INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS

Received: 26-03-2024

Accepted: 16-07-2024

Published: 07-08-2024

Citation: Derle M, Patil D (2024) Applications of The Double General Rangaig Integral Transform in Integro-Differential Equations. Indian Journal of Science and Technology 17(31): 3258-3271. https://doi.org/10.17485/IJST/v17i31.922

* **Corresponding author**.

derlemonalishivaji@gmail.com

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Applications of The Double General Rangaig Integral Transform in Integro-Differential Equations

Monali Derle[1]*, Dinkar Patil[2]

1 Research Scholar, Department of Mathematics, K.T.H.M. College, Nashik, 422002, Maharashtra, India
2 Professor, Department of Mathematics, K.T.H.M. College, Nashik, 422002, Maharashtra, India

Abstract

Objectives : To solve integral differential equations. Method: The convolution theory and double general Rangaig integral transform was used to solve integral differential equations, precisely. **Findings:** The present study derives the existence condition of the double general Rangaig integral transform. Theorems proved in this study, deals with popular properties of the double general Rangaig integral transform. The double general Rangaig integral transform of Bessel's function and modified Bessel's function are calculated. The convolution theorem has been stated and demonstrated using the unit step Heaviside function and finally, the double general Rangaig integral transform is applied to linear volterra integral equations of the first and second kind and subsequently to volterra integro-partial differential equations and partial integro-differential equations. **Novelty:** On an independent domain, the double general Rangaig integral transform is defined. This is the double-general Rangaig integral transform's specialty, in this study.

Keywords: Bessel's function; Convolution theorem; Rangaig transform; Double general Rangaig integral transform; Integral equations; Integral-differential equations

1 Introduction

The double Laplace transform was proposed by Lokenath Debnath in 2015 [(1)] and finds applications in each domain of functional, integral, and partial differential equations. A key component of linear integral equations is the integral transform. Abdallah, A. M., et al. discussed the important role of the RG transform and its applications in 2023 [(2)]. The fundamental features of the Ramadan group integral transform and some of its dualities are defined by Abdallah, A. M., et al. in 2022 [(3)]. Many double integral transforms, such as the double Shehu transform [(4)] and the double Ramadan group integral transform [(5)], have been developed by researchers and are used to solve linear integral equations. A new general double integral transform called the Jafari-Yang double integral transform, which is a generalization of the other double integral transform types mentioned above was defined by M. Meddahi et al. in 2021 [(6)]. Dr. D. P. Patil discovered dualities between double integral transforms in 2020[7] and he used the double Mahgoub transform[8] to solve parabolic boundary value issues. Dr. D. P. Patil and colleagues used the Anuj transform in 2022[9] to solve the first-kind Volterra integral equations. Furthermore, Dr. D. P. Patil and colleagues[10] utilize the Kushare transform to solve first-kind Faltung-type volterra-integro differential equations. Ahmed S. A., et al.[11] applied Laplce Sumudu transform in integral differential equations.

Eman A. Mansour and Emad A. Kuffi created a generalization of the Rangaig integral transform in 2021[12], in which they used to solve differential equations in several areas. The double general Rangaig integral transform was introduced in 2022[13] by M. S. Derle et al. and used to solve partial differential equations. Qazza, A., achieved a two-dimensional integral equation solution with the Laplace-Ara transform in 2023[14]. The double Ara transform was used in 2023[15] by Saadeh R. to solve integro-differential equations in two dimensions. The double Ara Sumudu transform was also used in partial differential equations and integral equations by Saadeh R. et al.[16]. Saadeh Rania et al. defined the double Ara formable transform in 2023[17] and used a number of PDEs, including the Klein-Gordon equation and the advection-diffusion equation. In [2022], Soliman, A.A., et.al[18], analyzed the solution for class of linear and nonlinear Caputo fractional Volterra Fredholm integro-differential equations with nonlinear time varying delay. In this study, we solve linear volterra integral equations of first and second kind as well as linear and partial integro-differential equations, by applying the double general Rangaig integral transform.

1.1. Preliminaries

In this section, we introduced some basic concepts that are required.

Rangaig integral transform[12]

The Rangaig integral transform can be written as:

$$\eta\left[h(t)\right]=\Lambda\left(\mu\right)=\frac{1}{\mu}\int_{t=-\infty}^{0}e^{-\mu t}h\left(t\right)dt,\frac{1}{\lambda_1}\le\mu\le\frac{1}{\lambda_2}$$

General Rangaig integral transform[12]

The set of functions, an exponential order is

$$Hg=\{\,h\left(t\right)\,:there\ exist\ N,\ \lambda 1\ and\ \lambda 2\ >\ 0\,,\ \left|h\left(t\right)\right|\ >Ne^{\lambda_j\left|t\right|},t\in(-1)^{j-1}\times(-\infty,0),\ where\,j\ =\ 1,2\}$$

The general Rangaig integral transform can be written as:

$$\eta_g\left\{h(t)\right\}=\wedge_g\left(\mu\right)=\frac{1}{\mu^n}\int_{t=-\infty}^{0}e^{p(\mu)t}h\left(t\right)dt$$

Double general Rangaig integral transform[13]

The set of functions, of an exponential order is defined as:

$$H_{2g}=\{\,h\left(t_1,t_2\right)\,:\,\exists\,N\,,M,\,\lambda_1,\lambda_2,\chi_1,\chi_{2_1}>0,\left|h\left(t_1,0\right)\right|>N.e^{\,\lambda_i t_1},\,t_1\in(-1)^{i-1}\times(-\infty,0)\,,\left|h\left(0,t_2\right)\right|>M.e^{\chi_j t_2},t_2\in(-1)^{j-1}\times(-\infty,0)\,,\ where\ i,j=1,2\}$$

Here, $N,M\equiv$ finite constants

$\lambda_1,\lambda_2,X_1,X_2$ are finite or infinite constants.

Then the double general Rangaig integral transform for the set H_{2g} can be written as:

$$\eta_{2g}\left\{h\left(t_1,t_2\right)\right\}=\wedge_{2g}\left(\,\mu_2\right)=\frac{1}{\mu_1{}^{\,n_1}.\mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0}e^{p(\mu_1\,)t_1+q\,(\mu_2)t_2}\,h\left(t_1,t_2\right)\,dt_1dt_2$$

Where $\wedge 2g$ denote the double general Rangaig integral transform of function $h\left(t_1,t_2\right)\in H_{2g}$

$$\frac{1}{\lambda_1}\le\mu_1,\ \ \le\frac{1}{\lambda_2}\,,\ \frac{1}{X_1}\le\ \mu_2\le\frac{1}{X_2}$$

$p(\mu_1),\ q\,(\mu_2)$ are functions of parameters $\mu_1\ \ and\,\mu_2$

The inverse of double general Rangaig transform $_{2g}\,\{h\,(t_1,t_2)\}$ is defined as

$$\eta_{2g}^{-1}\,\{\wedge_D(\mu_1,\mu_2)\} = \eta_{t_1}^{-1}\eta_{t_2}^{-1}\,\{\wedge_D(\mu_1,\mu_2)\} = \ h\,(t_1,t_2)$$

Double general Rangaig integral transform for partial derivatives.

Theorem 2.1 [13]

Consider $\Lambda_D(\mu_1,\mu_2)$ as the double general Rangaig integral transform of the function $h(t_1,t_2)$ and $\Lambda_g(0,\mu_2)$ is the general Rangaig integral transform of the function $h(0,t_2)$. then

$$i.\ \eta_{2g}\left\{\frac{\partial h(t_1,t_2)}{\partial t_1}\right\} = \frac{1}{\mu_1^{n_1}}\Lambda_g\,(0,\mu_2) - p\,(\mu_1)\,\Lambda_D(\mu_1,\mu_2)$$

$$ii.\ \eta_{2g}\left\{\frac{\partial^2\,h\,(t_1,t_2)}{\partial t_1^2}\right\} = \frac{1}{\mu_1^{n_1}}\frac{\partial\,\Lambda_g\,(0,\mu_2)}{\partial\,t_1} - p\,(\mu_1)\,.\frac{1}{\mu 1^{n_1}}\Lambda_g\,(0,\mu_2) + [\,p\,(\mu_1)]^2\,\Lambda_D(\mu_1,\mu_2)$$

$$iii.\ \eta_{2g}\left\{\frac{\partial^n\,h\,(t_1,t_2)}{\partial t_1^n}\right\} = \frac{1}{\mu_1^{n_1}}\sum\nolimits_{k=0}^{m-1}(-1)^k\,[p\,(\mu_1)]^{\ k}\frac{\partial^{\,(m-1-k)}\Lambda_g\,(0,\mu_2)}{\partial^{(m-1-k)}t_1}(-1)^m\,[\,p\,(\mu_1)]^m\,\Lambda_D(\mu_1,\mu_2)$$

Theorem 2.2 [13]

Consider $\Lambda_D(\mu_1,\mu_2)$ as the double general Rangaig integral transform of the function $h(t_1,t_2)$ and $\Lambda_g(\mu_1,0)$ is the general Rangaig integral transform of the function $h(t_1,0)$. then

$$i.\ \eta_{2g}\left\{\frac{\partial h(t_1,t_2)}{\partial t_2}\right\} = \frac{1}{\mu_2^{n_2}}\Lambda_g(\mu_1,0) - q\,(\mu_2)\,\Lambda_D(\mu_1,\mu_2)$$

$$ii.\ \eta_{2g}\left\{\frac{\partial^2\,h(t_1,t_2)}{\partial t_2^2}\right\} = \frac{1}{\mu_2^{n_2}}\frac{\partial\,\Lambda_g(\mu_1,0)}{\partial\,t_2} - q\,(\mu_2)\,.\frac{1}{\mu_2^{n_2}}\Lambda_g(\mu_1,0) + [\,q\,(\mu_2)]^2\,\Lambda_D(\mu_1,\mu_2)$$

$$iii.\ \eta_{2g}\left\{\frac{\partial^n\,h\,(t_1,t_2)}{\partial t_1^n}\right\} = \frac{1}{\mu_2^{n_2}}\sum\nolimits_{k=0}^{m-1}(-1)^k\,[q\,(\mu_2)]^{\ k}\frac{\partial^{\,(m-1-k)}\Lambda_g(\mu_1,0)}{\partial^{(m-1-k)}t_2} + (-1)^m\,[q\,(\mu_2)]^{\ m}\Lambda_D(\mu_1,\mu_2)$$

Double General Rangaig Integral Transform of Some Fundamental Functions.

The double general Rangaig transforms of some common functions are given in following Table 1.[13]

Table 1.

$h(\mathbf{t_1},\mathbf{t_2})$	$\wedge_{\mathbf{2g}}(\boldsymbol{\mu_1},\ \boldsymbol{\mu_2})$
1	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\,\frac{1}{p(\mu_1)q\,(\mu_2)}$
t_1t_2	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\,\frac{1}{[p(\mu_1)]^2[q\,(\mu_2)]^2}$
$t_1{}^mt_2{}^n\ m,n>0$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\,\frac{(-1)^{m+n}m!\ n!}{[p(\mu_1)]^{\ m+1}[q\,(\mu_2)]^{n+1}}$
$t_1{}^mt_2{}^n,m,n>-1$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\,\frac{(-1)^{m+n}\Gamma(m+1)\Gamma(n+1)}{[p(\mu_1)]^{\ m+1}[q\,(\mu_2)]^{n+1}}$
$e^{at_1+bt_2}$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\,\frac{1}{(p(\mu_1)+a)}\,\frac{1}{(q(\mu_2)+b)}$
$cos(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\left(\frac{p(\mu_1\ )q(\ \mu_2)-ab}{(([p(\mu_1)]^2+a^2))([q(\mu_2)]^2+b^2)}\right)$
$sin(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\left(\frac{-a\,q(\ \mu_2)-bp(\mu_1\ )}{(([p(\mu_1)]^2+a^2))([q(\mu_2)]^2+b^2)}\right)$
$cos\,h(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\left(\frac{p(\mu_1\ )q(\ \mu_2)+ab}{(([p(\mu_1)]^2-a^2))([q(\mu_2)]^2-b^2)}\right)$
$sin\,h(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\,.\,\mu_2^{n_2}}\left(\frac{-a\,q(\ \mu_2)-bp(\mu_1\ )}{(([p(\mu_1)]^2-a^2))([q(\mu_2)]^2-b^2)}\right)$

2 Methodology

In this section we explained the methodology of applying the double general Rangaig integral transform.

2.1. Double General Rangaig Integral Transform of Bessel's Function

Let $J_0(t_1,t_2) = J_0(c\sqrt{t_1 t_2})$ be Bessel's function of zero order, then.

The double general Rangaig integral transform is,

$$\eta_{2g}\left[J_0\left(c\sqrt{t_1 t_2}\right)\right] = \frac{4}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left(4\left[p(\mu_1)\right]\left[q(\mu_2)\right]+c^2\right)}$$

Let $J_0(c\sqrt{t_1 t_2}) = 1 - \frac{(c\sqrt{t_1 t_2})^2}{2^2} + \frac{(c\sqrt{t_1 t_2})^4}{2^2 4^2} - \frac{(c\sqrt{t_1 t_2})^6}{2^2.4^2.6^2} + \ldots$

Apply double general Rangaig integral transform

$$\eta_{2g}[J_0(c\sqrt{t_1 t_2})] = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{[p(\mu_1)][q(\mu_2]}\left[1-\left(\frac{c^2}{4[p(\mu_1)][q(\mu_2)]}\right)+\left(\frac{c^2}{4[p(\mu_1)][q(\mu_2)]}\right)^2-\left(\frac{c^2}{4[p(\mu_1)][q(\mu_2)]}\right)^3+\ldots.\right]$$
$$= \frac{4}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[4p(\mu_1)\,q(\mu_2)+c^2\right]}$$

Where $J_0(z)$ is the Bessel function of order zero.

Similarly,

If $I_0(t_1,t_2) = I_0(c\sqrt{t_1 t_2})$

then $_{2g}[I_0(c\sqrt{t_1 t_2})] = \frac{4}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{(4[p(\mu_1)][q(\mu_2)]-c^2)}$

where $I_0(z)$ is the modified Bessel function of order zero.

2.2. Existence condition for the Double General Rangaig Integral Transform

If a function $\emptyset(t_1,t_2)$is an exponential order with c& d as $t_1 \rightarrow -\infty,\ t_2 \rightarrow -\infty$ and if $\exists$ a positive constant k such that $\forall\, t_1 < T_1 \& t_2 < T_2$, then $|\emptyset(t_1,t_2)| = K\,e^{ct_1+dt_2}$& we write $|\emptyset(t_1,t_2)| = M\,e^{ct_1+dt_2}$ as

$t_1 \rightarrow -\infty,\ t_2 \rightarrow -\infty$or equivalently $\lim_{t_1\rightarrow-\infty, t_2\rightarrow-\infty} e^{p(\mu_1,\ )(t_1)+q(\mu_2)(t_2)}\left|\Phi(t_1,t_2)\right|$

$$= \lim_{t_1\rightarrow-\infty, t_2\rightarrow-\infty} e^{(p(\mu_1,\ ))+c)(t_1)+q((\mu_2)+d)(t_2)}$$

$$= 0 \qquad -p(\mu_1,\ ) < c\ ,\ -q(\mu_2) < d$$

The function $\emptyset(t_1,t_2)$is called an exponential order as $t_1 \rightarrow -\infty,\ t_2 \rightarrow -\infty$& clearly, it does not grow faster than $K\,e^{ct_1+dt_2}\ as t_1 \rightarrow -\infty,\ t_2 \rightarrow -\infty$

Now, we state and prove following theorems.

2.2.1. Theorem 1

If a function $\emptyset(t_1,t_2)$is a continuous function in every finite interval $(T_1,0)\,\&\,(T_2,0)$ of exponential order $e^{ct_1+dt_2}$, then the double general Rangaig integral transform of $\emptyset(t_1,t_2)$exist for all p$(\mu_1,\ )\,\&\,q(\mu_2)$ provided

$Re\ \left[-p(\mu_1,\ )\right] < c\ \&\ Re\left[-q(\mu_2)\right] <\ d.$

Proof: Using the definition of double general Rangaig integral transform, we have

$$\left|\Lambda_{2g}(\mu_1,\mu_2)\right| = \left|\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )(t_1)+q(\mu_2)(t_2)}\emptyset(t_1,t_2)\,dt_1 dt_2\right|$$

$$\leq \frac{K}{\mu_1^{n_1}.\mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{(p(\mu_1)+c)(t_1)+(q(\mu_2)+d)(t_2)}\emptyset(t_1,t_2)\,dt_1 dt_2$$

$$= \frac{K}{\mu_1^{n_1}.\mu_2^{n_2}} \frac{1}{(p(\mu_1)+c)(q(\mu_2)+d)} Re\,[-p(\mu_1)] < c \,\&\, Re\left[-q(\mu_2)\right] < d$$

Then From (1), we have

$$\lim_{t_1\to\infty,\, t_2\to\infty} \left|\wedge_{2g}(\mu_1,\,\mu_2)\right| = 0 \; or \lim_{t_1\to\infty,\, t_2\to\infty} \wedge_{2g}(\mu_1,\,\mu_2) = 0$$

2.2.2. Theorem 2

If $\Lambda_{2g}(\mu_1,\mu_2) =_{2g} [\emptyset(t_1,t_2)]$ then

$$\eta_{2g}[\Phi(t_1-\delta,t_2-\varepsilon)\,H(t_1-\delta,t_2-\varepsilon)] = e^{p(\mu_1,\ )\delta+q(\mu_2)\varepsilon}\Lambda_{2g}(\mu_1,\mu_2)$$

Where $H(t_1,t_2)$ is the Heaviside unit step function defined by

$$H(t_1-\delta,\,t_2-\varepsilon) = \begin{cases} 1,\; t_1<\delta,\, t_2<\varepsilon \\ 0,\; otherwise \end{cases}$$

Proof

$$\eta_{2g}[\Phi(t_1-\delta,t_2-\varepsilon)\,H(t_1-\delta,t_2-\varepsilon)] = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}} \int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )(t_1)+q(\mu_2)(t_2)}\Phi(t_1-\delta,t_2-\varepsilon)\,H(t_1-\delta,t_2-\varepsilon)\,dt_1dt_2$$

$$\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}} \int_{-\infty}^{\delta}\int_{-\infty}^{\varepsilon} e^{p(\mu_1,\ )(t_1)+q(\mu_2)(t_2)}\emptyset(t_1-\delta,t_2-\varepsilon)\,dt_1dt_2$$

Using substitution $t_1-\delta = q$, $t_2-\varepsilon = w$

$$= \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}} \int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )(q+\delta)+q(\mu_2)(w+\varepsilon)}\emptyset(q,w)\,dqdw$$

$$= \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}} e^{p(\mu_1,\ )(\delta)+q(\mu_2)(\varepsilon)} \int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )(q)+q(\mu_2)(w)}\emptyset(q,w)\,dqdw$$

$$= e^{p(\mu_1,\ )\delta+q(\mu_2)\varepsilon}\Lambda_{2g}(\mu_1,\mu_2)$$

2.2.3. Definition 1

The convolution of $f(x,y)$ and $h(x,y)$ is denoted by $(f**h)(x,y)$ and defined by,

$$(f**h)(x,y) = \int_0^x\int_0^y f(x-\tau,y-\mu)\,h(\tau,\mu)d\tau d\mu$$

2.2.4. Theorem 3: (Convolution Theorem)

If $\eta_{2g}\{f(t_1,t_2)\} = \Lambda_{2g_f}(\mu_1,\mu_2)$ and $\eta_{2g}\{h(t_1,t_2)\} = \Lambda_{2g_h}(\mu_1,\mu_2)$

then $\eta_{2g}((f**h)(t_1,t_2)\} = \mu_1^{n_1}\mu_2^{n_2}\Lambda_{2g_f}(\mu_1,\mu_2).\Lambda_{2g_h}(\mu_1,\mu_2)$

where, $\Lambda_{2g_f}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )t_1+q(\mu_2)t_2} f(t_1,t_2)\ dt_1 dt_2$

$\Lambda_{2g_h}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )t_1+q(\mu_2)t_2} h(t_1,t_2)\ dt_1 dt_2$

Proof: By Def. 1

$$(f**h)(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} f(t_1-\tau,t_2-\mu)\,h(\tau,\mu)d\tau d\mu$$

$$\eta_{2g}\{(f**h)(t_1,t_2)\} = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )t_1+q(\mu_2)t_2}\{(f**h)(t_1,t_2)\}dt_1 dt_2$$

$$= \frac{1}{\mu_1{}^{n_1}.\mu_2{}^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1,\ )t_1+q(\mu_2)t_2}\left(\int_0^{t_1}\int_0^{t_2} f(t_1-\tau,t_2-\mu)\,h(\tau,\mu)d\tau d\mu\right)dt_1 dt_2$$

$$= \frac{1}{\mu_1{}^{n_1}.\mu_2{}^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0}\left(\int_0^{t_1}\int_0^{t_2} f(t_1-\tau,t_2-\mu)\,h(\tau,\mu)d\tau d\mu\right)e^{p(\mu_1,\ )t_1+q(\mu_2)t_2}dt_1 dt_2$$

substitute, $x = t_1-\tau$, $y = t_2-\mu$

$$= \frac{1}{\mu_1{}^{n_1}.\mu_2{}^{n_2}}\int_{-\infty}^{-\tau}\int_{-\infty}^{-\mu}\left(\int_0^{t_1}\int_0^{t_2} f(x,y)h(\tau,\mu)d\tau d\mu\right)e^{p(\mu_1,\ )(x+\tau)+q(\mu_2)(y+\mu)}dxdy$$

$$= \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\int_{-\infty}^{-\tau}\int_{-\infty}^{-\mu}\left(\int_0^{t_1}\int_0^{t_2} f(x,y)h(\tau,\mu)d\tau d\mu\right)s$$
$$e^{p(\mu_1,\ )(x)+p(\mu_1,\ )(\tau)+q(\mu_2)(y)+q(\mu_2)(\mu)}dxdy$$

$$= \int_{-\infty}^{-\tau}\int_{-\infty}^{-\mu} f(x,y)\left(\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\int_0^{t_1}\int_0^{t_2} h(\tau,\mu)e^{p(\mu_1,\ )(\tau)+q(\mu_2)(\mu)}d\tau d\mu\right)$$
$$e^{p(\mu_1,\ )(x)+q(\mu_2)(y)}dxdy$$

but $t_1 \in (-1)^{i-1}\times(-\infty,0)$, $t_2 \in (-1)^{j-1}\times(-\infty,0)$

$\tau\ and\ \mu$are very small quantity, i.e. tending to zero.

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} f(x,y)\left(\frac{1}{\mu_1{}^{n_1}.\mu_2{}^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} h(\tau,\mu)e^{p(\mu_1,\ )(\tau)+q(\mu_2)(\mu)}d\tau d\mu\right)e^{p(\mu_1,\ )(x)+q(\mu_2)(y)}dxdy$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{0} f(x,y) \left(\frac{1}{\mu_1^{n_1} . \mu_2^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )(\tau) + q(\mu_2)(\mu)} h(\tau,\mu) \, d\tau d\mu \right) e^{p(\mu_1, \ )(x) + q(\mu_2)(y)} dx dy$$

$$= \Lambda_{2g_h}(\mu_1,\mu_2) \left(\mu_1^{n_1} . \mu_2^{n_2} \left\{ \frac{1}{\mu_1^{n_1} . \mu_2^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )(x) + q(\mu_2)(y)} f(x,y) \, dx dy \right\} \right)$$

$$\eta_{2g}\{(f**h)(t_1,t_2)\} = \frac{1}{\mu_1^{n_1} . \mu_2^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )t_1 + q(\mu_2)t_2} \{(f**h)(t_1,t_2)\} \, dt_1 dt_2$$

Hence, proved.

Proof by another way

$$\eta_{2g}\{(f**h)(t_1,t_2)\} = \frac{1}{\mu_1^{n_1} . \mu_2^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )t_1 + q(\mu_2)t_2} \{(f**h)(t_1,t_2)\} \, dt_1 dt_2$$

$$= \frac{1}{\mu_1{}^{n_1} . \mu_2{}^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )t_1 + q(\mu_2)t_2} \left(\int_{0}^{t_1} \int_{0}^{t_2} f(t_1-\tau, t_2-\mu) h(\tau,\mu) d\tau d\mu \right) dt_1 dt_2$$

Using Heaviside unit step function

$$= \frac{1}{\mu_1{}^{n_1} . \mu_2{}^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )t_1 + q(\mu_2)t_2} \left(\int_{-\infty}^{0} \int_{-\infty}^{0} f(t_1-\tau, t_2-\mu) g(t_1-\tau, t_2-\mu) h(\tau,\mu) d\tau d\mu \right) dt_1 dt_2$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{0} h(\tau,\mu) d\tau d\mu \left\{ \frac{1}{\mu_1^{n_1} . \mu_2^{n_2}} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )t_1 + q(\mu_2)t_2} f(t_1-\tau, t_2-\mu) g(t_1-\tau, t_2-\mu) \, dt_1 dt_2 \right\}$$

by **theorem 2**

$$= \int_{-\infty}^{0} \int_{-\infty}^{0} h(\tau,\mu) d\tau d\mu \left\{ e^{p(\mu_1, \ )\tau + q(\mu_2)\mu} \Lambda_{2g_f}(\mu_1,\mu_2) \right\}$$

$$= \Lambda_{2g_f}(\mu_1,\mu_2) \int_{-\infty}^{0} \int_{-\infty}^{0} e^{p(\mu_1, \ )\tau + q(\mu_2)\mu} h(\tau,\mu) d\tau d\mu$$

$$= \mu_1{}^{n_1} . \mu_2{}^{n_2} \Lambda_{2g_f}(\mu_1,\mu_2) \Lambda_{2g_h}(\mu_1,\mu_2)$$

$${}_{2g}\{(f**h)(t_1,t_2)\} = \mu_1^{n_1} . \mu_2^{n_2} . \Lambda_{2g_h}(\mu_1,\mu_2) . \Lambda_{2g_f}(\mu_1,\mu_2)$$

Hence, proved.

3 Results

3.1. Application of double general Rangaig integral transform of linear

Volterra integral differential equations.

This section deals with linear Volterra integral equations and Volterra integro-differential equations using the double general Rangaig integral transform method.

3.1.1. Volterra Integral Equations

Study the linear Volterra integral equation in its usual form.

$$\emptyset(t_1,t_2) = g(t_1,t_2) + \lambda \int_0^{t_1}\int_0^{t_2} \Phi(t_1-x,t_2-y)\,\psi(x,y)\,dxdy \tag{1}$$

where $\emptyset(t_1,t_2)$ is the unknown function, λ is a constant and $\psi(t_1,t_2)$are two known functions.

Utilizing the convolution theorem (3) and applying the double general Rangaig integral transform to both sides of Equation (1), we obtain

$$\Lambda_{2g}(\mu_1,\mu_2) = \Lambda_{2g_g}(\mu_1,\mu_2) + \lambda\,\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g_\Phi}(\mu_1,\mu_2)\,\Lambda_{2g_\psi}(\mu_1,\mu_2)$$

$$\Lambda_{2g}(\mu_1,\mu_2) = \frac{\Lambda_{2g_g}(\mu_1,\mu_2)}{1-\lambda\,\mu_1{}^{n_1}.\mu_2{}^{n_2}\Lambda_{2g_\psi}(\mu_1,\mu_2)}$$

On both sides, apply the inverse double general Rangaig integral transform to obtain

$$\emptyset(t_1,t_2) =_{2g}^{-1}\left[\frac{\Lambda_{2g_g}(\mu_1,\mu_2)}{1-\lambda\,\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g_\psi}(\mu_1,\mu_2)}\right]$$

Consider the examples of linear volterra integral equations of the first kind.

$$1)\ \int_0^{t_1}\int_0^{t_2} \Phi(t_1-x,t_2-y)\,\psi(x,y)\,dxdy = t_1e^{t_1-t_2} - t_1e^{t_1} \tag{2}$$

Utilizing the convolution theorem (3) and applying the double general Rangaig integral transform to both sides of Equation (2), we obtain

$$\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g}(\mu_1,\mu_2)\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{[p(\mu_1)+1]\,[q(\mu_2)-1]} = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{-1}{[p(\mu_1)+1]^2\,[q(\mu_2)-1]} - \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{-1}{[p(\mu_1)+1]^2\,[q(\mu_2)]}$$

$$\Lambda_{2g}(\mu_1,\mu_2)\frac{1}{[p(\mu_1)+1]\,[q(\mu_2)-1]} = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{[p(\mu_1)+1]^2}\left[\frac{1}{[q(\mu_2)]} - \frac{1}{[q(\mu_2)-1]}\right]$$

$$\Lambda_{2g}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{-1}{[p(\mu_1)+1]\,[q(\mu_2)]}$$

Apply inverse double general Rangaig integral transform on both sides, We get

$$\emptyset(t_1,t_2) =_{2g}^{-1}\left[\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{-1}{[p(\mu_1)+1]\,[q(\mu_2)]}\right]$$
$$\emptyset(t_1,t_2) = -e^{t_1}$$

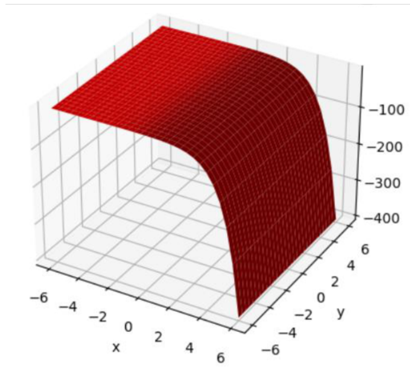

Fig 1. Contour graph for exact solution

$$2)\ a^2 t_2 = \int_0^{t_1}\int_0^{t_2} \emptyset(t_1 - x,\, t_2 - y)\, \emptyset(x,y)\, dxdy \tag{3}$$

Applying the double general Rangaig integral transform to both sides of Equation (4) and using the convolution theorem (3), we get

$$a^2 \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}} \frac{-1}{\left[p(\mu_1)\right]\left[q\left(\mu_2\right)\right]^2} = \mu_1^{n_1}.\mu_2^{n_2} \Lambda_{2g}^2\left(\mu_1,\mu_2\right)$$

After simplification, we get

$$\Lambda_{2g}\left(\mu_1,\mu_2\right) = \left[\frac{a^2}{\left[\mu_1^{n_1}.\mu_2^{n_2}\right]^2} \frac{-1}{\left[p(\mu_1)\right]\left[q\left(\mu_2\right)\right]^2}\right]^{\frac{1}{2}}$$

Apply inverse double general Rangaig integral transform on both sides, we get

$$\emptyset\left(t_1,t_2\right) = \frac{a}{\sqrt{\pi t_1}}$$

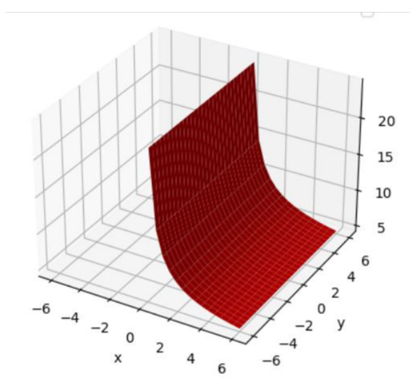

Fig 2. Contour graph for exact solution

Further, consider the linear volterra integral equations of the second kind.
Here, we solve some examples of linear volterra integral equations of the second kind**.**

$$3)\ \emptyset\left(t_1,t_2\right) = a - \lambda \int_0^{t_1}\int_0^{t_2} \Phi\left(x,y\right) dxdy \tag{4}$$

Where $a\ and\ \lambda$ are constant.

Applying the double general Rangaig integral transform to both sides of Equation (4) and using the convolution theorem (3), we get

$$\Lambda_{2g}(\mu_1,\mu_2) = \frac{a}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)\right]\left[q(\mu_2)\right]} - \lambda\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)\right]\left[q(\mu_2)\right]}\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g}(\mu_1,\mu_2)$$

$$\Lambda_{2g}(\mu_1,\mu_2) = \frac{a}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1).q(\mu_2)+\lambda\right]}$$

Apply inverse double general Rangaig integral transform on both sides, we get

$$\emptyset(t_1,t_2) = aJ_0\left(2\sqrt{\lambda t_1 t_2}\right)$$

Where $J_0(z)$ is Bessel's function of zero order.

$$4)\ \emptyset(t_1,t_2) = -1+e^{t_1}+e^{t_2}\int_0^{t_1}\int_0^{t_2}\Phi(x,y)\,dxdy \tag{5}$$

Utilizing the convolution theorem (3) and applying the double general Rangaig integral transform to both sides of Equation (5), we obtain

$$\begin{aligned}\Lambda_{2g}(\mu_1,\mu_2) = &\frac{-1}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)\right]\left[q(\mu_2)\right]} + \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)+1\right]\left[q(\mu_2)\right]} + \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)\right]\left[q(\mu_2)+1\right]}\\ &+ \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)\right]\left[q(\mu_2)\right]}\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g}(\mu_1,\mu_2)\end{aligned}$$

$$\left(1-\frac{1}{\left[p(\mu_1)\right]\left[q(\mu_2)\right]}\right)\Lambda_{2g}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\left(\frac{-1}{\left[p(\mu_1)\right]\left[q(\mu_2)\right]} + \frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)\right]} + \frac{1}{\left[p(\mu_1)\right]\left[q(\mu_2)+1\right]}\right]$$

$$\left(1-\frac{1}{\left[p(\mu_1)\right]\left[q(\mu_2)\right]}\right)\Lambda_{2g}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)+1\right]}\left(\frac{\left[p(\mu_1)\right]\left[q(\mu_2)\right]-1}{\left[p(\mu_1)\right]\left[q(\mu_2)\right]}\right)$$

$$\Lambda_{2g}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)+1\right]}$$

Apply inverse double general Rangaig integral transform on both sides, we get

$$\emptyset(t_1,t_2) = e^{t_1+t_2}$$

$$5)\ \emptyset(t_1,\,t_2) = e^{3t_1+2t_2}+e^{t_1+t_2}-\int_0^{t_1}\int_0^{t_2}2e^{3t_1+2t_2}\,\emptyset(x,y)\,dxdy \tag{6}$$

Applying the double general Rangaig integral transform to both sides of Equation (4) and using the convolution theorem (3), we get

$$\begin{aligned}\Lambda_{2g}(\mu_1,\mu_2) = &\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+3\right]\left[q(\mu_2)+2\right]} + \frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)+1\right]}\\ &-\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+3\right]\left[q(\mu_2)+1\right]}\\ &-2\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}\left[p(\mu_1)+3\right]\left[q(\mu_2)+1\right]}\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g}(\mu_1,\mu_2)\end{aligned}$$

$$\left(1+\frac{2}{\left[p(\mu_1)+3\right]\left[q(\mu_2)+1\right]}\right)\Lambda_{2g}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)+2\right]}\left(\frac{q(\mu_2)\left[p(\mu_1)+3\right]+p(\mu_1)+5}{\left[p(\mu_1)+3\right]}\right)$$

$$\left(1+\frac{2}{\left[p(\mu_1)+3\right]\left[q(\mu_2)+1\right]}\right)\Lambda_{2g}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)+2\right]}\left(q(\mu_2)+1+\frac{2}{\left[p(\mu_1)+3\right]}\right)$$

$$\Lambda_{2g}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]\left[q(\mu_2)+2\right]}$$

Apply inverse double general Rangaig integral transform on both sides, we get

$$\emptyset(t_1,t_2)=e^{t_1+2t_2}.$$

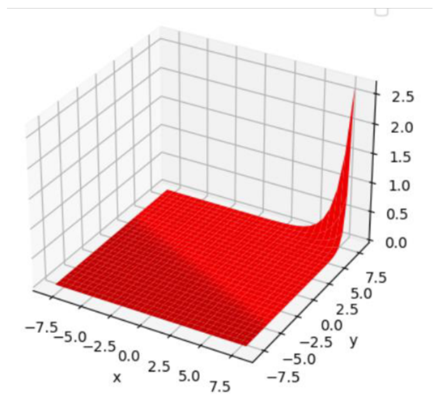

Fig 3. Contour graph for exact solution

3.1.2. Volterra Integro-Partial Differential Equations

6) Consider first-order volterra-integro partial differential equation

$$\frac{\partial\emptyset(t_1,t_2)}{\partial t_1}+\frac{\partial\emptyset(t_1,t_2)}{\partial t_2}=g(t_1,t_2)+\lambda\int_0^{t_1}\int_0^{t_2}\emptyset(t_1-x,t_2-y)\,\psi(x,y)\,dxdy \tag{7}$$

with conditions

$$\emptyset(t_1,0)=f_0(t_1),\ \emptyset(0,\,t_2)=h_0(t_2) \tag{8}$$

Where $\emptyset(t_1,t_2)$ is the unknown function, λ is a constant and $g(t_1,t_2)\&\psi(t_1,t_2)$ are two known functions.

Applying D.G.R.I.T. to both sides of Equation (7) and single G.R.I.T. to the initial conditions (Equation (8)), we get

$$\Lambda_{2g\emptyset}(\mu_1,\mu_2)=\frac{\Lambda_{2gg}(\mu_1,\mu_2)-\frac{1}{\mu_1^{n_1}}\Lambda_{gh_0}(\mu_2)-\frac{1}{\mu_2^{n_2}}\Lambda_{gf_0}(\mu_1)}{-p(\mu_1)-q(\mu_2)-\lambda\,\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g\psi}(\mu_1,\mu_2)}$$

$$\emptyset(t_1,t_2)=\ {}_{2g}^{-1}\left[\frac{\Lambda_{2gg}(\mu_1,\mu_2)-\frac{1}{\mu_1^{n_1}}\Lambda_{gh_0}(\mu_2)-\frac{1}{\mu_2^{n_2}}\Lambda_{gf_0}(\mu_1)}{-p(\mu_1)-q(\mu_2)-\lambda\,\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g\psi}(\mu_1,\mu_2)}\right] \tag{9}$$

Consider one example

$$\frac{\partial \emptyset (t_1,t_2)}{\partial t_1}+\frac{\partial \emptyset (t_1,t_2)}{\partial t_2}=-1+e^{t_1}+e^{t_2}+e^{t_1+t_2}+\int_0^{t_1}\int_0^{t_2}\emptyset (t_1-x,t_2-y)\,dxdy \tag{10}$$

with conditions

$$\emptyset (0,t_2)=e^{\ t_2}=h_0(t_2)\ \emptyset (0,t_2)=e^{\ t_2}=h_0(t_2) \tag{11}$$

Taking a single general range integral transform of Equation (11) and applying the D. G. R. I. transform to both sides of Equation (10),

$$\Lambda_{gf_0}(\mu_1)=\frac{1}{\mu_1^{\ n_1}}\frac{1}{p(\mu_1)+1}\ ,\ \Lambda_{gh_0}(\mu_2)=\frac{1}{\mu_2^{\ n_2}}\frac{1}{q(\mu_2)+1}\ ,$$

$$\Lambda_{2gg}(\mu_1,\mu_2)=\frac{-1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{p(\mu_1)q(\mu_2)}+\frac{1}{\mu_1^{n_1}.\mu_2^{n_2}}\frac{1}{\left[p(\mu_1)+1\right]q(\mu_2)}+\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{\left[q(\mu_2)+1\right]p(\mu_1)}$$
$$+\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{\left[q(\mu_2)+1\right]\left[p(\mu_1)+1\right]}$$

Substitute,

In Equation (9), we simply get the solution to Equation (10).

$$\emptyset (t_1,t_2)={}_{2g}^{-1}\left[\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{\left[q(\mu_2)+1\right]\left[p(\mu_1)+1\right]}\right]$$
$$=e^{t_1+t_2}$$

3.1.3. Partial Integro-Differential Equations

Consider the linear partial-integro differential equation of the form

$$\frac{\partial^2 \emptyset (t_1,t_2)}{\partial t_2^2}-\frac{\partial^2 \emptyset (t_1,t_2)}{\partial t_1^2}+\emptyset (t_1,t_2)+\int_0^{t_1}\int_0^{t_2}\psi (t_1-x,t_2-y)\emptyset (x,y)\,dxdy=g(t_1,t_2) \tag{12}$$

with conditions

$$\emptyset (t_1,0)=f_0(t_1)\ ,\frac{\partial \emptyset (t_1,0)}{\partial t_2}=f_1(t_1)\ \ ,\emptyset (0,t_2)=h_0(t_2)\,\&\frac{\partial \emptyset (0,t_2)}{\partial t_1}=h_1(t_2) \tag{13}$$

Applying double general Rangaig integral transform to both sides of Equation (12) & single general Rangaig integral transform to Equation (13) we get,

$$\Lambda_{2g\emptyset}(\mu_1,\mu_2)=\frac{\Lambda_{2gg}(\mu_1,\mu_2)-\frac{1}{\mu_2^{n_2}}\Lambda_{gf_1}(\mu_1)+\frac{q(\mu_2)}{\mu_2^{n_2}}\Lambda_{gf_0}(\mu_1)+\frac{1}{\mu_1^{n_1}}\Lambda_{gh_1}(\mu_2)-\frac{p(\mu_1)}{\mu_1^{n_1}}\Lambda_{gh_0}(\mu_2)}{\left[p(\mu_1)\right]^2+\left[q(\mu_2)\right]^2+1+\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g\psi}(\mu_1,\mu_2)}$$

$$\emptyset (t_1,t_2)={}_{2g}^{-1}\left[\frac{\Lambda_{2gg}(\mu_1,\mu_2)-\frac{1}{\mu_2^{n_2}}\Lambda_{gf_1}(\mu_1)+\frac{q(\mu_2)}{\mu_2^{n_2}}\Lambda_{gf_0}(\mu_1)+\frac{1}{\mu_1^{n_1}}\Lambda_{gh_1}(\mu_2)-\frac{p(\mu_1)}{\mu_1^{n_1}}\Lambda_{gh_0}(\mu_2)}{\left[p(\mu_1)\right]^2+\left[q(\mu_2)\right]^2+1+\mu_1^{n_1}.\mu_2^{n_2}\Lambda_{2g\psi}(\mu_1,\mu_2)}\right] \tag{14}$$

Consider one example

$$\frac{\partial^2 \emptyset (t_1,t_2)}{\partial t_2^2}-\frac{\partial^2 \emptyset (t_1,t_2)}{\partial t_1^2}+\emptyset (t_1,t_2)+\int_0^{t_1}\int_0^{t_2}e^{(t_1-x)+(t_2-y)}\emptyset (x,y)\,dxdy=e^{t_1+t_2}+t_1t_2e^{t_1+t_2} \tag{15}$$

with conditions $\emptyset(t_1,0) = e^{t_1} = f_0(t_1)$

$$\frac{\partial\emptyset(t_1,0)}{\partial t_2} = e^{t_1} = f_1(t_1), \emptyset(0,t_2) = e^{t_2} = h_0(t_2) \& \frac{\partial\emptyset(0,t_2)}{\partial t_1} = e^{t_2} = h_1(t_2) \tag{16}$$

D. G. R. I. transform applied to both sides of Equation (15); Consider only one general Rangaig integral transform of Equation (16).

$$\Lambda_{gf_0}(\mu_1) = \Lambda_{gf_1}(\mu_1) = \frac{1}{\mu_1^{n_1}} \frac{1}{\left[p(\mu_1)+1\right]}$$

$$\Lambda_{gh_0}(\mu_2) = \Lambda_{gh_1}(\mu_2) = \frac{1}{\mu_2{}^{n_2}} \frac{1}{q(\mu_2)+1}$$

$$\Lambda_{2gg}(\mu_1,\mu_2) = \frac{1}{\mu_1^{n_1}\mu_2^{n_2}} \frac{1}{\left[q(\mu_2)+1\right]\left[p(\mu_1)+1\right]} + \frac{1}{\mu_1^{n_1}\mu_2^{n_2}} \frac{1}{\left[q(\mu_2)+1\right]^2\left[p(\mu_1)+1\right]^2}$$

substitute, in Equation (14), we simply get the solution to Equation (15).

$$\emptyset(t_1,t_2) = {}_{2g}^{-1}\left[\frac{1}{\mu_1^{n_1}\mu_2^{n_2}} \frac{1}{\left[q(\mu_2)+1\right]\left[p(\mu_1)+1\right]}\right] = e^{t_1+t_2}$$

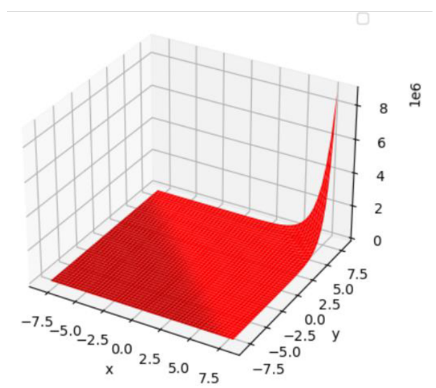

Fig 4. Contour graph for exact solution of example 4, 6, 7

4 Discussion

The double general Rangaig integral transform was effectively employed to solve linear integral and partial-integro differential equations, yielding an exact solution. View the exact solution graph in Figures 1, 2 and 3 and Figure 4. The double general Rangaig integral transform is defined on a novel domain in comparison to other double general integral transforms that are currently in use. Therefore, in comparison to the current double general Rangaig integral transform, we may simply change an integral equation into an algebraic equation to obtain an exact solution with less processing effort.

5 Conclusion

This study has successfully defined the existence condition for the convolution theorem and the double general Rangaig integral transform in this study. This study effectively applies the double general Rangaig integral transform to partial integro-differential equations and linear volterra integral equations, achieving accurate solutions. Compared to other integral transforms previously in use, it is found that employing the double general Rangaig integral transform made it easier to acquire exact solutions for integral differential equations.

6 Acknowledgement

The authors are thankful to the Department of Mathematics and Library, K.R.T. Arts, B.H. commerce and A. M. Science College, Nashik for providing reference resources and internet facility.

References

1) Debnath L, Bhatta D. Integral transform and their application second Edition. Chapman & Hall/CRC. 2006. Available from: https://doi.org/10.1201/9781420010916.
2) Soliman AM, Raslan K, Abdallah AM. Importance of RG Transform and Its Various applications. *Progress in Fractional Differentiation and Applications*. 2023;9:655–663. Available from: https://doi.org/10.18576/pfda/090409.
3) Soliman AM, Raslan K, Abdallah AM, Ramadan. Ramadan Group Transform Fundamental Properties and Some its Dualities. 2022. Available from: https://doi.org/10.1007/978-3-031-15784-4_23.
4) Ramadan MA, Hadhoud AR. Double Ramadan group integral transform: definition and properties with applications to partial differential equations. *Appl Math Inf Sci*. 2018;12:389–396. Available from: http://dx.doi.org/10.18576/amis/120213.
5) Alfaqeih S, Misirli E. On Double Shehu Transform and Its Properties with Applications. *International Journal of Analysis and Applications*. 2020;18:381–395. Available from: https://doi.org/10.28924/2291-8639.
6) Meddahi M, Jafari H, Yang XJ. Towards new general double integral transform and its applications to differential equations. *Math Meth Appl Sci*. 1916;45. Available from: http://dx.doi.org/10.1002/mma.7898.
7) Patil DP. Dualities between Double Integral Transforms. *International Advanced research Journal in Science, Engineering and Technology*. 2020;7:74–82. Available from: http://dx.doi.org/10.17148/IARJSET.2020.7610.
8) Patil DP. Double Mahgoub transform for the solution of parabolic boundary value problems. *Journal of Engineering Mathematics and Statistics*. 2020;4:28–36. Available from: https://dx.doi.org/10.2139/ssrn.4145866.
9) Patil DP, Shinde PD, Tile GK. Volterra integral equations of first kind by using Anuj transform. *International Journal of Advances in Engineering and Management*. 2022;4:917–920. Available from: https://dx.doi.org/10.35629/5252-0405917920.
10) Patil DP, Nikamand PS, Shinde PD. Kushare transform in solving Faltung Type Volterra Integro-Differential Equation of first kind. *International Advanced Research Journal in Science, Engineering and Technology*. 2022;8:84–91. Available from: http://dx.doi.org/10.17148/IARJSET.2022.91013.
11) Ahmed SA, Elzaki TM, Hassan AA. Solution of Integral Differential Equations by New Double Integral Transform (Laplace-Sumudu Transform), . *Hindawi Abstract and Applied Analysis*. 2020;2020:1–07. Available from: http://dx.doi.org/10.1155/2020/4725150.
12) Mansoura EA, Kuffi EA. Generalization of Rangaig transform. *Int J Nonlinear Anal Appl*. 2022;13:2227–2231. Available from: http://dx.doi.org/10.22075/ijnaa.2022.5919.
13) Derle MS, Patil DP, Rahane NK. On Generalized Double Rangaig Intergral Transform and Applications. *Stochastic Modelling& Applications*. 2022;26:533–545. Available from: https://www.researchgate.net/publication/361902166_On_Generalized_Double_Rangaig_Integral_Transform_and_Applications.
14) Qazza A. Solution of integral equations via Laplace ARA transform. *European Journal of pure and Applied mathematics*. 2023;16:919–933. Available from: https://dx.doi.org/10.29020/nybg.ejpam.v16i2.4745.
15) Saadeh R. Application of the ARA Method in Solving Integro-Differential Equations in Two Dimensions. *Computation*. 2023;11:1–11. Available from: https://doi.org/10.3390/computation11010004.
16) Saadeh R, Qazza A, & A Burqan. On the Double ARA-Sumudu transform and its applications. *Mathematics*. 2023;p. 1–19. Available from: https://doi.org/10.3390/math10152581.
17) Rania S, Mustafa MM. The Double ARA-Formable Transform with Applications. *Appl Math Inf Sci*. 2023;17:685–697. Available from: https://doi.org/10.18576/amis/170417.
18) Soliman AA, Raslan KR, Abdallah AM. On Fractional Integro-differential Equation with Nonlinear Time Varying Delay,. *Sound & Vibration*. 2022;56:147–163. Available from: https://doi.org/10.32604/sv.2022.015882.