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Odd and Co – Odd Sum Degree Domination in Graphs with Algorithm

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Abstract

Objectives: The main aim of this study is to introduce two new domination parameters, namely odd and co-odd sum degree domination numbers, and to provide an algorithm for finding the odd sum degree dominating set of a graph. **Methods**: The odd and co-odd sum degree domination numbers of a graph G are introduced by verifying the following two conditions on dominating and co-dominating sets of G. (i) The sum of degree of vertices of the dominating and co-dominating sets are odd numbers. (ii) The cardinality of the above two sets are minimum. Further, obtained the degree sum values of the above two sets are referred as odd and co-odd sum degree domination values of G. **Findings:** The odd and co-odd sum degree domination numbers and values are defined and their exact values are obtained for some standard classes of graphs. The bounds of the above numbers are found in terms of basic graph theoretical terminologies. Further, the above said domination parameters are characterized. The relationships with other domination parameters are determined. **Novelty:** Domination is a wide area with more applications in real-life situations. It has numerous applications in modern science and engineering. The odd and even numbers play a vital role in the following areas like Number Theory in Mathematics, Sorting Algorithms in Computer Science, Arrangements of atoms in Molecular Chemistry, etc. The degree-based domination also has plenty of applications like ensuring redundancy and reliability in Communication Networks, identifying sets of nodes or links that provide fault tolerance in Electrical Networks, etc. Motivated by the notion of the above applications in this paper, the odd and co-odd sum degree dominations are introduced. Further, the step-by-step algorithm for finding the newly introduced parameter is also presented with an illustration.

Keywords: Odd sum degree dominating set; Odd sum degree domination number; Odd sum degree domination value; Co-odd sum degree dominating set; Co-odd sum degree domination number; and Co-odd sum degree domination value.

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1 Introduction

Let $G = (V, E)$ be a simple, undirected, and connected graph. A set $R \subset V(G)$ is a dominating set. If every vertex in $V(G)–R$ is adjacent to at least one vertex in R, the vertex set V-R is said to be a co-dominating set of G. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The maximum and minimum degree of graph G is denoted by $\triangle(G)$ and $\delta(G)$ respectively. A graph with p vertices and q edges is called (p,q) graph. A regular graph is a graph where each vertex has the same number of neighbors. All the graphs used in this paper are referred from Joshep A. Gallian [1]. The basic terminologies, namely the degree of a vertex r *(degr)*, the distance between two vertices r_1 & r_2 $(d(r_1,r_2))$, and the diameter of a graph G *(diam(G))* are referred from Kulli V.R [2]. A set of vertices D is called an odd dominating set if for every vertex $v \in V(G)$, $|N[v] \cap D| \equiv (mod\ 2)$. The odd domination number of a graph was introduced by Yair Caro and William F. Klostermeyer in the year 2003 [3]. The odd and even dominating sets with open neighborhoods are introduced by John L. Gold Wasser and William F. Klostermeyer in the year 2007 [4]. The set D is called a co-even dominating set if v has even neighbors for all $v \in V - D$. This set is denoted by a co-even dominating set. The co-even domination number is introduced by M.M.Shalaam and A.A. Omran in the year 2020 [5,6]. The set D is called an even sum dominating set (ESDS) if, $\forall\ u \in V$-D there is a vertex $v \in D$ such that $deg(v) + deg(u)$ is even. In 2022 [7,8] Ishan M. Rasheed, A. A Omran, Ahllam A. Alfatlawi, introduced the even sum domination number. Motivated by the notion of the above parameters and their applicability, in this paper, the odd and co-odd sum degree dominations are introduced and studied for some standard classes of graphs such as Path, Cycle, Wheel, Comb, Star, Complete, Crown, Book, Friendship and Fan graphs. Further, the bounds of the above parameters are determined and the relationships between some of the existing domination parameters are found. Also, the odd and co-odd sum degree dominating sets are characterized. A. Nagoorgani and V.T. Chandrasekaran introduced an algorithm for finding the domination number of a graph [9]. In 2021 [10], Richard E. Korf was introduced to a breadth-first search algorithm for finding the exact diameter of a graph. Following this, in this paper, an algorithm for finding an odd sum degree dominating set of graph G is presented and explained with an example.

2 Methodology

- **Definition 2.1**

A dominating set $R \subseteq V(G)$ of a graph $G = (V, E)$ is said to be an odd sum degree dominating set (osd-set) R_o of G if the sum of the degree of all vertices in R_o is an odd number. The odd sum degree domination number $\gamma_{osd}(G)$ is the minimum cardinality taken over all odd sum degree dominating sets of G and is defined as zero if no such odd sum degree dominating set exists in G. The odd sum degree dominating set with cardinality $\gamma_{osd}(G)$ is denoted by γ_{osd}- set of G.

- **Definition 2.2**

Let R_o be a γ_{osd}set of G. Then the minimum sum value of the degree of the vertices of the set R_o is said to be the odd sum degree domination value and it is denoted by $S_{osd}\ (G)$.

A graph G is said to be an osd – graph if it has at least one osd – set. For example, P_3 is an osd – graph with $\gamma_{osd}(P_3) = 2$.

- **Definition 2.3**

A dominating set R $\subseteq$V(G) of a graph $G = (V, E)$ is said to be a co-odd sum degree dominating set(cosd–set) R_{co} of G if the sum of degree of all vertices in V- R_{co} is an odd number. The co-odd sum degree domination number - $\gamma_{cosd}(G)$ is the minimum cardinality taken over all cosd - set of G and it is defined as zero if no such cosd – set exists in G. The co-odd sum degree dominating set with cardinality $\gamma_{cosd}(G)$ is denoted by γ_{cosd}– set of G.

- **Definition 2.4**

Let R_{co} be a γ_{cosd}– set of G. Then the minimum sum value of the degree of the vertices of the set V-R_{co} is said to be the co-odd sum degree domination value and it is denoted by $S_{cosd}\ (G)$ of G.

A graph G is said to be a co-odd sum degree dominating graph (cosd - graph) if it has at least one co-odd sum degree dominating set of G.

- **Example 2.5**

For the below graph G, $R_1 = \{r_2, r_5, r_9\}$ and $R_2 = \{r_3, r_6, r_{10}\}$ are two γ_{osd}– set of G and hence $\gamma_{osd}(G) = 3$. But $\sum_{r \in R_2} deg(r)$ is minimum gives $S_{osd}(G)=7$. Similarly, $R_3 = \{r_1, r_4, r_8, r_9\}$ and $R_4 = \{r_2, r_5, r_7, r_{10}\}$ are two $\gamma_{cosd}(G)=4$. But $\sum_{r \in R_4} deg(r)$ is minimum gives S $_{cosd}$ $(G)=11$.

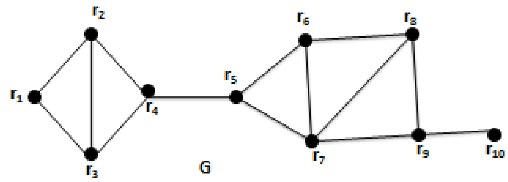

Fig 1. Odd and Co-odd sum degree domination numbers and values

- **Remark 2.6**

Since γ_{osd} – set and $\gamma_{\cos d}$–set does not exist in cycle C_n and friendship graph F_n, $n \geq 3$ gives in general, every graph need not have osd–sets and cosd–sets.

The following algorithm for finding an odd sum degree dominating set of graph G is presented below.

- **Algorithm 2.7**

Input: Given a graph $G = (V, E)$. Obtain a graph G with vertex set $V = \{r_1, r_2, \ldots, r_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_n\}$

Step 1: Initially, set the dominating set $R = \varphi$

Step 2: Find the diameter($diam(G)$) of G by using Breadth-First Search (BFS) algorithm for finding the diameter of a graph G. Let a path r_1 - r_n, be corresponded to the $diam(G)$.

Step 3: If $diam(G)=1$, then take $N(r_1) = \{r_{11}, r_{12}, \ldots, r_{1k}\}$. If $deg(r_1) < deg(r_{11})$, then include r_{11} in R, else include r_1 in R. Then go to step (8).

Step 4: If $diam(G)=2$, then take $N(r_1) = \{r_{11}, r_{12}, \ldots, r_{1k}\}$. If $deg(r_1) < deg(r_{11})$, then include r_{11} in R, else include r_1 in R. Now take $N(r_n) = \{r_{n1}, r_{n2}, \ldots, r_{nk}\}$. If $deg(r_n) < deg(r_{n1})$, then include r_{n1} in R, else include r_n in R. Then go to step (8).

Step 5: If $diam(G) \geq 3$, then take $N(r_1) = \{r_{11}, r_{12}, \ldots, r_{1k}\}$. Choose r_{1p} such that $deg(r_{1p})$ is maximum among all r_{1i} and $d(r_{1p}, r_n) \geq d(r_{1i}, r_n)$ for $i = 1,2,\ldots,k$, $i \neq p$. Then compare the degree of r_1 and r_{1p}. If $deg(r_1) < deg(r_{1p})$, include r_{1p} in R else include r_1 in R. Let the included vertex be θ_1. If a tie occurs in the above selection, choose a vertex not in the selected path as θ_1. Next take $N(r_n) = \{r_{n1}, r_{n2}, \ldots, r_{nm}\}$. Choose r_{nt} such that $deg(r_{nt})$ is maximum among all r_{ni}, where $d(r_1, r_{nt}) \geq d(r_1, r_{ni})$. For $i=1,2,\ldots,m$ and $i \neq t$ compare the $deg(r_n)$ and $deg(r_{nt})$. If $deg(r_n) < deg(r_{nt})$, then include r_{nt} in R, provided $r_{nt} \neq \theta_1$, else include r_n in R. Let the included vertex be θ_2. If a tie occurs in the above selection, choose a vertex not in the selected path as θ_2.

Step 6: Let $M = N(\theta_1) \cup N(\theta_2)$. Delete all the edges having either one end at θ_1 or θ_2. Also, delete all the edges having both ends in M. In this process, if any vertex becomes isolated, delete it from the graph

Step 7: Delete the pendent vertex in M from the graph. Repeat the steps 3-8, until the graph becomes empty.

Step 8: The resulting set R is the required dominating set. If the sum of R is odd then R is the required odd sum degree dominating set of G, else add one odd degree vertex to R which gives the required odd sum degree dominating set of G.

Output: Odd sum degree dominating set R.

- **Example 2.8**

Let us explain the algorithm with the help of the following example.

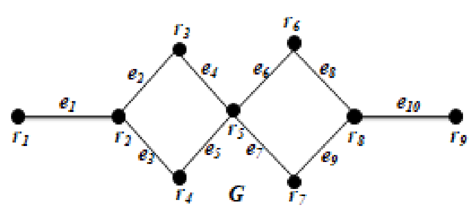

Fig 2. Illustration for finding the odd sum degree dominating set

Input: Given a graph $G=(V, E)$, with $V = \{r_1, r_2, \ldots, r_n\}$ and edge set $E=\{e_1, e_2, \ldots, e_n\}$

Step 1: Initially set the dominating set $R = \varphi$

Step 2: Using the BFS algorithm, the diameter of the given graph is 6 and the corresponding path is r_1 - r_2 – r_4 – r_5 – r_7 – r_8 – r_9. Since $diam(G)=6$, go to step 5.

Step 5: Inspecting the neighbors of r_1 and r_9, it is easy to find that $\theta_1 = r_2$ and $\theta_2 = r_8$, so to get $R=\{r_2, r_8\}$.

Step 6: Now $M = N(r_2) \cup N(r_8) = \{r_1, r_3, r_4, r_6, r_7, r_9\}$. Delete all the edges having either one end at r_2 or r_8. Also delete the edges $\{e_1, e_2, e_3, e_8, e_9, e_{10}\}$ which having both ends in M. The resulting graph is shown in Figure 3.

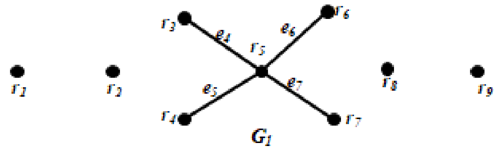

Fig 3. G_1 graph

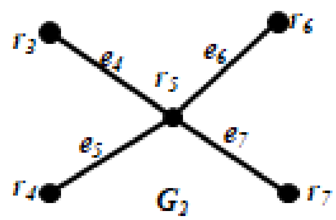

Fig 4. G_2 graph

Now, delete the isolated vertices r_1, r_2, r_8, and r_9 from the modified G_1 graph, and we get a new graph G_2 in Figure 4.

Step 7: In G_2, the vertices r_3, r_4, r_6 and r_7 have degree 1. Hence, they are removed from the graph. Now include the left-out vertex r_5 in R.

Step 8: Now the graph becomes empty. $R = \{r_2, r_5, r_8\}$ is the required dominating set of G. Now the sum of the degree of vertices of R is 10 (even). Hence, include an odd degree vertex r_9 in R. Now R is an odd sum degree dominating set.

Output: The odd sum degree dominating set $R = \{r_2, r_5, r_8, r_9\}$.

3 Results and Discussion

3.1 Exact Bounds

Theorem 3.1.1

For path graph P_n,

(i)

$$\gamma_{osd}(P_n) = \left[\frac{n}{3}\right] + 1,\ n \geq 2\,;\ S_{osd}(P_n) = 2\left[\frac{n}{3}\right] + 1,\ n \geq 3$$

(ii)

$$\gamma_{cosd}(P_n) = \left[\frac{n}{3}\right] + 1,\ n \geq 2\,;\ S_{cosd}(P_n) = \begin{cases} n-1,\ n \equiv 0(mod2) \\ n, n \equiv 1(mod2) \end{cases},\ n \geq 5$$

Proof: Let G be the path P_n with at least two vertices. Let $V(G) = \{r_1, r_2, r_3, \ldots, r_n\}$, $n \geq 2$ be the vertex set of G where $deg(r)=deg(r)=1$ and $deg(r)=2$, $i = 2, 3, 4, \ldots, n-1$.

Claim (i): Odd sum degree domination number and value

Suppose $R_1 = \{r_{3i-1} / i = 1,2,\dots \lfloor \frac{n}{3} \rfloor\}$and $R_2 = \{r_n\}$. Then $R = R_1 \cup R_2$ is a dominating set of G. Since all the vertices in R_1 have degree two and the vertex r_n has degree one, R is an odd sum degree dominating set of G. Therefore,

$$\gamma_{osd}(G) \le |R| = \left\lfloor \frac{n}{3} \right\rfloor + 1 \tag{1}$$

Let S be a γ_{osd} –set of G. If $n \equiv 0(mod\ 3)$ then S must contain at least $\frac{n}{3}$vertices and for odd sum degree domination it must contain r_n. Therefore,

$$\gamma_{osd}(G) = |S| \ge \frac{n}{3} + 1 \tag{2}$$

If $n \equiv 1(mod\,3)$ Smust contain atleast $\frac{n-1}{3}$vertices and for odd sum degree domination it must contain r_n. Therefore,

$$\gamma_{osd}(G) = |S| \ge \frac{n-1}{3} + 1 = \left\lfloor \frac{n}{3} \right\rfloor + 1 \tag{3}$$

If n$\equiv 2(mod\,3)$ S must contain atleast $\frac{n-2}{3}$ vertices and for odd sum degree domination it must have r_n. Therefore,

$$\gamma_{osd}(G) = |S| \ge \frac{n-2}{3} + 1 = \left\lfloor \frac{n}{3} \right\rfloor + 1 \tag{4}$$

Then it follows from Equations (1), (2) and (3) and Equation (4).

Further, if R is a γ_{osd} set of G then the odd sum degree domination value

$S_{osd}(G) = \sum_{r \in R} deg(r) = 2\lfloor \frac{n}{3} \rfloor + 1,\ n \ge 3$.

It completes (i).

Claim (ii): Co-odd sum degree domination number and value

Let $R_1=\{r_{3i-1} / i = 1,2,\dots \lfloor \frac{n}{3} \rfloor\}$ and $R_2=\{r_n\}$. Since every vertex in R_1 has degree 2 and the vertex r_n in R_2 has degree 1 gives $R = R_1 \cup R_2$ is a cosd – set of G. Therefore,

$$\gamma_{\cos d}(G) \le |R_1 \cup R_2| = \left\lfloor \frac{n}{3} \right\rfloor + 1,\ n \ge 2 \tag{5}$$

On the other hand, suppose R is a $\gamma_{\cos d-}$ set of G, then R must contain at least $\frac{n}{3}$ or $\frac{n-1}{3}$ or $\frac{n-2}{3}$number of vertices and the vertex $r_n$form a co-odd sum degree dominating set. If n is congruent to *0, 1,* and *2* under *mod 3* respectively.

$$\gamma_{\cos d}\,(G) = |R| \ge \begin{cases} \frac{n}{3} + 1,\ n \equiv 0(mod\,3) \\ \frac{n-1}{3} + 1,\ n \equiv 1(mod\,3) = \lfloor \frac{n}{3} \rfloor + 1\, n \ge 2 \\ \frac{n-2}{3} + 1,\ n \equiv 2(mod\,3) \end{cases}$$

That is,

$$\gamma_{\cos d}\,(G) \ge \left\lfloor \frac{n}{3} \right\rfloor + 1,\ n \ge 2 \tag{6}$$

Therefore, Equation (5) and Equation (6), we have $\gamma_{\cos d}\,(G) = \lfloor \frac{n}{3} \rfloor + 1$.

Further, if R is a $\gamma_{\cos d}$- set of G then the co-odd sum degree domination value

$S_{\cos d}(G) = \sum_{r \in V-R} deg(r) = \begin{cases} n-1,\ n \equiv 0\ (mod\ 2) \\ n,\ n \equiv 1\ (mod\ 2) \end{cases}$

It completes claim *(ii).*

Theorem 3.1.2

For book graph B_n, $n \ge 2$,

$$\gamma_{osd}(B_n) = \begin{cases} n+1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases} ;\ S_{osd}(B_n) = \begin{cases} 3n+1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}$$
$$\gamma_{cosd}(B_n) = \begin{cases} n+1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases} ;\ S_{cosd}(B_n) = \begin{cases} 6\left(\frac{n}{2}\right)+1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}$$

Proof: Let G be a book graph B_n with at least four vertices. Let $V(G) = \{r, s, r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_n\}$ be the vertex set of G, in which $deg(r) = deg(s) = n+1$ and $deg(r_i) = deg(s_i) = 2$, $\forall i$, $i = 1, 2, \dots, n$, where $r_1, r_2, \dots, r_n$ are adjacent to r and $s_1, s_2, \dots, s_n$ are adjacent to s. Further, r_i is adjacent to s_i, $\forall i$, $i = 1, 2, \dots, n$.

Claim (i): Odd sum degree domination number

When $n \equiv 0 (mod\ 2)$, the vertex set $R = \{r\} \cup \{s_i\ /\ i = 1,2,..,n\} \subseteq V(G)$ is the minimum osd–set of G and hence, $\gamma_{osd}(G) = n+1$, when $n \equiv 0 (mod\ 2)$.

When $n \equiv 1 (mod\ 2)$, every vertex in G has an even degree given there is no osd–set exists for G. Therefore, $\gamma_{osd}(G) = 0$, when $n \equiv 1 (mod\ 2)$.

Further, if R is a γ_{osd}- set of G then the odd sum degree domination value

$$S_{osd}\ (G) = \sum_{r \in R} deg(r) = \begin{cases} 3n+1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}$$

It completes claim *(i)*.

Claim (ii): Co-odd sum degree domination number and value

when $n \equiv 1 (mod\ 2)$, the vertex set $R = \{r\} \cup \{s_i\ /\ i = 1,2,..,n\} \subseteq V(G)$ is the minimum $\gamma_{\cos d}$–set of G and hence, $\gamma_{\cos d}(G) = n+1$, when $n \equiv 0 (mod\ 2)$.

When $n \equiv 0(mod2)$ every vertex in G has given there is no osd–set exists for G. Therefore, $\gamma_{\cos d}(G) = 0$ when $n \equiv 0 (mod\ 2)$.

Further, if R is a $\gamma_{\cos d}$-set of G then the co–odd sum degree domination value.

$$S_{\cos d}\ (G) = \sum_{\gamma \in V-R} deg(r) = \begin{cases} 6\left(\frac{n}{2}\right)+1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}$$

It completes Claim(ii).

By similar arguments, the exact bounds of odd and co-odd sum degree domination numbers and values for some more classes of graphs are given below.

Proposition 3.1.3

(i) For complete graph $K_n, n \geq 2$

$$\gamma_{osd}(K_n) = \begin{cases} 1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}; S_{osd}(K_n) = \begin{cases} n-1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}$$

$$\gamma_{cosd}(K_n) = \begin{cases} 1,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}; S_{cosd}(K_n) = \begin{cases} (n-1)^2,\ n \equiv 0(mod2) \\ 0,\ n \equiv 1(mod2) \end{cases}$$

(ii) For crown graph C_n^+, $n \geq 2$

$$\gamma_{osd}(C_n^+) = \begin{cases} n+1,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}; S_{osd}(C_n^+) = \begin{cases} 2\lfloor\frac{n}{2}\rfloor+3,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}$$

$$\gamma_{cosd}(C_n^+) = \begin{cases} n+1,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}; S_{cosd}(C_n^+) = 6\lceil\frac{n}{2}\rceil-3,\ n \geq 2$$

(iii) For comb graph $P_n^+, n \geq 2$

$$\gamma_{osd}(P_n^+) = n; S_{osd}(P_n^+) = \begin{cases} 2\lfloor\frac{n}{2}\rfloor+3,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}$$

$$\gamma_{\cos d}(P_n^+) = \begin{cases} n+1,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}; S_{\cos d}(P_n^+) = \begin{cases} 6\lfloor\frac{n}{2}\rfloor-5,\ n \equiv 0(mod2) \\ 6\lfloor\frac{n}{2}\rfloor+1,\ n \equiv 1(mod2) \end{cases}$$

(iv) For fan graph f_n, $n \geq 3$

$$\gamma_{osd}(f_n) = \begin{cases} 2,\ n \equiv 0(mod2) \\ 1,\ n \equiv 1(mod2) \end{cases}; S_{osd}(f_n) = \begin{cases} 2\lfloor\frac{n}{2}\rfloor+3,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}$$

$$\gamma_{cosd}(f_n) = \begin{cases} 2,\ n \equiv 0(mod2) \\ 1,\ n \equiv 1(mod2) \end{cases}; S_{cosd}(f_n) = \begin{cases} 6\lfloor\frac{n}{2}\rfloor-5,\ n \equiv 0(mod2) \\ 6\lfloor\frac{n}{2}\rfloor+1,\ n \equiv 1(mod2) \end{cases}$$

(v) For wheel graph W_n, $n \geq 3$

$$\gamma_{osd}(W_n) = \begin{cases} 2,\ n \equiv 0(mod2) \\ 1,\ n \equiv 1(mod2) \end{cases}; S_{osd}(W_n) = \begin{cases} 2\lfloor\frac{n}{2}\rfloor+3,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}$$

$$\gamma_{cosd}(W_n) = \begin{cases} 2,\ n \equiv 0(mod2) \\ 1,\ n \equiv 1(mod2) \end{cases}; S_{cosd}(W_n) = \begin{cases} 6\lceil\frac{n}{3}\rceil+5,\ n \geq 3 \\ 6\lfloor\frac{n}{3}\rfloor+3,\ n = 4) \end{cases}$$

(vi) For star graph S_n, $n \geq 2$

$$\gamma_{osd}(S_n) = \begin{cases} 1,\ n \equiv 0(mod2) \\ 2,\ n \equiv 1(mod2) \end{cases} ;\ S_{osd}(S_n) = \begin{cases} n-1,\ n \equiv 0(mod2) \\ n,\ n \equiv 1(mod2) \end{cases}$$
$$\gamma_{cosd}(S_n) = \begin{cases} 1,\ n \equiv 0(mod2) \\ 2,\ n \equiv 1(mod2) \end{cases} ;\ S_{cosd}(S_n) = \begin{cases} n-1,\ n \equiv 0(mod2) \\ n-2,\ n \equiv 1(mod2) \end{cases}$$

3.2 Bounds and characterization of odd and co-odd sum degree domination and values

Since every o dd sum degree dominating set and co-odd sum degree dominating set of a graph G is a dominating set of G, it gives the following results .

Theorem 3.2.1

For any graph *G*,

(a) $\gamma(G) \leq \gamma_{osd}(G)$

(b) $\gamma(G) \leq \gamma_{cosd}(G)$

The bounds of *(a)* and *(b)* are sharp in paths P_2 and P_4 respectively.

The odd sum degree domination number in terms of regularity of the graph is given below .

Theorem 3.2.2

For the *m*-regular graph when *m* is even there are no such osd-sets exits and hence $\gamma_{osd}(G)$and S_{osd}(G). Therefore, the following result exists, when *m* is odd. If *G* is a *m*-regular graph and *m* is odd,

$$\gamma_{osd}\,(G) = \begin{cases} \gamma(G)\ is\ odd,\ m \equiv 1(mod\ 2) \\ \gamma(G)\ is\ even,\ m \equiv 1(mod\ 2) \end{cases} ;\ S_{osd}\,(G) = \begin{cases} m.\gamma(G)\ is\ odd,\ m \equiv 1(mod\ 2) \\ m.\gamma(G)\ is\ even,\ m \equiv 1(mod\ 2) \end{cases}$$

Proof: Let *G* be a *m*-regular graph with *m* is odd. Since *m* is odd and the domination number γ(G) is also odd. Since the sum of an odd number of odd degrees is odd, when γ*(G)* is odd. The γ*(G)* itself is γ_{osd}*(G)*–set. Similarly, degrees is even giving the addition of one vertex with γ–set is an γ_{osd}–set. The above two statements prove the odd sum degree domination number. Since every vertex has the same degree *m* proves the odd sum degree domination value.

The Characterization of an odd sum degree dominating set of graph G is given below

Theorem 3.2.3

If R_o and R_{co} are osd and cosd–sets of a graph *G* then R_o and R_{co} have only an odd number of odd-degree vertices.

Proof: Suppose R_o and R_{co} have an odd number of even degree (or) an even number of odd degree or even number of even degree vertices then their summation is an even number. It contradicts the definition of odd sum degree domination sets and co-odd sum degree domination sets. Hence, R_o, R_{co} must have an odd number of odd-degree vertices. The converse is obvious.

The necessary and sufficient conditions for the existence of an odd sum degree dominating and co-odd sum degree dominating sets are presented below .

Theorem 3.2.4

Let *G* be a graph with at least two vertices. Let *R* be a dominating set of *G*. Then *R* is an osd-set if and only if *R* has an odd number of odd-degree vertices.

Proof: The above result can be extended to co-odd sum degree dominating set and hence we have the following theorem.

T heorem 3.2.3 and Theorem 3.2.4, give the following Necessary and sufficient conditions for the existence of an odd sum degree domination graph and co-odd sum degree domination- graph .

Theorem 3.2.5

Let *G* be a graph with at least two vertices.

(i) *G* is an osd–graph if and only if *G* has at least one odd degree vertex.

(ii) *G* is a cosd–graph if and only if *G* has at least one odd degree vertex.

Proof: Let *G* be an osd–graph, then there exists an osd–set R_o of *G*. Suppose *G* has no odd degree vertex. Then there is no osd–set existing in *G* which contradicts *G* is an osd–graph. Therefore, *G* has at least one odd-degree vertex. The converse is true, as follows from theorem 3.2.3.

Let G be a cosd–graph, there exists a cosd–set R_{co} of G. Suppose G has no odd degree vertex, then G has only vertices of even degree. Hence, there is no cosd–set that exists in G which contradicts G is a cosd–graph. Therefore, G has at least one odd-degree vertex. The converse is true, as follows from theorem 3.2.4.

The following two results characterize the odd and co-odd sum degree domination value of a graph.

Theorem 3.2.6

Let G be a *(p,q)* graph with an osd – set R_o and cosd– set R_{co} . Then

(i) $\sum_{r\in R_o} deg(r)\, is\, odd$

(ii) $\sum_{r\in R_{co}} deg(r)\, is\, odd$

Proof: For any graph G, we have $\sum_{r\in R_o} deg(r) \;= 2q$ =*2q*

Therefore, $\sum_{r\in R_o} deg(r) + \sum_{r\in R_{co}} deg(r) \;= 2q$an even number. But R_o is an osd–set of G gives $\sum_{r\in R_o} deg(r)$is an odd number.

Similarly, one can prove $\sum_{r\in R_o} deg(r)$ is also an odd number.

Theorem 3.2.7

Let G be a graph with *δ(G)* $\neq \triangle$ *(G)*. If $R_o$and $R_{co}$are γ_{osd} – *set* and $\gamma_{\cos\, d}$ — *set* of G then

(i)

$$\frac{1}{\Delta(G)} \sum_{r\in R_o} deg(r) \leq \gamma_{osd}\,(G) \leq \frac{1}{\delta(G)} \sum_{r\in R_o} deg(\, r)$$

(ii)

$$\frac{1}{\Delta(G)} \sum_{r\in R_{co}} deg(r) \leq p - \gamma_{osd}\,(G) \leq \frac{1}{\delta(G)} \sum_{r\in R_{co}} deg(\, r)$$

Proof: Since G is connected, *δ(G)* ≥*1* and $\triangle(G) \geq 1$. Further the minimum and maximum degree of graph G, gives *δ(G)* $\leq \sum_{r\in R} deg(r) \leq \triangle(G)$

Therefore, in R_o, $|R_o|\,\delta(G) \leq \sum_{r\in R_o} deg(r) \leq |R_o|\triangle(G)$

But $|R_o| = \gamma_{osd}\,(G)\, gives\, \frac{1}{\triangle(G)} \sum_{r\in R_o} deg(r) \leq \gamma_{osd}(G) \leq \frac{1}{\delta(G)} \sum_{r\in R_o} deg(r)$ By the similar arguments, one can prove the second result.

For *(ii),* the lower and upper bounds are sharp for the caterpillar graph Sn_3 and bistar graph $B_{2,2}$.

4 Discussion

The domination concept in graph theory has traveled nearly 66 years in the research area. In which, the odd domination number flourished in the year 2003. The practical significance of odd and co-odd degree domination numbers appears in various domains involving networks, such as communication, social, computer, and molecule networks, and system design. Domination numbers can help in bonds critical to the stability of molecules in molecule networks. Studying the odd domination numbers contributes to understanding the structural properties of graphs, particularly regarding their connectivity and resilience against node removals. The odd and co-odd degree domination numbers are highly applied in designing fault-tolerant networks. Analyzing domination numbers, especially those related to specific node degrees, contributes to algorithm design and complexity theory. The computational complexity of finding optimal dominating sets is essential for developing efficient algorithms.

5 Conclusion

In this study, the odd and co-odd sum degree domination number and value are found for some standard classes of graphs, and to provide the algorithm for finding the odd sum degree dominating set of graphs. The bounds of the odd and co-odd sum degree domination numbers are obtained. The odd and co-odd sum degree dominating sets are characterized. Further, the Relationship between some of the existing domination parameters is obtained.

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