two pebbles out of the first vertex and place one on the vertex next to it. The loss of a pebble during a locomotive determines the transportation cost. In 2020, Amitav Doley et al. talked about the energy graphs that match to the misbalance degree matrix [2]. Acharya B.D et al. described the energy of a set of vertices in a network. Sakunthala Srinivasan and Vimala Shanmugavel proposed the concept of adjacency energy to pebbling graphs in 2020, while the same authors explored seidel energy to pebbling graphs with indices in 2021 [3] [4]. The energy was used to analyze bipolar fuzzy graphs [5].

In 2023, Harishchandra S. Ramane et al. discovered a relationship between graph energy and Seidel energy [6]. Following the COVID-19 pandemic, Rupesh Atram and Sharad A. Barde talked about gathering research on graph energy [7]. In 2023, Hassan J. AI Hwaeer and Shahad H. Hadi provided a description of the energy and resolved energy in multi-monad graphs [8]. The pebbling concept was added to the Goldberg Snark Graph by Sagaya Suganya A and Saibulla A in 2023 [9], and Noor A'lawiah Abd Aziza et al. introduced new bounds for the graph's energy in 2024 [10]. Sakunthala Srinivasan and Vimala Shanmugavel utilized the idea of minimal degree energy to pebbling graphs in 2024 [11]. In the existing literature, the energy of path [12], tree, star and cycle and their bounds were presented. The concept of graph pebbling was explained briefly by Herman Bergwerf [13]. The authors applied the pebbling concept to the energy of path, tree, star, and cycle graphs and further their bounds were also evaluated.

Using pebbling path, star, cycle, and tree graphs, the author of this study illustrated the connection between lower bound, upper bound, and energy.

2 Methodology

Preliminaries

Pebbling cycle graph

A pebbling move can be demonstrated by evicting two pebbles from the first vertex and accommodating one pebble on the neighboring vertex in a cycle network with two vertices termed a pebbling cycle graph.

Energy of pebbling path, star, cycle, and tree graph

The energy of a pebbling path graph $E_{PP}(G)$, the energy of a pebbling star graph $E_{PS}(G)$, the energy of a pebbling cycle graph $E_{PC}(G)$ and energy of the pebbling tree graph $E_{PT}(G)$ were calculated by adding the eigenvalues of a pebbling path graph, pebbling star graph, pebbling cycle graph, and pebbling tree graph that meets the criterion $E_{PP}(G) = \sum_{i=1}^{n} |\lambda_i|$, $E_{PS}(G) = \sum_{i=1}^{n} |\lambda_i|$, $E_{PC}(G) = \sum_{i=1}^{n} |\lambda_i|$ and $E_{PT}(G) = \sum_{i=1}^{n} |\lambda_i|$ respectively.

3 Results and Discussion

3.1 Basic Properties of adjacency energy of pebbling graphs

Let's start by defining the letter B as $B = \sum_{i<j} e(v_i, v_j)$

Proposition 1

Let G be a pebbling graph with vertices and edges. Let $\phi(A_P(G), \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of $A_P(G)$. Then

- $a_0 = 1$
- $a_1 = 0$
- $a_2 = -B$

Proof:

- By the definition of $\phi(A_P(G), \lambda)$, it follows that $a_0 = 1$.
- The trace of $A_P(G)$ is equal to the sum of determinants of all the 1×1 principal submatrices of $A_P(G)$, implying $a_1 = (-1)^1 \times trace\,of\, A_P(G) = 0$.
- From the definition of $\phi(A_P(G), \lambda)$,

$$(-1)^2 a_2 = \sum_{1 \le i \le j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$
$$= \sum_{1 \le i \le j \le n} [a_{ii}a_{jj} - a_{ij}a_{ji}]$$
$$= -B$$

Proposition 2

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_P(G)$, then $\sum_{i=1}^{n} \lambda_i^2 = 2B$.

Proof:

$\sum_{i=1}^{n}\lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}$

$= 2\sum_{i<j}a_{ji}^2 + \sum_{i=1}^{n}a_{ii}^2$

= 2B.

The lower and upper bounds for pebbling graph adjacency energy can be established using this finding.

Theorem 3 [14]

Allow G to be an n-verticed pebbling graph. Then E (G) = $\sqrt{2nB}$.

Proof:

Let $\lambda_1 \geq \lambda_2 \geq \ldots\ldots \geq \lambda_n$ be the eigenvalues of $A_P(G)$. Now, according to Cauchy-Schwartz inequality,

$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$

Putting a_1 = 1 and $b_1 = |\lambda_i|$ in the above inequality,

We get $\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\lambda_i|^2\right)$

$[E(G)]^2 \leq n.2B$

Hence, $E(G) \leq \sqrt{2nB}$

This is an upper bound on pebbling graphs' adjacency energy.

Theorem 4

Let n-verticed pebbling graph. If $D = detA_P(G)$then $E(G) \geq \sqrt{2B + n\left(n-1\right)\left(D\right|^{\left(\frac{2}{n}\right)}}$

Proof:

Consider,

$[E(G)]^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$

$= \left(\sum_{i=1}^{n}|\lambda_i|\right)\left(\sum_{j=1}^{n}|\lambda_j|\right)$

$= \left(\sum_{i=1}^{n}|\lambda_i|^2\right) + \sum_{i\neq j}|\lambda_i||\lambda_j|$

Now by arithmetic mean and geometric mean inequality, the result becomes

$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$

Thus,

$[E(G)]^2 \geq \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$

$\geq \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 + n(n-1)\left(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$

$\geq \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 + n(n-1)D^{\left(\frac{2}{n}\right)}$

$= 2B + n(n-1)D^{\left(\frac{2}{n}\right)}$, **by Proposition 2**

Therefore, $E(G) \geq \sqrt{2B + n\left(n-1\right)\left(D\right|^{\left(\frac{2}{n}\right)}}$

This is a lower bound on pebbling graphs' adjacency energy.

3.2 Adjacency pebbling energy and bounds of path, cycle, tree, and star graphs

Theorem 5

Let G be a pebbling path graph with p vertices and q edges, then the pebbling path graph's bounds and energy values are

1. Lower bound < Energy < Upper bound for p= 2, 3, 4, 5,6,7,8 and q= 1, 2, 3,4,5,6,7
2. Lower bound < Energy < Upper bound for p = even, q = odd
3. Energy < Lower bound < Upper bound for p = odd, q = even

Proof:

- The vertices and edge values of a pebbling path graph can be obtained using preliminaries 2.1.2. Evicting two pebbles from the first vertex and accommodating one pebble on the neighbouring vertex is an example of a pebbling move. Then, using the formula $\sqrt{2B+n(n-1)(D|^{(\frac{2}{n})}}$ and $\sqrt{2nB}$, $E_{PP}(G)=\sum_{i=1}^{n}|\lambda_i|$ one can determine the lower bound and upper bound, energy. After calculating the upper bound, energy, and lower bound, the relation that, Lower bound < Energy < Upper bound is true for p= 2, 3, 4, 5,...,8 and q= 1, 2, 3,...,7.
- Similarly, using the above formulas, the relation Lower bound < Energy < Upper bound for p= even q= odd was obtained.
- Using preliminaries 2.1.2, the relation that, Energy < Lower bound < Upper bound for p =odd (where = 9, 11…) and q= even was obtained.

Note 1: The above theorem is applicable to the pebbling cycle graph.

Example:

A path graph with eight vertices and seven edges is shown in Figure 1. The transit originates at 'V_1' and should terminate at 'V_8'. Initially, one twenty-eight pebbles are present at vertex 'V_1'. The transit takes place from 'V_1' to 'V_2' losing sixty-four pebbles as fuel and reaches 'V_2' with sixty-four pebbles, likewise, the transit takes place from 'V_2' to 'V_3' losing thirty-two pebbles as fuel and reaches 'V_3' with thirty-two pebbles. During transit at 'V_3'to 'V_4', sixteen pebbles are lost as fuel and reach 'V_4'with sixteen pebbles, and the transit between 'V_4'to 'V_5', eight pebbles are lost as fuel and eight pebbles reach 'V_5'. Similarly, during the transit among 'V_5'to 'V_6', 'V_6'to 'V_7' and 'V_7'to 'V_8' four pebbles are lost as fuel at 'V_5'and four pebbles reaches 'V_6', two pebbles are lost as fuel at 'V_6' and two pebbles reaches 'V_7', one pebble lost as fuel at 'V_7' and one pebble reaches 'V_8' respectively.

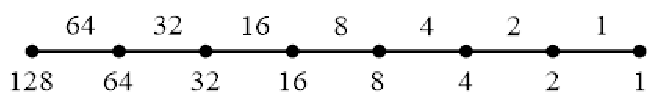

Fig 1. Pebbling path graph

The octane structure in chemistry is shown in Figure 2. The pebbling path graph was compared with the hydrogen-depleted molecular octane structure in Figure 3. The Adjacency matrix of the pebbling path graph is

$$A_{pp}(G)=\begin{bmatrix}0&64&0&0&0&0&0&0\\64&0&32&0&0&0&0&0\\0&32&0&16&0&0&0&0\\0&0&16&0&8&0&0&0\\0&0&0&8&0&4&0&0\\0&0&0&0&4&0&2&0\\0&0&0&0&0&2&0&1\\0&0&0&0&0&0&1&0\end{bmatrix}$$

Eigenvalues are –71.9306, 71.9306,–16.4330, 16.4330,–4.0257, 4.0257,-0.8608, 0.8608.

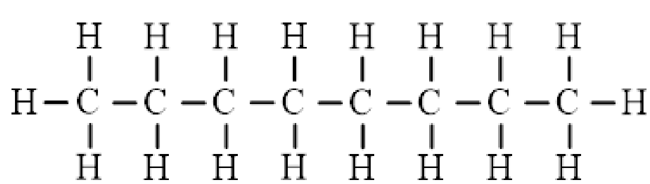

Fig 2. Octane Structure

The computed expression is Lower bound $\leq$ $E_{PP}(G)\leq$. The upper bound for the pebbling Path graph is true for 3,4,5,6,7,8,10 and $E_{PP}(G)$ < =9, as depicted in Table 1.

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Fig 3. Hydrogen depleted molecular octane structure

Table 1. Energy, Upper bound and Lower bound values of pebbling path graphs

Graph Name	Chemical Name	Energy	Lower Bound	Upper Bound
P_2	Ethane	2	0	2
P_3	Propane	4.4722	3.16228	5.47722
P_4	Butane	11.0992	9.48683	12.96148
P_5	Pentane	22.06	13.03840	29.15476
P_6	Hexane	45.9136	−200.69	63.96874
P_7	Heptane	92.357	52.2494	138.24
P_8	Octane	186.500	120.44085	295.594
P_9	Nonane	19.547	209.02	627.06
P_{10}	Decane	40.152	−4448707.6	1321.976
P_{11}	Undecane	19.4004	836.09	2773

Theorem 6

Let G be a pebbling star graph with $|V| = p$vertices and $|E| = q$ edges. Then $\sqrt{2B + n(n-1)(D|^{\left(\frac{2}{n}\right)}} \leq\ E_{PS}(G) \leq \sqrt{2nB}$.

Proof.

One can find the vertices and edge values of a pebbling star graph using preliminaries 2.1.2 of the energy of a pebbling star graph. Evicting two pebbles from the first vertex and accommodating one stone on the adjacent vertex is an example of a pebbling move. The lower and upper bounds are then calculated using the equations $\sqrt{2B + n(n-1)\,|D|^{\left(\frac{2}{n}\right)}}$ and $\sqrt{2nB}$ respectively. The pebbling star graph's energy can be calculated using $E_{PS}(G) = \sum_{i=1}^{P} |\lambda_i|$. After calculating the values, the relation $\sqrt{2B + n(n-1)\,|D|^{\left(\frac{2}{n}\right)}} \leq\ E_{PS}(G) \leq \sqrt{2nB} exists.$

Note 2: For the pebbling tree, the above theorem holds.

Example:

A Star graph with eleven vertices and ten edges is shown in Figure 4.

The eigenvalues of Figure 4 are 0, 0, 0, 0, 0, 0, 0, 0, 0, 3.1623, and −3.1623.

$|A_{PS}(G)|$= 0. Energy, $E_{PS}(G)$= 6.3246

The authors described the energy bounds in the order of Lower bound $\leq\ E_{PS}(G) \leq$ Upper bound in Table 2 and this is valid for $S_{1,n}$ where n = 1, 2,...n.

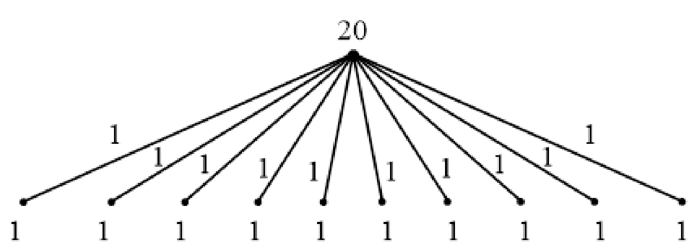

Fig 4. Pebbling star graph

Table 2. Energy, Upper bound and Lower bound values of pebbling star graphs

Graph name	Energy	Lower bound	Upper bound
S_2	2	1.4142	2
$S_{1,2}$	2.8284	2	3.4641
$S_{1,3}$	3.4642	2.4495	4.8989
$S_{1,4}$	4	2.8284	6.3245
$S_{1,5}$	4.4722	3.1623	7.7459
$S_{1,6}$	4.899	3.4641	9.1651
$S_{1,7}$	5.2916	3.7416	9.7979
$S_{1,8}$	5.6568	4	12
$S_{1,9}$	6	4.2426	13.4164
$S_{1,10}$	6.3246	4.4721	14.8324
$S_{1,11}$	6.6332	4.6904	16.2481
$S_{1,12}$	6.9282	4.8989	17.6635
$S_{1,13}$	7.2112	5.0990	19.0788
$S_{1,14}$	7.4834	5.2915	20.4939

Example 7

A cycle graph with seven vertices and six edges is shown in Figure 5. The eigenvalues of the Pebbling cycle graph are 0, -35.9653, 35.9653, –8.2165, 8.2165, –1.9967, and 1.9967. Energy, $E_{PC}(G)$= 92.357.

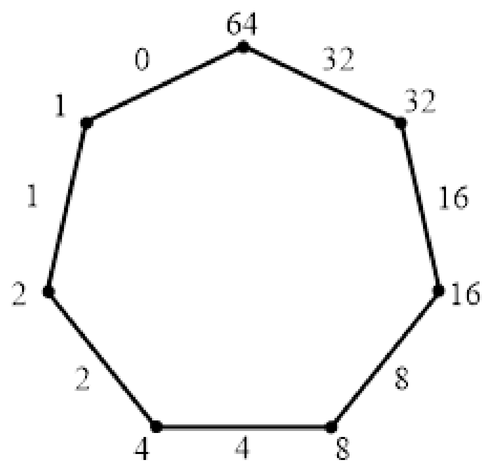

Fig 5. Pebbling cycle graph

The authors obtained a lower-bound $E_{PC}(G)$. The upper bound for pebbling cycle=2, 3, 4,5,6,7,8, $E_{PC}(G)$< Lower bound < Upper bound=9, 11 which is presented in Table 3.

Table 3. Energy, Upper bound and Lower bound values of pebbling cycle graphs

Graph name	Energy	Lower Bound	Upper Bound
C_2	2	0	2
C_3	4.4722	3.16228	5.47722
C_4	11.0992	9.48683	12.9615
C_5	22.06	13.03840	29.1547
C_6	45.9136	–200.69	63.9687
C_7	92.357	52.2494	138.24
C_8	186.5002	120.440857	295.594
C_9	19.547	209.02	627.06
C_{10}	40.152	–4448707.6	1321.97
C_{11}	19.4004	836.09	2773

Example 8

A tree with ten vertices and nine edges is shown in Figure 6. The eigenvalues of the Pebbling tree graph are 11.8492, −11.8492, −3.8842, 3.8842, −1.5165, 1.5165, −1.0000, 1.0000, 0.4585, −0.4585. Energy, $E_{PT}(G)$ = 37.4168.

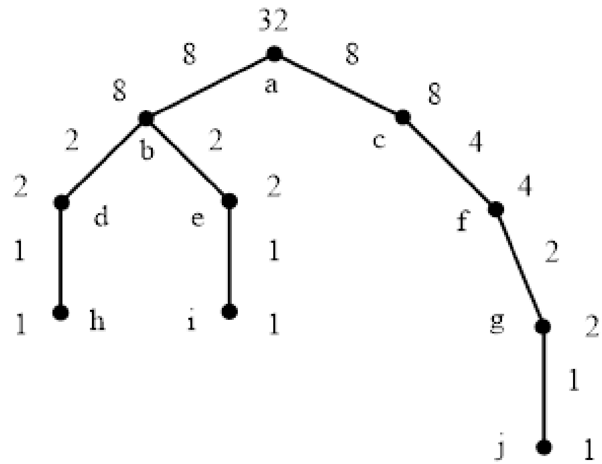

Fig 6. Pebbling tree graph

The authors described the energy bounds in the order of Lower bound $\leq E_{PT}(G) \leq$ The upper bound in Table 4 and this is valid for all values of T_n where n = 1, 2,...n.

Table 4. Energy, Upper bound and Lower bound values of pebbling tree graphs

Graph name	Energy	Lower Bound	Upper Bound
T_2	2	0	2
T_3	2.8284	2	3.4641016
T_4	8.4852	6	12
T_5	9.3442	6.164414	13.784049
T_6	11.524	6.782329	16.613248
T_7	16.671	8.831761	23.366643
T_8	22.058	15.42725	13.749016
T_9	27.410	13.78405	41.352146
T_{10}	37.417	25.88436	56.391489
T_{11}	47.544	22.58318	74.899933
T_{12}	57.631	27.67670	95.874919
T_{13}	57.964	27.71281	99.919968
T_{14}	58.146	27.74887	103.82678
T_{15}	58.301	27.78489	107.61041

The graphs in this paper are hydrogen-depleted molecular graphs created by the authors. Every vertex in an energy graph represents a carbon atom, and every edge represents a carbon-carbon bond. Carbon is required for life to exist. It can be found in food, clothing, cosmetics, and gasoline. Carbon is found in anthracite, graphite, and diamond in nature. It has a significant impact on life's chemistry.

Many computer science disciplines, including distributed computing, communication difficulty, and algorithm design, have applications for energy pebbling. Also, emphasized are its connections to other theoretical concepts like space-time tradeoffs in computation and reversible computing. Because it provides a framework for analyzing the computational and resource constraints of particular graph-based issues, energy pebbling is often a helpful tool in theoretical computer science research.

4 Conclusion

The results of energy in the pebbling path graph, pebbling star graph, pebbling cycle graph, and energy in the pebbling tree graph were discovered using the information provided above. The author expands on their work by examining various types of pebbling graphs and their applications. In theoretical computer science, specifically in the area of computational complexity theory, there is a concept known as "energy pebbling" that involves a graph-based game where the player's goal is to distribute energy (represented by "pebbles") in a way that allows for the achievement of a particular configuration or objective.

Open problems:

1. Determine the graph class in which the energy of adjacency pebbling is equal to the normal energy.
2. Identify and characterize the class of hyperenergetic, cospectral graphs with respect to adjacency pebbling energy.

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