INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS

Received: 28-06-2024

Accepted: 15-07-2024

Published: 02-08-2024

Citation: Islam DU, Shirivatasava A (2024) Solving Linear Second Kind Non-Homogenous Volterra Integral Equations By Numerical Methods VIM, ADM And TSM. Indian Journal of Science and Technology 17(31): 3165-3173. https://doi.org/10.17485/IJST/v17i31.2099

[*] **Corresponding author**.

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Solving Linear Second Kind Non-Homogenous Volterra Integral Equations By Numerical Methods VIM, ADM And TSM

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Abstract

Objectives: For the purpose of evaluating and determining the solution of the integral equations, various numerical methods is used to obtain the optimum solution of such equations. **Methods:** We developed numerical methods VIM, ADM and TSM to solve the linear second kind non-homogenous Volterra Integral Equations. **Findings:** Based on the findings of the study, implementing numerical methods VIM, ADM, and TSM are very helpful to obtain the solutions of linear second kind non-homogenous Volterra Integral Equations. Further numerical examples illustrate the accuracy of the Adomian Decomposition Method, Variational Iteration Method and Taylors Series Method. **Novelty:** Our research highlights the importance of the numerical methods VIM, ADM add TSM as these methods are quick convergence of solutions. Furthermore, our analysis led us to the determination that any integral equations can be solved by these methods easily and all the methods led us approximately at one point.

Keywords: Volterra Integral Equation; NonHomogenous Volterra Integral Equations; Adomian Decomposition Method; Variational Iteration Method; Taylors Series Method

1 Introduction

Integral equations form one of the most appropriate techniques in many branches of pure analysis such as the theories of functional analysis and stochastic process. It is one of the most significant branches of mathematical analysis, predominantly on account of its importance in boundary value problems, in the theory of ordinary and partial differential equations. Integral equations arise in many fields of mechanics and mathematical physics. They are also linked with the problems in mechanical vibrations, theory of analytic functions, orthogonal system, and theory of quadratic forms of infinitely many variables.

The analysis of these numerical methods is a multifaceted endeavor important for understandings and solving a wide range of glitches across abundant scientific disciplines. Through theoretical examination and computational methods, researchers unravel the workings of these equations, shedding light on fundamental principles governing real-world phenomena. By linking theory and application, this research enables advancement in fields as diverse as physics, biology, engineering and economics, offering solution to complex tests and driving innovation forward. Ultimately the study of numerical methods stands as anevidence to the power of mathematical analysis in shaping our understandings of the world and evolving scientific knowledge.

The Adomian Decomposition Method (ADM) and Modified Adomian Decomposition Method(MADM) are introduced to provide the approximate solution for Volterra-Fredholm integro-differential equations. Moreover it proves unique results and convergence of the solution [1]. Novel mixture of Aboodh Transform and Adomian Decomposition Method (ADM)are used to solve the nonlinear Integro-differential equations and the solution are given as rapidly expanding series of terms [2]. Numerical methods ADM and VIM are used to solve Fredholm Integral Equations [3]. Volterra Integral Equation of the first kind is transformed into Volterra Integral Equation of the second kind by using Leibnitz rule and then apply Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) to solve Volterra Integral Equation of the second kind [4]. Volterra Integral Equations, characterized by their lower integration limit being variable, often model systems with memory effects, such as population dynamics and viscoelastic materials. Conversely, Fredholm Integral Equations have fixed integration limits and are instrumental in resolving problems related to potential theory, quantum mechanics, and boundary value problems [5] .Variational Iteration Method is applied to find the approximation of analytical solution for linear and nonlinear integral equations [6]. Adomian Decomposition Method (ADM) is used to solve the Volterra Integral type, firstly analytical points are clarified and then used the operator form obtained to solve problem of Volterra Integral type [7]. Numerical solution of Fredholm Integral Equations are obtained by using the analytical method Adomian Decomposition Method (ADM). Accuracy and effectiveness of this method is also illustrated by some numerical examples [8]. ADM and MADM is used to solve the nonlinear Volterra Integral Equations, also some differences between the two methods is done for same class of numerical equations [9]. Volterra Integral Equation has been decomposed into an infinite series of components via Adomian Decomposition Method (ADM) [10].

2 Methodology

2.1 Volterra Integral Equation

An integral equation is an equation in which an unknown function to be determined, appears under one or more integral signs.

An integral equation is titled linear if only linear operations are accomplished in it upon the unknown functions, i.e., the equation in which no non-linear function of the unknown function are involved e.g.

$$\varphi(z) = f(z) + \lambda \int_a^b K(z,\xi)\, \varphi\xi d\xi \tag{1}$$

$$\varphi(z) = \lambda \int_a^b K(z,\xi)\, \varphi\xi d\xi \tag{2}$$

is a Linear integral equation.

And

$$\varphi(\mathrm{z}) = \mathrm{f}(\mathrm{z}) + \lambda \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{K}(\mathrm{z},\xi)\varphi(\xi)^3\, \mathrm{d}\xi \tag{3}$$

is a non-linear integral equation

An integral equation is supposed to be a Volterra Integral Equation if the upper limit of integration is a variable e.g.$a(z)\,\varphi(z) = f(z) + \lambda \int_a^z K(z,\xi)\, \varphi\xi d\xi$

• When $a = 0$, the unidentified function φ appears only under the integral sign and nowhere else in the equation , then

$$f(z) = \lambda \int_a^z K(z,\xi)\, \varphi\xi d\xi,\ a > -\infty$$

is called the Volterra's integral equation of first kind

• When $a = 1$,the equation contains the unknown function $\varphi, both$ inside as well as the outside the integral sign, then

$$\varphi(z) = f(z) + \lambda \int_a^z K(z,\xi)\, \varphi\xi d\xi$$

is called the Volterra's Integral Equation of the second kind

• When $a = 1$,$F(z) = 0$, the equation decreases to

$$\varphi(z) = \lambda \int_a^z K(z,\xi)\varphi\xi d\xi$$

is called the homogeneous Volterra's integral equation of the second kind

2.2 Proposed Adomian Decomposition Method for Solving Linear Second Kind Non-Homogenous Volterra Integral Equations

Consider the VolterraIntegral equation of the first kind

$$g(x) = \gamma \int_0^x K_1(x,t)\, u(t)\, dt$$

Applying Leibnitz's rule,

$$\frac{g'(x)}{\gamma} = \int_0^x \frac{d}{dx}(K_1(x,t)\, u(t)]\, dt + (K_1(x,x)\, u(x)]\frac{d}{dx}(x) - (K_1(x,0)\, u(0)] . \frac{d}{dx}(0)$$

we obtain Volterra Integral Equation of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)\, u(t)\, dt$$

Where$f(x) = \frac{g'(x)}{\gamma . K(x,x)}$and $K(x,t) = \frac{\frac{d}{dx}K(x,t)}{K(x,x)}$.

Adomian Decomposition Method consists of decomposing the unknown function $u(x)$ of any Volterra Integral Equation into a sum of an infinite number of components by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{5}$$

Where the constituents$u_n(x) n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $u_o(x), u_1(x), u_2(x), \cdots$independently. The determination of these components can be completed in an easy way through a recurrence relation that usually comprises simple integrals that can be easily calculated. To establish the recurrence relation, we substitute (5) into the Volterra Integral Equation.

$$u(x) = f(x) + \gamma \int_0^x K(x,t)\, u(t)\, dt \tag{6}$$

We achieve

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \gamma \int_0^x K(x,t) \sum_{n=0}^{\infty} u_n t dt \tag{7}$$

With u_0 identified as all terms out of the integral sign. Consequently, the components $u_n(t), n \geq 1$ of the unidentified function $u(x)$ are completely determined in a recurrent manner if we set

$$u_o(x) = f(x),$$

$$u_{n+1}(x) = \gamma \int_0^x K(x,t)\, u_n(t)\, dt, n \geq 0 \tag{8}$$

That is equivalent to

$$u_0(x) = f(x)$$

$$u_1(x) = \gamma \int_0^x K(x,t)\, u_0(t)\, dt$$

$$u_2(x) = \gamma \int_0^x K(x,t) u_1(t) dt, \tag{9}$$

$$u_3(x) = \gamma \int_0^x K(x,t) u_2(t) dt$$

$$\vdots$$

and so on for the other constituents. Thus, the components $u_0(x), u_1(x), u_2(x), \cdots$

are completely determined. As a result the solution $u(x)$ (6)in a series form is obtained by using the series (5)

2.3 Proposed Variational Iteration Method for Solving Linear Second Non-Homogenous Volterra Integral Equations

Consider the Volterra Integral Equation of the first kind given as:

$$f(x) = \gamma \int_0^x K(x,t) u(t) dt \tag{10}$$

Applying Leibnitz‘s rule, we obtain

$$\frac{f'(x)}{\gamma} = \int_0^x \frac{d}{dx} (K(x,t) u(t)] dt + (K(x,x) u(x)] \frac{d}{dx}(x) - (K(x,0) u(0)] . \frac{d}{dx}(0)$$

After simplification, we obtain Volterra Integral Equation of the second kind given as

$$u(x) = g(x) + \int_0^x K_1(x,t) u(t) dt \tag{11}$$

where $g(x) = \frac{f'(x)}{\gamma . K(x,x)}$

and $K_1(x,t) = \frac{\frac{d}{dx} K(x,t)}{K(x,x)}$

which is the integral equation of the second kind. Then, apply the Variational Iteration Method on equation (11) which provides the solution of the equation (10) as discussed in above Section.

2.4 Proposed Taylors Series Method for Solving Linear Second Kind Non-Homogenous Volterra Integral Equations

The linear second kind non-homogenous V.I.E. is given as

$$u(x) = f(x) + \int_0^x K(x,t) u(t) dt \tag{12}$$

Suppose the solution $u(x)\ of\ equation$ (12) is analytic so it can be characterize in the form of Taylor's series as

$$u(x) = \sum_{n=0}^{\infty} \beta_n x^n \tag{13}$$

Use equation (13) in equation (12) we get

$$\sum_{n=0}^{\infty} \beta_n x^n = T(f(x)) + \int_0^x K(x,t) \sum_{n=0}^{\infty} \beta_n x^n dt \tag{14}$$

Where $T(f(x))$ is the Taylor series expansion of the Function $f(x)$

Equation (14) can be written as

$$[\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \ldots] = T(f(x)) + \int_0^x K(x,t) \sum_{n=0}^{\infty} \beta_n x^n dtdt \tag{15}$$

On simplifying, (15) gives a system of algebraic equations in terms of$\beta_0, \beta_1, \beta_2, \beta_3 \ldots$. After solving this system we get a chain of coefficients namely $\beta_o, \beta_1, \beta_2, \beta_3 \ldots$. The required solution of equation (12) may be acquired by using these coefficients in equation (13)

3 Result and Discussion

Example 1. Consider a linear second kind non-homogenous Volterra Integral Equation

$$u(x) = 1 + x + \int_0^x (x-t)\, u(t)\, dt \tag{16}$$

with an exact solution $u_n(x) = e^x$

By Adomian Decomposition Method

By applying the Adomian Decomposition Method on equation (16) we get

$u_0(x) = 1 + x$

$u_1(x) = \int_0^x [(x-t)1 + t]dt$

$u_1(x) = \int_0^x (x + xt - t - t^2]\, dt$

$u_1(x) = \frac{x^2}{2} + \frac{x^3}{6}$

$u_2(x) = \int_0^x [(x-t)\frac{t^2}{2} + \frac{t^3}{6}]dt$

$u_2(x) = \int_0^x \left(\frac{xt^2}{2} + \frac{xt^3}{6} - \frac{t^3}{2} - \frac{t^4}{6}\right]\, dt$

$u_2(x) = \frac{x^4}{24} + \frac{x^5}{120}$

and so on

$u_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$

$u_n(x) = e^x$

which is the exact solution of the equation (16).

By Variational Iteration Method

By applying the Variational Iteration Method on equation (16) we get $u_0(x) = 1 + x$

$u_1(x) = 1 + x + \int_0^x [(x-t)1 + t]dt$

$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$u_2(x) = 1 + x + 1 + x + \int_0^x (x-t)1 + t + \frac{t^2}{2} + \frac{t^3}{6}dt$

$u_2(x) = 1 + x + \int_0^x x + xt + x\frac{t^2}{2} + x\frac{t^3}{6} - t - t^2 - \frac{t^3}{2} - \frac{t^4}{6}dt$

$u_2(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$

and so on

$u_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$

$u_n(x) = e^x$

which is the exact solution of the equation (16).

By Taylors Series Method

Suppose the solution of equation (16) is analytic so it can be represent in the form of Taylors series as

$$u(x) = \sum_{n=0}^{\infty} \beta_n x^n \tag{17}$$

Use equation (16) in equation (17) we get

$$\sum_{n=0}^{\infty} \beta_n x^n = 1 + x + \int_0^x (x-t) \sum_{n=0}^{\infty} \beta_n t^n\, dt \tag{18}$$

Equation (18) can be written as

$$\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \ldots = 1 + x + \int_0^x (x-t)\left(\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \ldots\right) dt \tag{19}$$

Now on simplification, (19) gives a system of following algebraic equations

$$\left.\begin{array}{c} \beta_0 = 1 \\ \beta_1 = 1 \\ \beta_2 = \beta_0 - \dfrac{\beta_0}{2} \\ \beta_3 = \dfrac{\beta_1}{2} - \dfrac{\beta_1}{3} \\ \beta_4 = \dfrac{\beta_2}{3} - \dfrac{\beta_2}{4} \\ \beta_5 = \dfrac{\beta_3}{4} - \dfrac{\beta_3}{5} \end{array}\right\} \tag{20}$$

After solving the above equation we get

$$\left.\begin{array}{l}\beta_0 = 1\\ \beta_1 = 1\\ \beta_2 = \frac{1}{2}\\ \beta_3 = \frac{1}{6}\\ \beta_4 = \frac{1}{24}\\ \beta_5 = \frac{1}{120}\end{array}\right\} \quad (21)$$

Using equation (21) in equation (17) we get the required solution of equation (16) given by

$$u_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$$

$u_n(x) = e^x$

which is the exact solution of equation (16)

Example 2. Consider a linear second kind non-homogenous Volterra Integral Equation

$$u(x) = x - \int_0^x (x-t)\, u(t) dt \quad (22)$$

With the exact solution $u_n(x) = sinx$

By Adomian Decomposition Meyhod

Applying Adomian Decomposition Method on Equation (22) we observe

$u_0(x) = x$

$u_1(x) = -\int_0^x (x-t)\, t dt$

$u_1(x) = \int_0^1 -xt + t^2 dt$

$u_1(x) = -\frac{x^3}{6}$

$u_2(x) = -\int_0^x [(x-t)(-t^3)] dt$

$u_2(x) = -\int_0^x -\frac{xt^3}{6} + \frac{t^4}{6} dt$

$u_2(x) = \int_0^x \frac{xt^3}{6} - \frac{t^4}{6} dt$

$u_2(x) = \frac{x^5}{24} - \frac{x^5}{30}$

$u_2(x) = \frac{x^5}{120}$

and so on

$u_n(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$

$u_n(x) = sinx$

which is the exact solution of the equation (22)

By Variational Iteration Method

By applying Variational Iteration Method on equation (22) we observe

$u_0(x) = x$

$u_1(x) = x + \int_0^x (x-t)\, t dt$

$u_1(x) = x - \int_0^x (xt - t^2)\, dt$

$u_1(x) = x - \frac{x^3}{6}$

$u_2(x) = x - \int_0^x [(x-t)(t - \frac{x^3}{6})] dt$

$u_2(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$

and so on

$u_n(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$

$u_n(x) = sinx$

which is the exact solution of the equation (22)

By Taylors Series Method

Suppose the solution of equation (22) is analytic so it can be represent in the form of Taylors series as

$$u(x) = \sum_{n=0}^{\infty} \beta_n x^n \tag{23}$$

Use equation (22) in equation (23) we get

$$\sum_{n=0}^{\infty} \beta_n x^n = x - \int_0^x (x-t) \sum_{n=0}^{\infty} \beta_n t^n dt \tag{24}$$

Equation (24) can be written as

$$\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \ldots = x - \int_0^x (x-t)\left(\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \ldots\right) dt \tag{25}$$

Now on simplification, (25) gives a system of following algebraic equations

$$\left.\begin{array}{c} \beta_0 = 0 \\ \beta_1 = 1 \\ \beta_2 = -\beta_0 + \dfrac{\beta_0}{2} \\ \beta_3 = -\dfrac{\beta_1}{2} + \dfrac{\beta_1}{3} \\ \beta_4 = -\dfrac{\beta_2}{3} + \dfrac{\beta_2}{4} \\ \beta_5 = -\dfrac{\beta_3}{4} + \dfrac{\beta_3}{5} \end{array}\right\} \tag{26}$$

After solving the above equation we get

$$\left.\begin{array}{l} \beta_0 = 0 \\ \beta_1 = 1 \\ \beta_2 = 0 \\ \beta_3 = -\dfrac{1}{6} \\ \beta_4 = 0 \\ \beta_5 = \dfrac{1}{120} \end{array}\right\} \tag{27}$$

Using equation (27) in equation (23) we get the required solution of equation (22) given by
$u_n(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$

$$u_n(x) = sinx$$

which is the exact solution of equation (22)

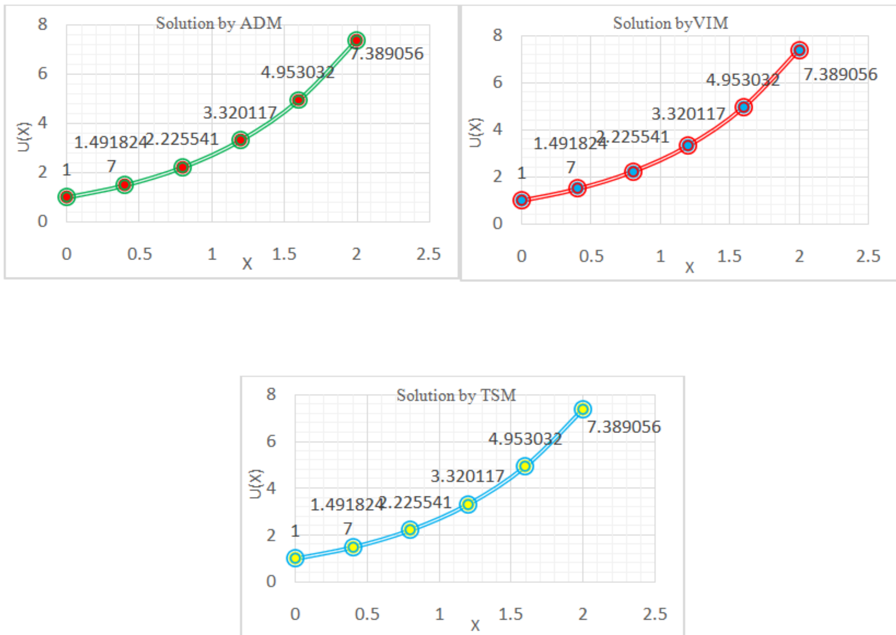

Fig 1. Graphical representation of Numerical Example 1

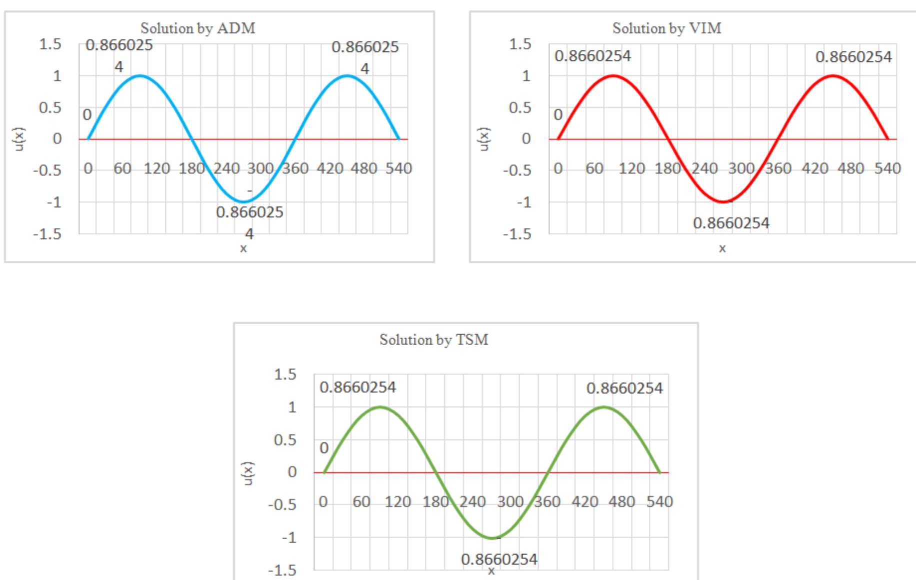

Fig 2. Graphical representation of Numerical Example 2

It can be clearly seen from the graphical representation of numerical examples mentioned above that all the numerical methods ADM, VIM and TSM converge to approximately at one point.

4 Conclusion

In this work, ADM, TSM, and VIM were efficaciously functional to solve the linear second kind non-homogenous Volterra Integral Equations. It is overt that VIM, ADM and TSM are very powerful and effective method for finding the solution for wide classes of problems. These numerical methods needs much less computational effort as matched with the traditional methods. Some of the benefits of the schemes over the existing methods are that, it is easy to use, it is precise, and effective also. It is worth noting that these numerical methods are rapid and converge exactly at one point.

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